We obtained an integro-differential equation in Ch 8 (Eq8.23)
\[
\frac{dl_a}{ds} = \mathbf{s} \cdot \nabla I_{\eta} = \kappa \eta I_{\eta} - \beta \eta I_{\eta} (\mathbf{s}) + \frac{\sigma_{\eta}}{4\pi} \int I_{\eta}(\mathbf{s}', \Phi_{\eta}) (\mathbf{s}', \mathbf{s}) d\Omega,
\]

Which depends on 3 space coordinates \((\mathbf{r})\) and two direction coordinates \((\theta, \phi)\).

Additional complexity may come, if:
- Medium is non gray (spectral variable)
- Other modes of heat transfer exist (simultaneous soln and nonlinear on temperature)

Exact analytical solutions exist for only a few extremely simple situations.

The simplest case: 1-D plane parallel gray medium with radiative equilibrium or known temperature field.

Governing equation for the intensity field in an absorbing, emitting and scattering medium (Eq8.23):
\[
\mathbf{s} \cdot \nabla I = \kappa I - \beta I + \frac{\sigma_s}{4\pi} \int I(\mathbf{s}') \Phi(\mathbf{s}, \mathbf{s}') d\Omega,
\]

with a solution:
\[
I(\mathbf{r}, \mathbf{s}) = I_w(\mathbf{s}) e^{-\tau} + \int_0^{\tau} S(\tau', \mathbf{s}) e^{-(\tau - \tau')} d\tau',
\]

Plane parallel medium:

Both plates are isothermal and isotropic
Only \(\theta\) angle dependence (1-D medium)

Radiative source at position \(Q\) : \(S(\tau', \theta)\) where \(\tau' = \int_0^\tau \beta dz\)

Radiative source at position \(Q_s\) : \(S(\tau', \theta)\)
Here \(S(\tau', \theta)\) and \(S(\tau', \theta)\) are the same. Therefore \(S(\tau, \theta)\) and \(I(\tau, \theta)\) depend only on \(\tau\) and \(\theta\).

Therefore \(S(\tau', \mathbf{s}) \rightarrow S(\tau', \theta)\)
\[
S(\tau', \theta) = (1 - \omega)I_b(\tau') + \frac{\omega}{4\pi} \int_{\phi_i = 0}^{2\pi} \int_{\theta_i = 0}^{\pi} I(\tau', \theta_i) \Phi(\theta, \phi, \theta_i, \phi_i) \sin \theta_i d\theta_i d\phi_i.
\]

This source term may further simplify depending on the nature of \(\Phi\).
DEPENDING ON TYPE OF SCATTERING:

- **Isotropic scattering**: \( \Rightarrow \Phi = 1 \) therefore
  \[
  S(\tau') = (1 - \omega)I_b(\tau') + \frac{\omega}{4\pi}G(\tau'). \tag{12.5}
  \]

- **Anisotropic scattering**: \( \Rightarrow \)
  \[
  \Phi(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_i) = 1 + \sum_{n=1}^{M} A_n P_n(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_i). \tag{12.6}
  \]

  \[
  \hat{\mathbf{s}} = \sin \theta (\cos \psi \hat{i} + \sin \psi \hat{j}) + \cos \theta \hat{k}, \tag{12.7}
  \]

  \[
  \hat{\mathbf{s}}_i = \sin \theta_i (\cos \psi_i \hat{i} + \sin \psi_i \hat{j}) + \cos \theta_i \hat{k}, \tag{12.8}
  \]

  and

  \[
  \Phi(\theta, \psi, \theta_i, \psi_i) = 1 + \sum_{n=1}^{M} A_n P_n[\cos \theta \cos \theta_i + \sin \theta \sin \theta_i \cos(\psi - \psi_i)]. \tag{12.9}
  \]

  where:

  \[
  P_n[\cos \theta \cos \theta_i + \sin \theta \sin \theta_i \cos(\psi - \psi_i)] = P_m(\cos \theta)P_m(\cos \theta_i) + 2 \sum_{n=1}^{m} \frac{(m - n)!}{(m + n)!} P^m_n(\cos \theta)P^m_n(\cos \theta_i) \cos m(\psi - \psi_i). \tag{12.10}
  \]

  therefore:

  \[
  \Phi(\theta, \psi, \theta_i, \psi_i) = 1 + \sum_{n=1}^{M} P_n(\cos \theta)P_n(\cos \theta_i) + 2 \sum_{m=1}^{M} \sum_{n=1}^{m} \frac{A_n}{(m + n)!} P^m_n(\cos \theta)P^m_n(\cos \theta_i) \cos m(\psi - \psi_i). \tag{12.11}
  \]

