MATH538 - ALGEBRAIC TOPOLOGY-II - SPRING 2018
Take-Home Final Exam

Due: June 1, 2018

Q1. The *join* of two topological spaces X and Y is the space

$$X * Y \doteq X \times Y \times [0,1] \diagup_{\sim}$$

where the identifications are as follows: $(x, y, 0) \sim (x, y', 0)$ for all $x \in X$ and $y, y' \in Y$, and $(x, y, 1) \sim (x', y, 1)$ for all $x, x' \in X$ and $y \in Y$. Show that for any $n > 1$ the homology groups satisfy

$$H_n(X * Y) \cong H_{n-1}(X \times Y, X \vee Y).$$

Q2. Compute the cohomology ring structure of Σ_g, the orientable surface of genus $g \geq 2$.

Q3. Assume M is a closed and connected manifold of dimension n. Show that:

(a) If M is orientable, then $H_{n-1}(M, \mathbb{Z})$ is torsion free.

(b) If M is non-orientable, then $H_n(M, \mathbb{Z}_k) = 0$ if k is odd.

(c) If M is non-orientable, then the torsion subgroup of $H_{n-1}(M, \mathbb{Z})$ is isomorphic to $\mathbb{Z}_2$.

(d) If M is non-orientable, $H_1(M, \mathbb{Z}_2)$ is non-trivial.

[Hint: What is an R-*orientation* on a manifold for a given commutative ring R with identity? Our textbook might be useful!]

Q4. Prove that $\mathbb{CP}^{538}$ can not be the boundary of a compact manifold of (real) dimension 1077.