Q1. Problem 3-3 (from the textbook).

Q2. Problem 3-12.

Q3. Consider the optimization problem

$$
\begin{aligned}
V_N(x) \quad &= \quad \min_{u_0,\,\ldots,\,u_{N-1}} x_N^T H x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k \\
&\qquad \text{subject to} \begin{cases} x_0 &= x \\ x_{k+1} &= Ax_k + Bu_k \end{cases}
\end{aligned}
$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, the symmetric positive definite matrices H, Q, R are of appropriate dimensions, the pair (A, B) is controllable, and $N \geq 0$ is an integer.

(a) Write the recursive relation that allows one to compute Q_{N+1} from Q_N where $V_N(x) = x^T Q_N x$. What does Q_0 equal?

(b) Let $P = \lim_{N\to\infty} Q_N$. Show that P must satisfy

$$
x^T P x = x^T Q x + \min_u \left\{ u^T R u + (Ax + Bu)^T P (Ax + Bu) \right\} \tag{1}
$$

for all x.

(c) Show that (1) is equivalent to the Riccati equation

$$
A^T P A - P - A^T P B (B^T P B + R)^{-1} B^T P A + Q = 0\,.
$$

(d) Show that the trajectories of the system $x^+ = Ax + Bu$ where u solves (1) converge to the origin, i.e., $x_k \to 0$ as $k \to \infty$.

Q4. Consider the feedback law $\kappa_N(x) = u_0^*(x)$ where

$$
\begin{aligned}
(u_0^*(x),\,\ldots,\,u_{N-1}^*(x)) \quad &= \quad \arg \min_{(u_0,\,\ldots,\,u_{N-1})} \sum_{k=0}^{N-1} x_k^T Q x_k \\
&\qquad \text{subject to} \begin{cases} x_0 &= x \\ x_{k+1} &= Ax_k + Bu_k \\ x_N &= 0 \end{cases}
\end{aligned}
$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $Q \in \mathbb{R}^{n\times n}$ is symmetric positive definite, the pair (A, B) is controllable, and $N \geq n$ is an integer. Show that the origin of the closed-loop system $x^+ = Ax + B\kappa_N(x)$ is exponentially stable.