# Optimal Control Problem

**System** : $\dot{x} = f(t,x,u)$ , $x(t_0) = x_0$

→ $x \in \mathbb{R}^n$ is the state ($x(t)$ is the solution (trajectory); depends on $x_0$ & $u(t)$)

→ $u \in U \subset \mathbb{R}^m$ is the control ($U$ is the control set)

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}, \quad f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ f_2(t,x,u) \\ \vdots \\ f_n(t,x,u) \end{bmatrix}$$

**Cost** :

$$J(u) := h(t_f, x(t_f)) + \int_{t_0}^{t_f} g(t, x(t), u(t))\,dt$$

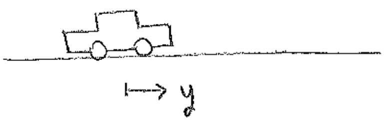
→ $g$ and $h$ are given functions (running cost & terminal cost, respectively)

→ final time $t_f$ is either free or fixed

→ final state $x(t_f) = x_f$ is either free or fixed or belongs to some target set

**Problem** : find the function $u(\cdot)$ which minimizes $J(u)$ over all admissible controls.

**Example**




$\ddot{y} = u$ , $u \in [-1, 1]$

**goal** : bring the car from the initial state ($y(0) = 0$, $\dot{y}(0) = 0$) to final state ($y(t_f) = 10$, $\dot{y}(t_f) = 0$) as quickly as possible.

**System?** let $x_1 = y$ & $x_2 = \dot{y}$. Define $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ as the state

Then $\dot{x} = \underbrace{\begin{bmatrix} x_2 \\ u \end{bmatrix}}_{f(t,x,u)}$ , $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $(t_0 = 0)$

is the system with $x \in \mathbb{R}^2$ and $u \in [-1, 1] = U \subset \mathbb{R}$

Cost? $J(u) = \int_0^{t_f} 1\, dt \qquad (h \equiv 0,\ g \equiv 1)$

subject to $x(t_f) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$ ($t_f$ is free)

Examples of applications

→ Send a rocket to the moon with minimal fuel consumption

→ Produce a given amount of chemical in minimal time and/or with minimal amount of catalyst used

→ Bring sales of a new product to a desired level while minimizing the cost of advertising

→ Communication: maximize throughput / accuracy for given bandwidth / capacity

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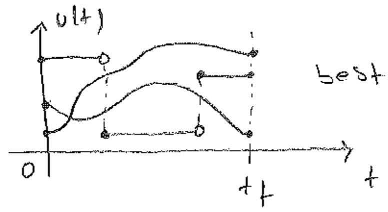
optimal control theory studies dynamic, infinite-dimensional optimization.
Roughly, it is on how to choose the best path among all feasible paths.




It is related to but different from static, finite-dimensional optimization where we search for a point that minimizes $f: \mathbb{R}^n \to \mathbb{R}$.

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Example (Resource allocation)

system: $\dot{x}(t) = \gamma u(t) x(t) \qquad (\gamma > 0 : \text{fixed})$

$x \in \mathbb{R}$ : production rate

$u \in [0,1]$ : portion of production rate to be allocated to reinvestment

$1-u$ : portion to be allocated to production of a storable good

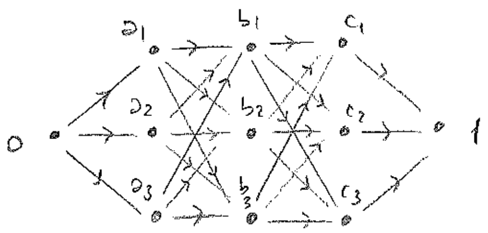
goal : maximize $\int_0^{T} (1-u(t))x(t)\,dt$ (the total amount of product stored)

## Dynamic Programming

Dynamic programming is a computational technique in which an optimal decision is reached through a sequence of optimal subdecisions. It is based on the (Bellman's) Principle of Optimality.

Principle of optimality: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

Example (Shortest Path) Find the shortest path from the node 0 to node 1.



