Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{k\times n}$. Recall the eigenvector tests:

Theorem 1 *The pair* $(A,\, B)$ *is controllable if and only if no eigenvector of* A^T *belongs to* $\operatorname{null} B^T$.

Theorem 2 *The pair* $(A,\, B)$ *is stabilizable (in the continuous-time sense) if and only if no eigenvector of* A^T *with eigenvalue* $\operatorname{Re}(\lambda) \geq 0$ *belongs to* $\operatorname{null} B^T$.

Theorem 3 *The pair* $(C,\, A)$ *is observable if and only if no eigenvector of* A *belongs to* $\operatorname{null} C$.

Theorem 4 *The pair* $(C,\, A)$ *is detectable (in the continuous-time sense) if and only if no eigenvector of* A *with eigenvalue* $\operatorname{Re}(\lambda) \geq 0$ *belongs to* $\operatorname{null} C$.

Q1. Using the proper eigenvector test, prove the following statements.

(a) The pair $(A,\, B)$ is controllable if and only if

$$\operatorname{rank}\,[A - \lambda I \quad B] = n \quad \forall \lambda \in \mathbb{C}\,.$$

(b) The pair $(A,\, B)$ is stabilizable if and only if

$$\operatorname{rank}\,[A - \lambda I \quad B] = n \quad \forall \lambda \in \{\eta \in \mathbb{C} : \operatorname{Re}(\eta) \geq 0\}\,.$$

(c) The pair $(C,\, A)$ is observable if and only if

$$\operatorname{rank}\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C}\,.$$

(d) The pair $(C,\, A)$ is detectable if and only if

$$\operatorname{rank}\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \quad \forall \lambda \in \{\eta \in \mathbb{C} : \operatorname{Re}(\eta) \geq 0\}\,.$$

Q2. Show that the pair $(C,\, A)$ is observable if and only if the pair $(C^T C,\, A)$ is observable.

Q3. Given a controllable pair $(A,\, B)$ suppose there exists $P = P^T > 0$ such that $A^T P + PA \leq 0$. Prove that the matrix $[A - BB^T P]$ is Hurwitz.

Q4. Suppose there exists $P = P^T > 0$ such that $A^T P + PA - C^T C < 0$.

(a) Show that the pair $(C,\, A)$ is detectable.

(b) Propose an observer gain $L \in \mathbb{R}^{n\times k}$ such that the matrix $[A - LC]$ is Hurwitz.

Q5. Show that if $\operatorname{rank}\,[B \;\, AB \;\, \cdots \;\, A^{n-1}B] = n$ then $\operatorname{rank}\,[A - \lambda I \quad B] = n$ for all $\lambda \in \mathbb{C}$.

Q6. Let the subspace $\mathcal{S} \subset \mathbb{C}^n$ with nonzero dimension be invariant under A. Show that $\mathcal{S}$ must contain an eigenvector of A.