Q1. Consider the first-order LTI system $\dot{x} = u$ (without any bound on the control) and the associated optimization problem

$$J^*(x(0)) = \min_{u(\cdot)} \int_0^\infty \big(qx(t)^2 + ru(t)^2\big)\, dt$$

where q and r are positive constants.

(a) Obtain and solve the Riccati equation.

(b) Find the optimal control.

(c) Verify $J^*(x(0)) = px(0)^2$ (where p denotes the solution of the Riccati equation) by carrying out the integration.

(d) Verify $J^*(x(0)) = px(0)^2$ using the HJB equation.

Q2. Problem 5-15.

Q3. Consider the LTI system $\dot{x} = Ax + Bu$ (without any bound on the control) and the associated optimization problem

$$J^*(t_0,\, x(t_0)) = x(t_f)^T H x(t_f) + \min_{u(\cdot)} \int_{t_0}^{t_f} \big(x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)\big)\, d\tau$$

where the symmetric matrices $H \geq 0$, $Q \geq 0$, and $R > 0$ are of appropriate sizes. Recall that the HJB equation for this case reads

$$0 = \nabla_t J^*(t,\, x) + \min_u \big\{x^T Q x + u^T R u + \nabla_x J^*(t,\, x)^T [Ax + Bu]\big\}$$

where $J^*(t,\, x)$ enjoys the form $J^*(t,\, x) = x^T P(t) x$ for some symmetric matrix $P(t)$ satisfying the Riccati equation

$$\dot{P} = -Q + PBR^{-1}B^T P - A^T P - PA\,.$$

Let us now introduce the *Hamiltonian* as

$$\mathcal{H}(x,\, u,\, p) = x^T Q x + u^T R u + p^T [Ax + Bu]\,.$$

Show that along the optimal trajectories the Hamiltonian $\mathcal{H}(x^*(t),\, u^*(t),\, \nabla_x J^*(t,\, x^*(t)))$ is constant. *Hint:* $\dot{\mathcal{H}} = \nabla_x \mathcal{H}^T \dot{x} + \nabla_u \mathcal{H}^T \dot{u} + \nabla_p \mathcal{H}^T \dot{p}$.

Q4. Given a stabilizable pair $(A,\, B)$ let the symmetric positive definite matrix P solves the following Riccati equation

$$A^T P + PA + I - PBB^T P = 0\,.$$

Show that the system $\dot{x} = [A - \alpha BB^T P]x$ is exponentially stable for all $\alpha \geq 1$.