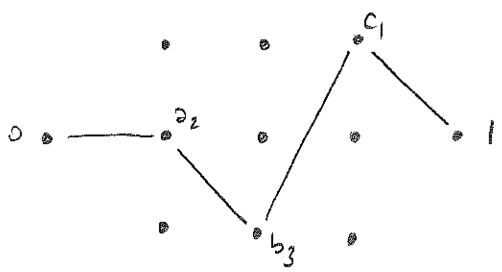

$\ell(a_i, b_j)$ : length of the edge from $a_i$ to $b_j$

$J^*(a_i, c_k)$ : length of the shortest path from $a_i$ to $c_k$

Problem; Find $J^*(0,1)$.

Note that $J^*(0,1) = \min\limits_{i,j,k=1,2,3} \ell(0,a_i) + \ell(a_i,b_j) + \ell(b_j,c_k) + \ell(c_k,1)$

Suppose $J^*(0,1) = \ell(0,a_2) + \ell(a_2,b_3) + \ell(b_3,c_1) + \ell(c_1,1)$. That is, the shortest path is :




Claim $J^*(b_3,1) = \ell(b_3,c_1) + \ell(c_1,1)$. (by Principle of optimality)

proof Suppose not. Then there exists $\bar{k} \in \{2,3\}$ such that

$$\ell(b_3, c_{\bar{k}}) + \ell(c_{\bar{k}}, 1) < \ell(b_3, c_1) + \ell(c_1, 1)$$

Then $J^*(0,1) > \ell(0,a_2) + \ell(a_2,b_3) + \ell(b_3,c_{\bar{k}}) + \ell(c_{\bar{k}},1)$. But this contradicts that $0 \to a_2 \to b_3 \to c_1 \to 1$ is the shortest path. ☒

Principle of optimality allows us to compute $J^*(0,1)$ in the following sequential way (which for large scale problems is much cheaper computationally)

<u>Step 1</u> Set $J^*(c_k,1) = \ell(c_k,1)$ $\quad k=1,2,3$ $\quad$ 3

<u>Step 2</u> $J^*(b_j,1) = \min_k \ \ell(b_j,c_k) + J^*(c_k,1)$ $\quad j=1,2,3$ $\quad$ 3×3

<u>Step 3</u> $J^*(a_i,1) = \min_j \ \ell(a_i,b_j) + J^*(b_j,1)$ $\quad i=1,2,3$ $\quad$ 3×3

<u>Step 4</u> $J^*(0,1) = \min_i \ \ell(0,a_i) + J^*(a_i,1)$ $\quad$ 3

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<u>Example</u> Consider the discrete-time LTI system

$$x_{k+1} = Ax_k + Bu_k \quad (1) \qquad x_k \in \mathbb{R}^n, \ u_k \in \mathbb{R}^m, \ k=0,1,2,\dots$$

Suppose the pair $(A,B)$ is controllable and $B$ is full column rank. Let $Q \in \mathbb{R}^{n\times n}$ be a positive definite matrix, $Q = Q^T > 0$. For an integer $N \geq 0$ define the cost

$$J_N(x, u_{0,\dots,N-1}) = \sum_{k=0}^{N} x_k^T Q x_k$$

where $x_{k+1} = Ax_k + Bu_k$ & $x_0 = x$.
& $u_{0,\dots,N-1} = (u_0, u_1, \dots, u_{N-1})$

Let us define the optimal cost

$$V_N(x) = \min_{u_{0,\dots,N-1}} J_N(x, u_{0,\dots,N-1})$$

<u>Problem</u>: Compute $V_N(x)$.

<u>Solution</u>: Let us use Dynamic Programming.

<u>Claim</u> $V_N(x) = x^T Q x + \min_u V_{N-1}(Ax+Bu)$ (1)

<u>Proof</u> Exercise. (Use principle of optimality)

<u>Claim</u> For each $N$ we can find an $n \times n$ matrix $Q_N = Q_N^T > 0$ such that

$V_N(x) = x^T Q_N x$ for all $x \in \mathbb{R}^n$.