  For 1-D plane parallel medium:

  \[
  \Phi(\theta, \theta_i) = \frac{1}{2\pi} \int_{0}^{2\pi} \Phi(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_i) \, d\psi_i = 1 + \sum_{m=1}^{M} A_m P_m(\cos \theta)P_m(\cos \theta_i). \tag{12.12}
  \]

  therefore:

  \[
  S(\tau', \theta) = (1 - \omega)I_b(\tau') + \frac{\omega}{2} \int_{0}^{\pi} I(\tau', \theta_i) \Phi(\theta, \theta_i) \sin \theta_i \, d\theta_i. \tag{12.13}
  \]

- **For linear anisotropic scattering**: \( \Phi(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_i) = 1 + A_1 P_1(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_i) = 1 + A_1 \hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_i, \quad M = 1, \tag{12.14} \)

  Therefore:

  \[
  S(\tau', \theta) = (1 - \omega)I_b(\tau') + \frac{\omega}{4\pi} \left[ G(\tau') + A_1 q(\tau') \cos \theta \right]. \tag{12.15}
  \]
Simplification of Radiative transfer equation for 1-D plane parallel medium:

Using $\tau_s = \tau / \cos \theta$ and $\tau_s' = \tau' / \cos \theta$:

$$\frac{1}{\beta} \frac{dI}{ds} = \frac{dI}{d\tau} = \cos \theta \frac{dI}{d\tau} = (1 - \omega)I_b - I + \frac{\omega}{2} \int_0^\pi I(\tau, \theta) \Phi(\theta, \theta) \sin \theta, d\theta. \quad (12.16)$$

Also expression for intensity:

$$I(\vec{r}, \hat{s}) = I_w(\hat{s}) e^{-\tau_t} + \int_0^\tau S(\tau_0', \hat{s}) e^{-(\tau_t - \tau_0')} d\tau_0'$$

may be simplified to get for intensities from lower wall in positive directions:

$$I^+(\tau, \theta) = I_1(\theta) e^{-\tau/\cos \theta} + \int_0^\tau S(\tau', \theta) e^{-(\tau - \tau')/\cos \theta} \frac{d\tau'}{\cos \theta}, \quad 0 < \theta < \frac{\pi}{2}. \quad (12.17)$$

For negative directions using $\tau_s = -(\tau_{L} - \tau) / \cos \theta$ and $\tau_s' = -(\tau_{L} - \tau') / \cos \theta$:

$$I^-(\tau, \theta) = I_2(\theta) e^{\tau_{L} - \tau/\cos \theta} + \int_0^\tau S(\tau', \theta) e^{-(\tau - \tau')/\cos \theta} \frac{d\tau'}{\cos \theta}$$

$$= I_2(\theta) e^{\tau_{L} - \tau/\cos \theta} - \int_0^\tau S(\tau', \theta) e^{(\tau - \tau')/\cos \theta} \frac{d\tau'}{\cos \theta}, \quad \frac{\pi}{2} < \theta < \pi. \quad (12.18)$$

Defining $\mu = \cos \theta$:

$$\mu \frac{dI}{d\tau} + I = (1 - \omega)I_b + \frac{\omega}{2} \int_1^\tau I(\tau, \mu) \Phi(\mu, \mu) \, d\mu. \quad S(\tau, \mu), \quad (12.19)$$

$$I^+(\tau, \mu) = I_1(\mu) e^{-\tau/\mu} + \int_0^\tau S(\tau', \mu) e^{-(\tau - \tau')/\mu} \frac{d\tau'}{\mu}, \quad 0 < \mu < 1. \quad (12.20a)$$

$$I^-(\tau, \mu) = I_2(\mu) e^{(\tau_{L} - \tau)/\mu} - \int_\tau^\tau S(\tau', \mu) e^{(\tau - \tau')/\mu} \frac{d\tau'}{\mu}, \quad -1 < \mu < 0. \quad (12.20b)$$
Therefore incident radiation:

\[
G(\tau) = \int_{0}^{2\pi} \int_{0}^{\pi} I(\tau, \theta) \sin \theta d\theta d\phi = 2\pi \int_{-1}^{+1} I(\tau, \mu) d\mu \\
= 2\pi \left[ \int_{-1}^{1} I^{-}(\tau, \mu) d\mu + \int_{0}^{1} I^{+}(\tau, \mu) d\mu \right] \\
= 2\pi \left[ \int_{0}^{1} \beta(\mu) e^{-(\tau^{+}/\mu)} d\mu + \int_{0}^{1} \beta(\tau) \frac{\mu}{(1\cdot\mu) + \sigma_{s} \omega} \right] \\
+ \int_{0}^{1} \int_{0}^{\pi} \beta(\tau) \int_{0}^{\tau} \beta(\tau) \frac{\mu}{(1\cdot\mu) + \sigma_{s} \omega} d\tau^{'} \right] d\mu. \\
\]  
(12.21)

Radiative heat flux:

\[
q(\tau) = \int_{0}^{2\pi} \int_{0}^{\pi} I(\tau, \theta) \cos \theta \sin \theta d\theta d\phi = 2\pi \int_{-1}^{+1} I(\tau, \mu) \mu d\mu \\
= 2\pi \left[ \int_{0}^{1} \beta(\mu) e^{-(\tau^{+}/\mu)} d\mu + \int_{0}^{1} \beta(\tau) \frac{\mu}{(1\cdot\mu) + \sigma_{s} \omega} \right] \\
+ \int_{0}^{1} \int_{0}^{\pi} \beta(\tau) \int_{0}^{\tau} \beta(\tau) \frac{\mu}{(1\cdot\mu) + \sigma_{s} \omega} d\tau^{'} \right] d\mu. \\
\]  
(12.22)

Now we will study the solution of these two equations for specific cases.

Also notice that for 1-D plane parallel medium

\[
\nabla \cdot q = \kappa (4\pi I_{b} - G), \\
\]  
(12.23)

becomes after dividing by \(\beta\):

\[
\frac{dq}{d\tau} = (1 - \omega)(4\pi I_{b} - G). \\
\]  
(12.24)

Note that \(\kappa' / \beta = (1 - \omega)\).

- If radiative equilibrium \(\Rightarrow \frac{dq}{d\tau} = 0\)
- If heat source is present

\[
\frac{dq}{d\tau} = \frac{\dot{Q}'(\tau)}{\beta}, \\
\]  
(12.25)

In that case combining this with Eq(12.24)

\[
\frac{dq}{d\tau} = \frac{\dot{Q}'(\tau)}{\beta} = \frac{\kappa}{\beta} (4\pi I_{b} - G) \\
\]  

therefore after canceling \(\beta\)’s:

\[
4\pi I_{b}(\tau) = G(\tau) + \frac{\dot{Q}'(\tau)}{\kappa}(\tau). \\
\]  
(12.26)

Incident radiation is related to \(I_{b}\) therefore \(T\).
Cases we will consider:

- Black Boundaries
  - In non-scattering medium
    - Radiative Equilibrium
  - Gray-Diffuse boundaries
- In scattering medium
  - Isotropic scattering
  - Anisotropic scattering

- Specified temperature field
Radiative Equilibrium in Non-scattering Medium:  

Note: still 1-D plane-parallel medium

Non-scattering $\Rightarrow \omega = 0$

**Enclosure with black bounding surfaces:**

$S(\tau', \hat{s}) = I_b(\tau'), \quad I_b(\theta) = I_{b1}$ and $I_b(\theta) = I_{b2}$

therefore

$$I^+(\tau', \theta) = I_{b1} e^{-\tau'/\cos \theta} + \int_0^{\tau} I_b(\tau') e^{-(\tau-\tau')/\cos \theta} \frac{d\tau'}{\cos \theta}, \quad 0 < \theta < \frac{\pi}{2}, \quad (12.27a)$$

$$I^-(\tau', \theta) = I_{b2} e^{(\tau-\tau')/\cos \theta} - \int_{\tau}^{\infty} I_b(\tau') e^{(\tau' - \tau)/\cos \theta} \frac{d\tau'}{\cos \theta}, \quad \frac{\pi}{2} < \theta < \pi. \quad (12.27b)$$

