Allpass variable fractional delay filters by pole loci interpolation

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ABSTRACT: A simple, accurate, and efficient variable fractional delay (VFD) IIR filter is presented. The method is based on the interpolation of pole loci of maximally-flat allpass fractional-delay filters. The responses of the proposed VFD filter for different fractional delay values are simulated and the approximation errors are shown to be very small.

Introduction: Fractional delay (FD) filters are digital filters that ideally have flat group delays emulating a non-integer delay. They are used in a variety of applications including sample rate conversion, synchronisation of telecommunications circuitry, and sound synthesis, where sub-sample accuracy is required [1]. Allpass IIR filters having maximally-flat group delays can be designed for this purpose [2]. Closed-form solutions exist for obtaining the coefficients of allpass IIR FD filters [1]. It is often necessary to have a variable fractional delay filter (VFD) that allows continuously varying the emulated delay. Different methods and allpass VFD filter structures have been proposed previously [3, 4, 5]. These methods suffer from one or more of the following problems: (i) The filter cannot be easily implemented using standard IIR filter structures making it impossible to use optimised DSP libraries, (ii) calculation of coefficients is computationally costly and not suitable for real-time operation, (iii) finite-precision arithmetic may cause stability issues, and (iv) transient errors are observed due to coefficient update. In order to overcome these, an allpass VFD filter using root displacement interpolation was proposed [6]. That method required the calculation and storage of the poles of two allpass FD filters modelling different fractional delay values D_1 and D_2. Intermediate fractional delay values could then be obtained by bilinear interpolation of the poles of two filters. In order for the emulated delay to be accurate, the original filters had to be selected in such a way that $|D_1 - D_2|$ is small. This requires the storage of the poles of several filters in order to get accurate results for a wide range of delays. It was also shown that implementing VFD filters as a cascade of second-order blocks significantly reduces transient errors due to filter coefficient update and also guarantees numerical stability [6]. In this letter, a new allpass VFD filter is proposed based on the interpolation of pole loci of reference allpass FD filters via fitting analytic, continuous functions of delay. The method allows emulating fractional delay values for a wide range of delays accurately.

Allpass IIR VFD Filter structure: The denominator coefficients of a maximally-flat group-delay allpass filter (also called Thiran filter), emulating a fractional delay D can be calculated using the closed-form expression [1]:

$$a_m = (-1)^m \binom{M}{m} \prod_{n=0}^{M} \frac{D+n}{D+n+m} \tag{1}$$

where $D > -1$ is the fractional part of the delay that the filter emulates. The complex poles, $r_i = R_i e^{\pm j\phi_i}$, of the filter are conjugate symmetric and have reciprocal zeros. Therefore, the M^{th}-order maximally-flat group delay allpass filter can be specified only with $M/2$ poles for M even or $(M-1)/2 + 1$ poles if M is odd.

Let us consider an even-order Thiran filter. The filter can be implemented as a cascade of

$M/2$ second-order allpass sections. In this case, the denominator can be expressed as $A(z) = \prod_{i=1}^{M/2} A_i(z) = \prod_{i=1}^{M/2}(1 - R_i e^{-j\phi_i} z^{-1})(1 - R_i e^{j\phi_i} z^{-1})$ where $A_i(z)$ is the denominator of a second-order section, and $R_i e^{\pm j\phi_i}$ is the pole pair associated with it. The denominator of the second-order section is a second-order polynomial of unit delay elements such that:

$$A_i(z) = 1 - 2R_i \cos(\phi_i) z^{-1} + R_i^2 z^{-2} \tag{2}$$

Here, the coefficients of the second order section are $a_{i,0} = 1$, $a_{i,1} = -2R_i \cos(\phi_i)$, and $a_{i,2} = R_i^2$. When $D \geq 0$ the loci of Thiran filter poles are smooth functions (see Fig. 1. in [6] for example). Within a given range, $D_{\min} < D < D_{\max}$ of delays, modulus R_i, and the cosine of the angle $\cos(\phi_i)$ of each pole can be approximated by analytic functions of the emulated delay, i.e. $\rho_i(D) = R_i$ and $\tau_i(D) = \cos(\phi_i)$. The coefficients of the corresponding second-order section can thus be obtained using these analytic functions without recourse to (1). For this purpose, poles of reference Thiran filters are calculated on a fine grid of delay values and functional approximations are obtained. Fitted functions must allow simple calculation, and approximate the pole loci well. It was observed that the ratio of cubic polynomials provides a good fit to both $\rho_i(D)$ and $\tau_i(D)$:

