   pp.161-162, problems 1,2,3,4,5.

2. Past TMS questions

3. Find the complete integrals and the *singular solutions*, if any, of the following partial differential equations.

   Notation : \( p = \frac{\partial u}{\partial x}, \) \( q = \frac{\partial u}{\partial y}. \)

   (a) \( u = px + qy + pq. \)
   (b) \( u^2(p^2 + q^2 + 1) = 1. \)
   (c) \( p^2 + pq = 4u. \)
   (d) \( x^4p^2 + y^2q = 0. \)


3. Let $\mu$ be a metric with coefficients $g_{ij}$ in a region $\Omega \subset \mathbb{R}^n$. Write the corresponding Laplace operator and verify that it is invariant under the isometries of $\mu$.

4. a) Show that for any region $\Omega \subset \mathbb{R}^n$, Green’s function is unique.
   
   b) Write the Green’s function for $\Omega = \{x \in \mathbb{R}^n : |x| > R > 0\}$. 

2. Review of the method of separation of variables:

Solve the following boundary value problem.

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{in} \quad \{(x,t) \in \mathbb{R}^2 : \pi > x > 0, \ t > 0\},
\]

\[u(0,t) = u(\pi,t) = 0, \quad u(x,0) = g(x),\]

where

\[g(x) = \begin{cases} 
  x & \text{if } 0 \leq x \leq \pi/2 \\
  \pi - x & \text{if } \pi/2 \leq x \leq \pi 
\end{cases}\]

3. Let \( G(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right). \)

a) Show that \( u(x,t) = 2 \int_0^t G(x,t-t')f(t')dt' \) is the solution of the following BVP

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{in} \quad \{(x,t) : x > 0, \ t > 0\}
\]

\[u(x,0) = 0 \quad \text{for } x > 0, \quad \frac{\partial u}{\partial x} \big|_{x=0} = -f(t) \quad \text{for } t > 0.
\]

b) Show that \( u(x,t) = 2 \frac{\partial}{\partial x'} \left( \int_0^t G(x-x',t-t')f(t')dt' \right) \bigg|_{x'=0} \) is the solution of the following BVP
\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{in} \quad \{(x,t) : x > 0, \ t > 0\} \]

\[ u(x,0) = 0 \quad \text{for} \quad x > 0, \quad u(0,t) = f(t) \quad \text{for} \quad t > 0, \]

\[ \lim_{x \to \infty} u = 0, \quad \lim_{x \to \infty} \frac{\partial u}{\partial x} = 0, \quad \text{for} \quad t > 0. \]

MATH 583
Partial Differential Equations I
Spring 2014
Problem Set 4

1. Past TMS questions:


3. Consider the PDE
   \[ x^2 \frac{\partial u}{\partial x^2} - y^2 \frac{\partial u}{\partial y^2} = 0 \quad (*) \]
   a) Determine the normal form of this equation.
   b) Show that the general solution of the normal form is given by
   \[ u(\xi, \eta) = f(\xi) + \sqrt{\xi}h(\eta) \]
   for suitable functions \( f, h \) (\( \xi, \eta \) are the characteristic variables).
   c) Solve the following boundary value problem for (*) and determine the domain of the solution.
   \[ u = 1 \quad \text{on} \quad y = x, \quad 0 \leq x \leq 1 \]
   \[ u = x \quad \text{on} \quad y = 1/x, \quad 1 \leq x \leq \infty. \]