Using $\mu$:

$$I^+(\tau', \mu) = I_{b1} e^{-\tau'/\mu} + \int_0^\tau I_b(\tau') e^{-(\tau-\tau')/\mu} d\tau', \quad 0 < \mu < 1, \quad (12.28a)$$

$$I^-(\tau', \mu) = I_{b2} e^{(\tau-\tau')/\mu} - \int_{\tau}^\infty I_b(\tau') e^{(\tau' - \tau)/\mu} d\tau', \quad -1 < \mu < 0. \quad (12.28b)$$

From Eq(12.21):

$$G(\tau) = 2\pi \left[ \int_{-1}^{0} I_b(\tau', \mu) \, d\mu + \int_{0}^{1} I^+(\tau', \mu) \, d\mu \right]$$

$$= 2\pi \left[ I_{b1} \int_{0}^{1} e^{-\tau'/\mu} \, d\mu + I_{b2} \int_{0}^{1} e^{(\tau'-\tau)/\mu} \, d\mu \right.$$

$$+ \int_0^{\tau} I_b(\tau') \left[ \int_{0}^{\tau} e^{-(\tau'-\tau')/\mu} \, d\mu \right] \, d\tau' + \int_{\tau}^{\infty} I_b(\tau') \left[ \int_{0}^{\tau} e^{-\tau'/\mu} \, d\mu \right] \, d\tau'. \quad (12.29)$$

and from Eq(12.22):

$$q(\tau) = 2\pi \left[ I_{b1} \int_{0}^{1} e^{-\tau'/\mu} \, d\mu - I_{b2} \int_{0}^{1} e^{-(\tau'-\tau)/\mu} \, d\mu \right.$$

$$+ \int_0^{\tau} I_b(\tau') \left[ \int_{0}^{\tau} e^{-(\tau'-\tau')/\mu} \, d\mu \right] \, d\tau' - \int_{\tau}^{\infty} I_b(\tau') \left[ \int_{0}^{\tau} e^{-\tau'/\mu} \, d\mu \right] \, d\tau'. \quad (12.30)$$

To take these integrals we will introduce “Exponential integral of order n”: 

$$E_n(\tau) = \int_1^\infty e^{-\tau t} \frac{dt}{t^n} = \int_{0}^{1} \mu^n e^{\tau \mu} \, d\mu. \quad (12.31)$$

$$E_n(0) = \int_1^\infty \frac{dt}{t^n} = \frac{1}{n-1}. \quad (12.32)$$

$$\frac{d}{dx} E_n(\tau) = -E_{n-1}(\tau); \quad \text{or} \quad E_n(\tau) = \int_{x}^{\tau} E_n(x) \, dx. \quad (12.33)$$

See Fig(12.2) and App. E.
Using Exponential Integrals we can write:

\[ G(\tau) = 2\pi \left[ I_{b1}E_2(\tau) + I_{b2}E_2(\eta - \tau) \right. \]
\[ \left. + \int_0^\eta I_b(\tau')E_1(\tau' - \tau) d\tau' + \int_0^\tau I_b(\tau')E_1(\tau' - \tau) d\tau' \right]. \]  
(12.34)

\[ q(\tau) = 2\pi \left[ I_{b1}E_3(\tau) - I_{b2}E_3(\eta - \tau) \right. \]
\[ \left. + \int_0^\tau I_b(\tau')E_3(\tau' - \tau) d\tau' - \int_0^\tau I_b(\tau')E_2(\tau' - \tau) d\tau' \right]. \]  
(12.35)

Gray medium \( \Rightarrow I_b(\tau) = \frac{n^2\sigma T^4(\tau)}{\pi} \)

Radiative equilibrium without vol. energy generation \( \Rightarrow \frac{dq}{d\tau} = 0 \Rightarrow q(\tau) = \text{const.} \)

From Eq(12.24) : \( 0 = 4\pi I_b(\tau) - G(\tau) \Rightarrow G(\tau) = 4\pi I_b(\tau) = 4n^2\sigma T^4(\tau) \)