<u>Proof</u> Suppose $V_N(x) = x^T Q_N x$, $Q_N = Q_N^T > 0$ for some $N$. Then by (1) we can write

$$V_{N+1}(x) = x^T Q x + \min_u (Ax+Bu)^T Q_N (Ax+Bu)$$

$$= x^T Q x + \min_u \{ \underbrace{x^T A^T Q_N A x + 2u^T B^T Q_N A x + u^T B^T Q_N B u}_{f(u)} \} \quad (2)$$

To perform the minimization in (2) we solve $\nabla f(u^*) = 0$

$$\nabla f = 2B^T Q_N A x + 2B^T Q_N B u \Rightarrow u^* = -(B^T Q_N B)^{-1} B^T Q_N A x =: -K_N x \quad (K_N \in \mathbb{R}^{m \times n})$$

Hence,

$$V_{N+1}(x) = x^T Q x + (Ax - BK_N x)^T Q_N (Ax - BK_N x)$$

$$= x^T \{ Q + (A - BK_N)^T Q_N (A - BK_N) \} x$$

$$= x^T Q_{N+1} x$$

$$Q_{N+1} := \underbrace{\underbrace{Q}_{\text{pos. def}} + \underbrace{(A-BK_N)^T Q_N (A-BK_N)}_{\text{pos. semidef}}}_{\Rightarrow \text{ pos. def.}}$$

We have shown that if $V_N(x) = x^T Q_N x$ then $V_{N+1}(x) = x^T Q_{N+1} x$, $Q_{N+1} = Q_{N+1}^T > 0$.
Since $V_0(x) = x^T Q x$ $(Q_0 = Q)$ the result follows by induction. ∎

<u>Remark</u> While proving the claim we've obtained the dynamic programming algorithm to compute $V_N(x)$, which is now equivalent to computing $n \times n$ matrix $Q_N$.

Summary

problem: solve $V_N(x) = \min_{u_0,\dots,u_{N-1}} \sum_{k=0}^{N} x_k^T Q x_k$ subj. to $x_{k+1} = Ax_k + Bu_k$ & $x_0 = x$

solution (algorithm): Set $Q_0 = Q$. Then sequentially compute for $k = 0,1,\dots,N-1$

$$K_k = (B^T Q_k B)^{-1} B^T Q_k A$$

$$Q_{k+1} = Q + (A - BK_k)^T Q_k (A - BK_k)$$

Exercise (Matlab) For various choices of $A, B, Q$

1) observe that $\lim_{N\to\infty} Q_N$ exists. (Therefore $\lim_{N\to\infty} K_N$ also exists.)

2) Take $G = \lim_{N\to\infty} K_N$ as the feedback gain. observe that the solutions of the closed-loop system $x_{k+1} = Ax_k - BGx_k$ satisfy $\|x_k\| \to 0$ as $k \to \infty$.

3) Can you prove your observations?

Exercise Solve the modified problem:

(1) $V_N(x) = x_N^T H x_N + \min_{u_0,\dots,u_{N-1}} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k$ subj. to $x_{k+1} = Ax_k + Bu_k$ & $x_0 = x$

where $Q, R, H$ are sym. pos. def. matrices. and $(A, B)$ is controllable.

Exercise Redo the matlab exercise for the modified problem.

Remark Consider the problem (1). Let $V_N(x) = x^T Q_N x$ and $P = \lim_{N\to\infty} Q_N$.
(why does $P$ exist?)

The matrix $P$ solves (do you see why?)

$$\boxed{A^T P A - P - A^T P B (B^T P B + R)^{-1} B^T P A + Q = 0}$$

→ this equation is called the DT algebraic Riccati equation. (related MATLAB command: "dare")

# The Hamilton Jacobi Bellman (HJB) Equation

System: $\dot{x} = f(t,x,u)$ , $x(t_0) = x_0$ , $u \in U$

Cost (to be minimized) : $J = h(t_f, x(t_f)) + \int_{t_0}^{t_f} g(\tau, x(\tau), u(\tau))\, d\tau$

For $t \in [t_0, t_f]$ define

$$J^*(t,x) = \min_{u(\cdot)} \; h(t_f, x(t_f)) + \int_t^{t_f} g(\tau, x(\tau), u(\tau))\, d\tau$$

subject to $x(t) = x$

Note that here $t$ is treated as the initial time and $x$ as the initial cond.

