MATH 420 “Point Set Topology”

Set #6

SOLUTIONS

Section 24:

1: (a) Before starting, let us note that $(0,1) - \{a\}$ is always disconnected if $0 < a < 1$. This is because $(0,1) - \{a\} = (0,a) \cup (a,1)$ and the sets $(0,a)$ and $(a,1)$ form a separation of $(0,1) - \{a\}$.

Now if $(0,1]$ were homeomorphic to $(0,1)$ under homeomorphism

$$f : (0,1] \to (0,1),$$

then by removing the point 1 from the domain we would get a homeomorphism $(0,1] - \{1\} = (0,1) \to (0,1) - \{f(1)\}$. However, this is impossible since $(0,1)$ is connected and $(0,1) - \{f(1)\}$ is disconnected. Thus, we see $(0,1]$ is not homeomorphic to $(0,1)$.
If $[0,1]$ were homeomorphic to $(0,1)$ under homeomorphism g, then

$$[0,1] - \{1\} = [0,1)$$

would be homeomorphic to $(0,1) - \{g(1)\}$. This is again impossible since $[0,1)$ is connected while $(0,1) - \{g(1)\}$ is not connected.
If $[0,1]$ were homeomorphic to $(0,1]$ under homeomorphism h, then

$$[0,1] - \{0,1\} = (0,1)$$

would be homeomorphic to $(0,1] - \{h(0), h(1)\}$. This is impossible since $(0,1)$ is connected but $(0,1] - \{h(0), h(1)\}$ is disconnected. (Note since h is bijective, $h(0)$ not equal to $h(1)$ and it is easy to check removing two points from $(0,1]$ always leaves a disconnected space.)
Thus, $(0,1), (0,1]$ and $[0,1]$ all have different homeomorphism types.

(b) Note $i : (0,1) \to [0,1]$, $i(t) = t$ is an obvious embedding. (It is a homeomorphism onto its image.) Also $j : [0,1] \to (0,1)$ given by $j(t) = 0.5t + 0.2$ maps $[0,1]$ homeomorphically to the subspace $[0.2, 0.7]$ of $(0,1)$, and hence is an embedding of $[0,1]$ into $(0,1)$. Thus, there are embeddings of $[0,1]$ into $(0,1)$ and $(0,1)$ into $[0,1]$ even though $[0,1]$ is not homeomorphic to $(0,1)$ by part (a).

(c) Fix $n > 1$. If $f : \mathbb{R} \to \mathbb{R}^n$ were a homeomorphism, then it would induce a homeomorphism from $\mathbb{R} - \{0\}$ to $\mathbb{R}^n - \{f(0)\}$. However, this is impossible since $\mathbb{R}-\{0\}$ is clearly disconnected while $\mathbb{R}^n-\{f(0)\}$ is path connected, and so connected. Thus, there is no homeomorphism from $\mathbb{R}$ to $\mathbb{R}^n$ when $n>1$.

2: $f : S^1 \to \mathbb{R}$ a continuous map. Let $g : S^1 \to \mathbb{R}$ be defined by $g(x) = f(x) - f(-x)$. [Note if $x \in S^1$, so is $-x$ so this makes sense.] Now, g is continuous since f was. Furthermore,

$$g(-x) = f(-x) - f(-(-x)) = f(-x) - f(x) = -[f(x) - f(-x)] = -g(x).$$

Thus, if there is x such that $g(x) > 0$, then $g(-x) < 0$ and vice versa. Thus, there are two cases:
Case(1): $g(x) = 0$ for all $x \in S^1$, in which case $f(x) = f(-x)$ for all $x \in S^1$ and we are done!
Case(2): There is $x \in S^1$ such that $g(x) > 0$ and $g(-x) < 0$. However, $g : S^1 \to \mathbb{R}$ is continuous and S^1 is connected so the intermediate value theorem guarantees a y in S^1 such that $g(y) = 0$. For this y, we have $f(y) = f(-y)$ [From definition of g], and so we are done here too.

3: Assume $f : [0,1] \to [0,1]$ is continuous. We want to show there exits x in $[0,1]$ such that $f(x) = x$. If $f(0) = 0$ or $f(1) = 1$ then we can use $x = 0$ or 1, respectively. So we can assume $f(0) > 0$ and $f(1) < 1$. Now, define $g : [0,1] \to \mathbb{R}$ by $g(x) = f(x) - x$. Note that g is continuous as f is continuous, and that $g(0) = f(0) - 0 > 0$ and $g(1) = f(1) - 1 < 0$. Thus, by the fact that $[0,1]$ is connected and the intermediate value theorem, we see there exits x in $[0,1]$ such that $g(x) = 0$. Thus, $f(x) - x = 0$ or $f(x) = x$ as desired.

9: Let A be a countable set and consider $X = \mathbb{R}^2 - A$. We want to show that X is path connected. So take b and c in X. Working in the plane, if L is the set of lines passing through b, then L is bijective to $\mathbb{R} \cup \{\infty\}$ since we can correspond the lines bijectively with their slope (note distinct lines have different slopes, and there is a line through b of any given slope in $\mathbb{R} \cup \{\infty\}$). Thus, there are uncountably many lines passing through b. Let S be the subset of these lines which intersect the set A. Note we can make a map $i : S \to A$, which sends each of these lines to a choice of a point in A on that line. Notice i is injective since no point of A lies on more that one such line. Thus, S is bijective to a subset of A, and hence is countable. Thus, only countably many of these lines will intersect the set A. Since there were uncountably many lines through b, we conclude that there are uncountably many lines through b, which do not intersect the set A. Choose one of these lines L_1. (Note if c happened to lie on L_1, then we could use L_1 as a path from b to c in X but we will assume that we are not so lucky and continue.) Doing a similar analysis at the point c, we conclude that there are uncountably many lines through c which do not intersect A. Out of these only 1 is parallel to L_1, so we can find

L_2 a line through c, which does not intersect A and which is NOT parallel to L_1. Thus, L_1 and L_2 will meet at a point d. Then the line segment along L_1 from b to d followed by the line segment along L_2 from d to c forms continuous path in X from b to c. Thus, we have shown that any two points in X can be joined by a continuous path in X, and so X is path connected.