Using this with Eq(12.34) we can get temperature field:

\[ T^4(\tau) = \frac{1}{2} \left[ T_1^4E_2(\tau) + T_2^4E_2(\eta - \tau) + \int_0^\tau T_1^4(\tau')E_1(\tau' - \tau) d\tau' \right]. \]  
(12.36)

Heat flux distribution:

\[ q(\tau) = 2n^2\sigma T_1^4E_3(\tau) - 2n^2\sigma T_2^4E_3(\eta - \tau) \]
\[ + \int_0^\tau n^2\sigma T_1^4(\tau')E_2(\tau' - \tau) d\tau' - \int_0^\tau n^2\sigma T_1^4(\tau')E_2(\tau' - \tau) d\tau'. \]  
(12.37)

Since \( q(\tau) \) is constant we can evaluate \( q \) at any convenient position (take \( \tau = 0 \)):

\[ q = n^2\sigma T_1^4 - 2n^2\sigma T_2^4E_3(\eta) - 2 \int_0^\eta n^2\sigma T_1^4(\tau')E_2(\tau') d\tau'. \]  
(12.38)

So we need to solve for \( T \) using Eq(12.36) then solve for \( q \) using Eq(12.38). But there is no close form solution for Eq(12.36), so we need to find a numerical solution or approximate solution.
Before doing that, let us define:
Non-dimensional emissive power (temperature):
\[ \Phi_b(\tau) = \frac{T^4(\tau) - T_2^4}{T_1^4 - T_2^4}, \]  
(12.39)

Non-dimensional radiative heat flux:
\[ \Psi_b = \frac{q}{n^2 \sigma (T_1^4 - T_2^4)}. \]  
(12.40)

Substitute these in Eq(12.36) and Eq(12.37) (or 12.38) to get
\[ \Phi_b(\tau) = \frac{1}{2} \left[ E_3(\tau) + \int_0^{\tau} \Phi_b(\tau') E_2(|\tau - \tau'|) d\tau' \right]. \]  
(12.41)
\[ \Psi_b(\tau) = 2 \left[ E_3(\tau) + \int_0^{\tau} \Phi_b(\tau') E_2(\tau - \tau') d\tau' \right] - \int_0^{\tau} \Phi_b(\tau') E_2(\tau - \tau)d\tau'. \]  
(12.42)

or
\[ \Psi_b = 1 - 2 \int_0^{\tau} \Phi_b(\tau') E_2(\tau') d\tau'. \]  
(12.43)

Note that \( E_3(\tau = 0) = 1/2 \) and \( E_3(\tau - \tau') = -E_2(\tau' - \tau) \).

These equations are Fredholm Integral Equations, so they can be solved numerically using Integral Kernels (see section 5.5).

Solution given in Fig(12.3) and Table(12.1):

\[ \text{FIGURE 12-3} \]
Non-dimensional temperature distribution for a gray medium at radiative equilibrium between isothermal plates.
Enclosure with gray-diffuse bounding surfaces:

Now we have radiosities leaving bottom and top surfaces: $J_1$ and $J_2$. We can relate wall radiosities to blackbody emissive powers:

$$q_{w} \cdot \hat{n} = \frac{e_{w}}{1 - e_{w}} \left( \pi I_{bw} - J_{w} \right), \quad (12.44)$$

or

$$\tau = 0: \quad q = \frac{e_{1}}{1 - e_{1}} \left[ n^{2} \sigma T_{1}^{4} - J_{1} \right], \quad (12.45a)$$

$$\tau = \tau_{L}: \quad -q = \frac{e_{2}}{1 - e_{2}} \left[ n^{2} \sigma T_{2}^{4} - J_{2} \right], \quad (12.45b)$$

In definition non-dimensional heat flux replace $\pi I_{\text{bi}}$ with $J_{i}$:

$$\frac{q}{J_{1} - J_{2}} = \Psi_{\text{b}} = 1 - 2 \int_{0}^{\tau_{L}} \Phi_{\text{b}}(\tau') E_{2}(\tau') d\tau', \quad (12.45)$$

and from equation (12.45)

$$J_{1} - J_{2} = n^{2} \sigma (T_{1}^{4} - T_{2}^{4}) - \left( \frac{1}{e_{1}} + \frac{1}{e_{2}} - 2 \right) q.$$