We can write

$$J^*(t,x) = \min_u \left\{ \int_t^{t+\Delta t} g\, d\tau + \int_{t+\Delta t}^{t_f} g\, d\tau + h(t_f, x(t_f)) \right\}$$

Principle of optimality requires

$$J^*(t,x) = \min_u \left\{ \int_t^{t+\Delta t} g\, d\tau + J^*(t+\Delta t, x(t+\Delta t)) \right\}$$

Note that $x(t+\Delta t)$ depends on $u(\cdot)$. Taylor series expansion of $J^*(t+\Delta t, x(t+\Delta t))$ at the point $(t,x)$ yields

$$J^*(t,x) = \min_u \left\{ \int_t^{t+\Delta t} g\, d\tau + \underbrace{J^*(t,x) + \nabla_t J^*(t,x)\, \Delta t}_{\text{do not depend on } u} + \nabla_x J^*(t,x)^T [x(t+\Delta t) - x(t)] + o(\Delta t) \right\}$$

we're minimizing over functions $u(\tau)$ in the interval $\tau \in [t, t+\Delta t]$

represents higher order terms, satisfies $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$

Canceling $J^*(t,x)$ on both sides yields

$$0 = \nabla_t J^*(t,x)\Delta t + \min_u \left\{ \int_t^{t+\Delta t} g\, d\tau + \nabla_x J^*(t,x)^T [x(t+\Delta t) - x(t)] + o(\Delta t) \right\}$$

$$\Rightarrow 0 = \lim_{\Delta t \to 0} \nabla_t J^*(t,x) + \min_u \left\{ \underbrace{\frac{\int_t^{t+\Delta t} g\, d\tau}{\Delta t}}_{g(t,x,u)} + \nabla_x J^*(t,x)^T \underbrace{\frac{[x(t+\Delta t) - x(t)]}{\Delta t}}_{f(t,x,u)} + \underbrace{\frac{o(\Delta t)}{\Delta t}}_{0} \right\}$$

Hence we've obtained the famous HJB equation:

$$\boxed{0 = \nabla_t J^*(t,x) + \min_{u \in U} \left\{ g(t,x,u) + \nabla_x J^*(t,x)^T f(t,x,u) \right\}} \qquad \text{(HJB)}$$

this u is no longer a function but a vector

Note that $J^*(t,x)$ is subject to the boundary condition:

$$\boxed{J^*(t_f, x) = h(t_f, x)} \qquad \text{(Boundary cond.)}$$

Theorem (Sufficiency thm.) Suppose $V(t,x)$ is a solution to the HJB eqn. That is, $V$ is continuously differentiable in $t$ and $x$, and is such that

$$0 = \nabla_t V(t,x) + \min_{u \in U} \left\{ g(t,x,u) + \nabla_x V(t,x)^T f(t,x,u) \right\} \quad \text{for all } t, x \qquad (1)$$

$$\& \quad V(t_f, x) = h(t_f, x) \quad \text{for all } x \qquad (2)$$

Suppose also that $\mu^*(t,x)$ attains the minimum in (1) for all $t, x$. Let $\{x^*(t) : t \in [t_0, t_f]\}$ be the state trajectory obtained from the given initial condition $x(t_0) = x_0$ when the control signal $u^*(t) = \mu^*(t, x^*(t))$, $t \in [t_0, t_f]$ is used. Then

$$V(t,x) = J^*(t,x) \quad \text{for all } t, x.$$

Furthermore, the control signal $\{u^*(t) : t \in [t_0, t_f]\}$ is optimal.

<u>proof</u> For notational convenience we prove $V(t_0, x_0) = J^*(t_0, x_0)$. The general case has a similar demonstration.