$$\rho_i(D) = \mathbf{f}_i \mathcal{D}^T / \mathbf{g}_i \mathcal{D}^T \tag{3}$$

$$\tau_i(D) = \mathbf{p}_i \mathcal{D}^T / \mathbf{q}_i \mathcal{D}^T \tag{4}$$

where $\mathcal{D} = [1 \ D \ D^2 \ D^3]$, is the delay vector, $\mathbf{f}_i = [f_{i,0} \ f_{i,1} \ f_{i,2} \ f_{i,3}]$, $\mathbf{g}_i = [g_{i,0} \ g_{i,1} \ g_{i,2} \ g_{i,3}]$, $\mathbf{p}_i = [p_{i,0} \ p_{i,1} \ p_{i,2} \ p_{i,3}]$, and $\mathbf{q}_i = [q_{i,0} \ q_{i,1} \ q_{i,2} \ q_{i,3}]$ are polynomial coefficient vectors for a second-order section. This way, coefficients of each second order section are determined by a total of 12 coefficients for each arbitrary $D_{\min} < D < D_{\max}$. Calculation of coefficients for a single second-order section is shown in Fig. 1. It may be observed that $\mathcal{D}$ can be calculated only once for a given fractional delay value D, and can be reused to obtain coefficients of all second-order sections. It may be noted that the proposed strategy is particularly well suited for single-instruction multiple-data (SIMD) processors.

[Figure 1 about here.]

Example: Performance of the proposed method was evaluated using 10^{th}-order reference Thiran filters. Poles of Thiran filters modelling delays between $0.01 \leq D \leq 2$ were calculated with 0.01 sample resolution. The functional forms for $\rho_i(D)$ and $\tau_i(D)$ were obtained as rational functions of cubic polynomials. It was observed that for all fitted functions the r^2 metric representing the goodness of fit was always greater than 0.99.

Fig. 2 shows the phase delays of the VFD filter for different D. While the designed filters do not strictly have maximally-flat phase delays, they provide an excellent approximation.

[Figure 2 about here.]

Let us define two error metrics, the frequency response error (FRE), and phase delay error $\epsilon(\omega; D)$:

$$\begin{aligned} \mathrm{FRE}(\omega; D) &= 20 \log_{10} |H_{TH}(\omega; D) - H_{VFD}(\omega; D)| && (5) \\ \epsilon(\omega; D) &= |\sphericalangle H_{TH}(\omega; D) - \sphericalangle H_{VFD}(\omega; D)| / \omega && (6) \end{aligned}$$

where $H_{TH}(\omega; D)$ and $H_{VFD}(\omega; D)$ are the frequency responses of the reference Thiran filter and the proposed VFD filter emulating a fractional delay of D, respectively, and $\sphericalangle H(\omega)$ denotes the phase response. FRE reflects the accuracy of the VFD filter in emulating the frequency response characteristics of the corresponding Thiran filter. The phase delay error, represents the accuracy of the phase delay response of the VFD filter in comparison with an actual Thiran filter.

Fig. 3 shows FRE$(\omega; D)$ for different values of D across all frequencies. It may be observed that FRE is always less than about -50 dB showing an excellent match between the original Thiran filters and the proposed VFD filter. Thiran filters and the proposed VFD filter have virtually the same frequency response at $\omega = 0$ and $\omega = \pi$. The average FRE across all the simulated delays and frequencies is -64.9 dB.

[Figure 3 about here.]

Fig. 4 shows $\epsilon(\omega; D)$ for different values of D across different frequencies. The maximum error is always less than 0.0045 samples showing an excellent approximation of the emulated delay. The mean error is 9.77×10^{-4} samples across all the simulated delays and frequencies.

[Figure 4 about here.]

Conclusions: A simple, accurate, and effective allpass IIR variable fractional delay (VFD) filter was proposed. The proposed filter is based on the interpolation of the pole loci of reference Thiran filters. A design example was provided and the accuracy of the proposed filter was compared to that of reference Thiran filters. It was shown that the proposed VFD filter provided almost identical phase delays with the reference Thiran filters.

