Q1. Consider the linear system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

where the triple $(C,\, A,\, B)$ is both observable and controllable.

(a) Show that the triple $(C,\, [A + \alpha I],\, B)$ is both detectable and stabilizable for all $\alpha \in \mathbb{R}$.

(b) Let $\alpha > 0$ and R be symmetric positive definite. Show that the below cost

$$J = \int_0^\infty e^{2\alpha t}(y^T y + u^T R u)dt$$

admits a minimizer in the form $u_{\min}(t) = -Kx(t)$ where K is a constant feedback gain. *Hint: Employ the change of variables $z := e^{\alpha t}x$ and $v := e^{\alpha t}u$.*

(c) What can be said about the eigenvalue locations of the closed-loop system matrix $A - BK$?

Q2. Consider the nonlinear system

$$\dot{x} = f(x) + g(x)u$$

with the unique equilibrium at the origin $x = 0$. The cost to be minimized is given as

$$J = \int_0^\infty (q(x) + u^T R(x) u)dt \tag{1}$$

where, for all $x \in \mathbb{R}^n$, $q(x) \geq 0$ with $q(0) = 0$ and $R(x)$ is symmetric positive definite. Suppose there exists a differentiable positive semidefinite function $V : \mathbb{R}^n \to \mathbb{R}$ with $V(0) = 0$ satisfying the HJB equation

$$\begin{aligned} 0 &= \min_u \{q + u^T R u + (\nabla V)^T (f + gu)\} \\ &= q + (\nabla V)^T f - \frac{1}{4}(\nabla V)^T g R^{-1} g^T \nabla V \end{aligned}$$

such that the feedback control

$$\hat{u}(x) = -\frac{1}{2}R^{-1}(x) g(x)^T \nabla V(x)$$

achieves $\lim_{t\to\infty} x(t) = 0$. Show that this $\hat{u}(x)$ is the optimal control which minimizes the cost (1) (over all $u(\cdot)$ guaranteeing the convergence $\lim_{t\to\infty} x(t) = 0$) and $J_{\min} = V(x(0))$. *Hint: Substitute $u = \hat{u} + v$ in (1) and obtain*

$$J = -\int_{t=0}^{t\to\infty} dV + \int_0^\infty v^T R(x) v \, dt \, .$$

Q3. Consider the second-order (normalized) antenna position system with friction

$$\ddot{y} + \dot{y} = u$$

where y is the (angular) position of the antenna and u is the torque applied to the system. Cost to be minimized is

$$J = \int_0^\infty (y^2 + u^2)dt \, .$$

(a) Find $J_{\min}$ in terms of $y(0)$ and $\dot{y}(0)$.

(b) Let the optimal control law be $u_{\min} = -k_1 y - k_2 \dot{y}$. Find k_1 and k_2.

(c) Consider the cost

$$\hat{J} = \int_0^\infty ((y - r)^2 + u^2)dt$$

where r is a constant reference position. Show that the optimal control law reads $\hat{u}_{\min} = -k_1(y - r) - k_2\dot{y}$, where k_1 and k_2 are same as in part (b).

Q4. For the controllable system $\dot{x} = Ax + Bu$ consider the problem of minimizing cost

$$J = \int_0^\infty (x^T Q x + u^T R u + \dot{u}^T M \dot{u})dt$$

where all the matrices Q, R, M are symmetric positive definite.

(a) Show that this problem reduces to a standard LQR problem for an augmented system.

(b) Show that the resulting optimal control law will have both state and integral feedback terms. *Note that the solution will depend on* $u(0)$ *which, depending on the application, can be fixed or chosen to further minimize* J.

Q5. Consider the collection of systems

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix} x + B_i u \\ y &= C_j x \end{aligned}$$

where

$$B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = [2 \ \ -1], \quad C_2 = [1 \ \ 0].$$

The cost to be minimized is

$$J = \int_0^\infty (y^2 + u^2)dt\,.$$

(a) For which of the four systems (C_1, A, B_1), (C_1, A, B_2), (C_2, A, B_1), (C_2, A, B_2) can you claim that the optimal stabilizing Riccati solution P exists? How about its positive definiteness?

(b) Solve the optimal control problem for the system (C_1, A, B_2). *Hint:* $\dot{y} = ?$