Thus,

$$q = \Psi_{\text{b}}(J_{1} - J_{2}) = \Psi_{\text{b}} \left[ n^{2} \sigma (T_{1}^{4} - T_{2}^{4}) - \left( \frac{1}{e_{1}} + \frac{1}{e_{2}} - 2 \right) q \right].$$

or

$$\Psi = \frac{q}{n^{2} \sigma (T_{1}^{4} - T_{2}^{4})} = \frac{\Psi_{\text{b}}}{1 + \Psi_{\text{b}} \left( \frac{1}{e_{1}} + \frac{1}{e_{2}} - 2 \right)}. \quad (12.46)$$

Similarly

$$\Phi(\tau) = \frac{T_{1}^{4}(\tau) - T_{2}^{4}}{T_{1}^{4} - T_{2}^{4}} = \frac{\Phi_{\text{b}}(\tau) + \left( \frac{1}{e_{2}} - 1 \right) \Psi_{\text{b}}}{1 + \Psi_{\text{b}} \left( \frac{1}{e_{1}} + \frac{1}{e_{2}} - 2 \right)}. \quad (12.47)$$
Radiative Equilibrium of a Scattering Medium:

Isotropic scattering:
Source function for isotropic scattering from Eq(12.5) and Eq(12.26)

$$S(\tau') = (1 - \omega)I_b(\tau') + \frac{\omega}{4\pi} G(\tau').$$  \hspace{1cm} (12.5)

$$4\pi I_b(\tau) = G(\tau) + \frac{Q''}{K}(\tau).$$  \hspace{1cm} (12.26)

Assuming no heat generation:

$$S(\tau) = (1 - \omega)I_b(\tau) + \frac{\omega}{4\pi} G(\tau) = I_b(\tau).$$  \hspace{1cm} (12.48)

So \( S(\tau) = I_b(\tau) \) that is exactly what we had for non-scattering medium. All equations derived for non-scattering medium are valid only \( \kappa \rightarrow \beta \).

Now \( \tau \equiv \int_0^z \beta dz \).

Comment: For a gray medium at radiative equilibrium there is no distinction between absorption and isotropic scattering.

If purely scattering medium then we will not have any \( I_b(\tau) \) in equations, then we need to replace \( T^4(\tau) \) in Eqs (12.39) and (12.47) with \( \frac{G(\tau)}{4\pi^2 \sigma} \).

Anisotropic scattering (Linear Anisotropic Scat.):

We also have radiative equilibrium and 1-D plane parallel medium. Therefore Eq(12.15)

$$S(\tau', \theta) = (1 - \omega)I_b(\tau') + \frac{\omega}{4\pi} \left[ G(\tau') + A_1 q(\tau') \cos \theta \right].$$  \hspace{1cm} (12.15)

becomes

$$S(\tau, \mu) = I_b(\tau) + \frac{A_1 \omega}{4\pi} q \mu.$$

\hspace{1cm} (12.49)

Therefore, equations (12.21) and (12.22) become

$$\frac{G(\tau)}{4\pi} = I_b(\tau) = \frac{J_1}{2\pi} E_2(\tau) + \frac{J_2}{2\pi} E_2(\tau_L - \tau)$$

$$+ \frac{1}{2} \int_0^\tau I_b(\tau')E_1(\tau - \tau') \, d\tau' + \frac{1}{2} \int_\tau^{\infty} I_b(\tau')E_1(\tau' - \tau) \, d\tau'$$

$$+ \frac{A_1 \omega}{8\pi} q \left[ E_3(\tau_L - \tau) - E_3(\tau) \right].$$

\hspace{1cm} (12.50)

$$q(\tau) = q = 2J_1 E_3(\tau) - 2J_2 E_3(\tau_L - \tau)$$

$$+ 2\pi \left[ \int_0^\tau I_b(\tau')E_2(\tau - \tau') \, d\tau' - \int_\tau^{\infty} I_b(\tau')E_2(\tau' - \tau) \, d\tau' \right]$$

$$+ \frac{A_1 \omega}{2} q \left[ \frac{2}{3} - E_4(\tau) - E_4(\tau_L - \tau) \right].$$

\hspace{1cm} (12.51)