References

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Figure 1:

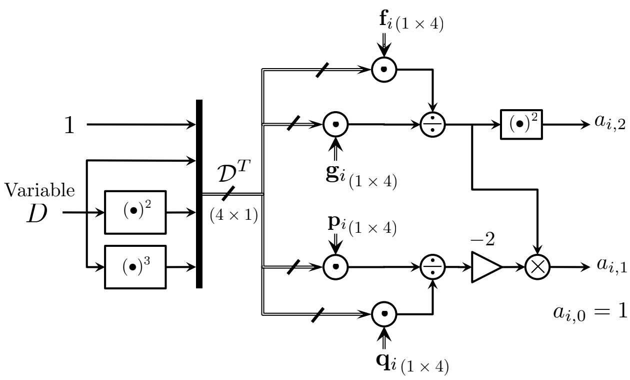

$\mathbf{f}_{i\,(1\times4)}$
1
$(\bullet)^2$
$a_{i,2}$
$\mathcal{D}^T$
$\mathbf{g}_{i\,(1\times4)}$
Variable
D
$(\bullet)^2$
(4×1)
$\mathbf{p}_{i\,(1\times4)}$
-2
$(\bullet)^3$
$a_{i,1}$
$a_{i,0}=1$
$\mathbf{q}_{i\,(1\times4)}$

Figure 2:

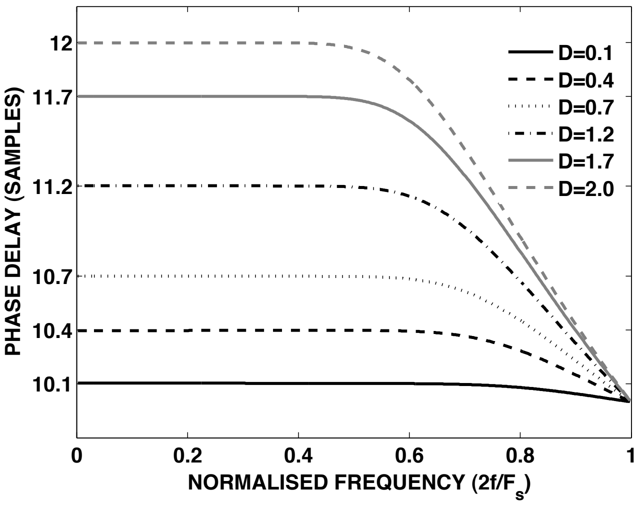
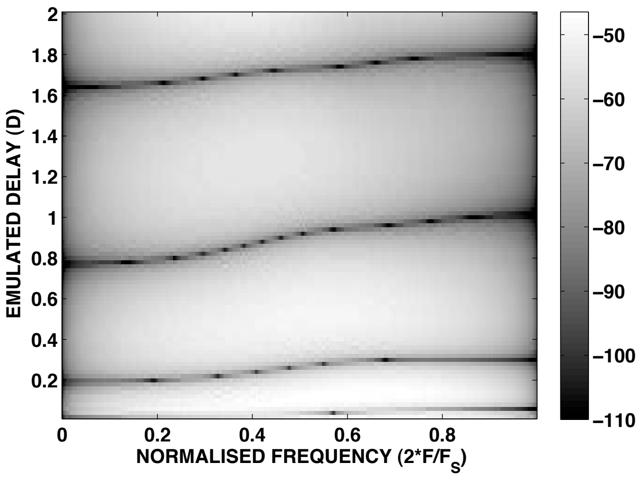
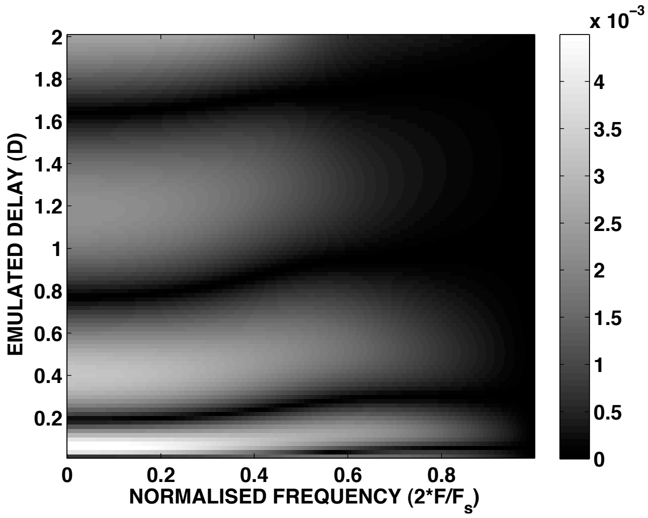

12
11.7
11.2
10.7
10.4
10.1
PHASE DELAY (SAMPLES)
0
0.2
0.4
0.6
0.8
1
NORMALISED FREQUENCY (2f/F_s)
D=0.1
D=0.4
D=0.7
D=1.2
D=1.7
D=2.0

Figure 3:



2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
EMULATED DELAY (D)
0
0.2
0.4
0.6
0.8
NORMALISED FREQUENCY (2*F/F$_S$)
−50
−60
−70
−80
−90
−100
−110

Figure 4:



x 10^-3
EMULATED DELAY (D)
NORMALISED FREQUENCY (2*F/F_s)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0.5
1.5
2.5
3
3.5
4