Q1. For the discrete-time LTI system

$$x^+ = Ax + Bu$$

we have shown in class that the feedback gain (assuming the inverse exists)

$$K_\mathrm{d} = B^T A^{(N-1)T} \left[\sum_{i=0}^{N-1} A^i B B^T A^{iT} \right]^{-1} A^N \tag{1}$$

can be used to exponentially regulate the origin, i.e., the closed-loop system matrix $[A - BK_\mathrm{d}]$ is Schur. Can you guess a continuous-time version of the gain (1), call it K_c, that regulates the origin of the continuous-time system $\dot{x} = Ax + Bu$? (I.e., the matrix $[A - BK_\mathrm{c}]$ is Hurwitz.) Check whether your guess works in MATLAB (and reguess if necessary) over various numerical instances. (For the answer see: D.L. Kleinman, "An easy way to stabilize a linear constant system," *IEEE Transactions on Automatic Control*, vol. 15, pp. 692-692, 1970.)

Q2. Consider the MATLAB code below.

```
function L = dbLfun(C,A)
X = null(C);
for i = 1:length(A)-2
X = null([C;null((A*X)')']);
end
L = A*A*X/(C*A*X);
```

(a) Verify numerically that the function `dbLfun` generates deadbeat observer gain for any observable pair $(C,\, A)$ with $C \in \mathbb{R}^{1\times n}$ and $A \in \mathbb{R}^{n\times n}$.

(b) Why does this algorithm work?

Q3. Consider the system

$$x^+ = f(x,\, u)$$

where $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$ with $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^m$. Suppose for all $x \in \mathcal{X}$ the following optimization problem admits a solution

$$\mathrm{Prob}(x,\, N):\ V_N(x) = \min_{(v_0,\, \ldots,\, v_{N-1})} h(\xi_N) + \sum_{k=0}^{N-1} g(\xi_k,\, v_k) \quad \text{subj. to} \quad \begin{cases} \xi_0 = x \\ \xi_{k+1} = f(\xi_k,\, v_k)\ \forall k \\ \xi_k \in \mathcal{X}\ \forall k \\ v_k \in \mathcal{U}\ \forall k \\ \xi_N \in \mathcal{X}_\mathrm{f} \end{cases}$$

where $\mathcal{X}_\mathrm{f} \subset \mathcal{X}$ is called the terminal set, which is assumed to contain the origin. Let the feedback law $\kappa_N : \mathcal{X} \to \mathcal{U}$ be such that for each $x \in \mathcal{X}$, $\kappa_N(x) = v_0^*$, where $(v_0^*,\, \ldots,\, v_{N-1}^*)$ is a minimizing control sequence for $\mathrm{Prob}(x,\, N)$. Assume that the following conditions hold.

A1. There exist positive constants c_1, c_2 such that

- $g(x, u) \geq c_1\|x\|^2$ for all $x \in \mathcal{X}$ and $u \in \mathcal{U}$,
- $V_N(x) \leq c_2\|x\|^2$ for all $x \in \mathcal{X}$.

A2. $h(x) \geq 0$ for all $x \in \mathcal{X}_{\rm f}$ and there exists a feedback law $\kappa_{\rm f} : \mathcal{X}_{\rm f} \to \mathcal{U}$ such that

- $f(x, \kappa_{\rm f}(x)) \in \mathcal{X}_{\rm f}$ for all $x \in \mathcal{X}_{\rm f}$,
- $h(f(x, \kappa_{\rm f}(x))) - h(x) \leq -g(x, \kappa_{\rm f}(x))$ for all $x \in \mathcal{X}_{\rm f}$.

Show that the origin of the closed-loop system $x^+ = f(x, \kappa_N(x))$ is asymptotically stable.

Remark. In MPC literature it is customary to use $V_N(x)$ as a Lyapunov function to establish stability. See, for instance, D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789-814, 2000.

Q4. Consider the system

$$x^+ = f(x, u)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Suppose that the origin of this system is exponentially stabilizable. That is, there exist positive constants c, α and for each initial condition $x_0 \in \mathbb{R}^n$ one can find an (infinite) input sequence $(u_0, u_1, u_2, \ldots)$ such that the resulting trajectory satisfies $\|x_k\| \leq c\|x_0\|e^{-\alpha k}$ for $k = 0, 1, 2, \ldots$ Consider the following optimization problem

$$\text{Prob}(x, N):\; V_N(x) = \min_{(v_0, \ldots, v_{N-1})} \sum_{k=0}^{N} \|\xi_k\|^2 \quad \text{subj. to} \quad \begin{cases} \xi_0 = x \\ \xi_{k+1} = f(\xi_k, v_k)\ \forall k \end{cases}$$

Let the feedback law $\kappa_N : \mathbb{R}^n \to \mathbb{R}^m$ be such that for each $x \in \mathbb{R}^n$, $\kappa_N(x) = v_0^*$, where $(v_0^*, \ldots, v_{N-1}^*)$ is a minimizing control sequence for $\text{Prob}(x, N)$. Show that there exists a (finite) horizon $\hat{N}$ such that the origin of the closed-loop system $x^+ = f(x, \kappa_N(x))$ is asymptotically stable for all $N \geq \hat{N}$.