In nondimensional form these relations reduce to

$$\Phi_b(\tau) = \frac{\pi I_b(\tau)L_2}{J_1 - J_2} = \frac{1}{2} \left[ E_2(\tau) + \int_0^\tau \Phi_b(\tau')E_1(\tau - \tau') \, d\tau' \right.$$

$$+ \frac{A_1 \omega}{4} \Psi_b \left[ E_3(\tau_L - \tau) - E_3(\tau) \right] \bigg],$$

\hspace{1cm} (12.52)

$$\Psi_b = \frac{q}{J_1 - J_2} = 2 \left[ E_3(\tau) + \int_0^\tau \Phi_b(\tau')E_2(\tau - \tau') \, d\tau' \right.$$

$$- \int_\tau^{\infty} \Phi_b(\tau')E_2(\tau' - \tau) \, d\tau' + \frac{A_1 \omega}{4} \Psi_b \left[ \frac{2}{3} - E_4(\tau) - E_4(\tau_L - \tau) \right] \bigg].$$

\hspace{1cm} (12.53)
Comment: It was observed that approximating a complex phase function with a linear anisotropic one always lead to accurate results in heat transfer problems.

**Plane Medium with Specified Temperature Field:**

If medium is not at radiative equilibrium, the problem of finding temperature field and heat fluxes is always nonlinear.

- Conduction and Convection $\propto T_1 - T_2$
- Radiation $\propto T_1^4 - T_2^4$

Therefore we need an iterative procedure:

- Guess $T(\tau)$
- Determine $G(\tau)$ and $\nabla \cdot q$
- Put the result into overall energy conservation to get estimate of $T(\tau)$
- Repeat the procedure

In some cases we may assume the knowledge of $T(\tau)$, for example swirling combustion chambers.

1-D plane parallel, gray medium (or spectral calculations) and diffuse bounding surfaces ($I_w = J_w / \pi$) and isotropic scattering $\Rightarrow$ we may rewrite Eqs(12.21) and (12.22) as

\[ G(\tau) = 2J_1E_2(\tau) + 2J_2E_3(\tau) - \tau \]
\[ q(\tau) = 2J_1E_3(\tau) - 2J_2E_2(\tau) + \tau \]

Radiative source from Eq(12.15):

\[ S(\tau) = (1 - \omega)I_\phi(\tau) + \frac{\omega}{4\pi}G(\tau). \]  

(12.56)

Radiative equilibrium can not be assumed but temperature field is given then heat flux is not constant across the medium. If $T(\tau)$ is given divergence of heat flux is desired, so from Eq(12.24):

\[ \frac{dq}{d\tau} = (1 - \omega)4\pi I_b - G. \]  

(12.57)

Non-scattering medium $\Rightarrow S(\tau) = I_b(\tau)$ so integrals in Eqs (12.54) and (12.55) readily evaluated. Heat flux $q$, incident radiation $G$ and divergence of radiative heat flux $dq/d\tau$ may be determined explicitly.

Isotropically scattering medium $\Rightarrow$ Eq (12.54) becomes an integral equation of unknown $G(\tau)$ defining a non-dimensional function similar to Eq (12.39):

\[ \Phi(\tau) = \frac{\pi S(\tau) - J_2}{J_1 - J_2}, \]  

(12.58)

Eq (12.54) simplifies to:

\[ \Phi(\tau) = (1 - \omega)\frac{\pi I_b(\tau) - J_2}{J_1 - J_2} + \frac{\omega}{2} \left[ E_2(\tau) + \int_0^{\tau} \Phi(\tau)E_1(|\tau' - \tau|) d\tau' \right]. \]  

(12.59)

If purely scattering medium ($\omega \to 1$) Eq(12.59) reduces to Eq(12.41). For such a medium thermal radiation is decoupled from temperature field (no emission) and radiative equilibrium prevails regardless of temperature field.
Similarly Eq(12.55) may be non-dimensionalized to get

\[ \Psi(\tau) = \frac{q(\tau)}{J_1 - J_2} \]

\[ = 2 \left[ E_3(\tau) + \int_0^\tau \Phi(\tau')E_2(\tau - \tau') d\tau' - \int_\tau^{\tau_0} \Phi(\tau')E_2(\tau' - \tau) d\tau' \right] \]  

(12.60)