Let $\{\hat{u}(t) : t \in [t_0, t_f]\}$ be any admissible control signal and let $\hat{x}(t)$ ($\hat{x}(t_0) = x_0$) denote the corresponding state trajectory. From eq. (1) we have for all $t \in [t_0, t_f]$

$$0 \le g(t, \hat{x}(t), \hat{u}(t)) + \underbrace{\nabla_t V(t, \hat{x}(t)) + \nabla_x V(t, \hat{x}(t))^T f(t, \hat{x}(t), \hat{u}(t))}_{\text{this equals } \frac{d}{dt}\{V(t,\hat{x}(t))\} \text{ (why?)}}$$

Hence,

$$0 \le g(t, \hat{x}(t), \hat{u}(t)) + \frac{d}{dt}\{V(t, \hat{x}(t))\}$$

Integrating both sides yields

$$0 \le \int_{t_0}^{t_f} g(t, \hat{x}(t), \hat{u}(t))dt + V(t_f, \hat{x}(t_f)) - V(t_0, \hat{x}(t_0))$$

Using eq. (2) and $\hat{x}(t_0) = x_0$ we have

$$V(t_0, x_0) \le h(t_f, \hat{x}(t_f)) + \int_{t_0}^{t_f} g(t, \hat{x}(t), \hat{u}(t))dt. \qquad (3)$$

Had we used $u^*(t)$ and $x^*(t)$ instead of $\hat{u}(t)$ and $\hat{x}(t)$, respectively, the preceding inequalities would have become equalities and we would have obtained

$$V(t_0, x_0) = h(t_f, x^*(t_f)) + \int_{t_0}^{t_f} g(t, x^*(t), u^*(t))dt \qquad (4)$$

Now, (3) $\Rightarrow$ $V(t_0, x_0) \le J^*(t_0, x_0)$ and (4) $\Rightarrow$ $V(t_0, x_0) \ge J^*(t_0, x_0)$. Hence

$$V(t_0, x_0) = J^*(t_0, x_0).$$

Then (4) $\Rightarrow$ $u^*(t)$ is optimal.

▨

Example (HJB)

Scalar system : $\dot{x} = u$ , $u \in [-1, 1] =: U$

Cost : $J = \frac{1}{2} x(T)^2$ ; given: initial state $x(0)$ & final time $T$ $(g \equiv 0)$

HJB equation : $0 = \nabla_t V(t,x) + \min_{|u| \le 1} \{ \nabla_x V(t,x) u \}$ for all $t, x$

with the terminal cond. $V(T,x) = \frac{1}{2} x^2$.

An evident optimal strategy: " Approach the origin $x=0$ as rapidly as possible and stay there (if $x(t) = 0$ can be attained for $t \le T$) once reached." Then the corresponding control policy is

$$\mu^*(t,x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x > 0 \end{cases}$$

$$= -\mathrm{sgn}(x)$$

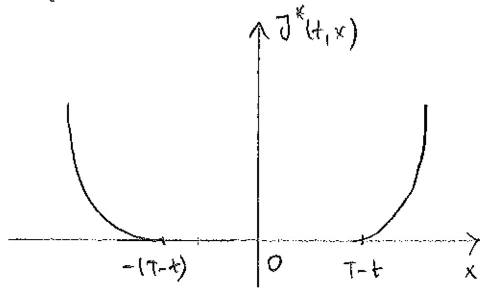
$J^*(t,x) = ?$

Note that for $u = \bar{u}$ constant $x(\tau) = x(t) + \bar{u}(\tau - t)$ $\tau \in [t, T]$

Since $|u| \le 1$ , $x(T) = 0$ is possible when $|x(t)| \le T - t$.

If $|x(t)| > T - t$ then the closest we can get to the origin is $|x(T)| = |x(t)| - (T-t)$. Hence

$$J^*(t,x) = \frac{1}{2} \left( \max\{0, |x| - (T-t)\} \right)^2$$




Let's check if this satisfies the HJB eqn. We have

$$\nabla_t J^*(t,x) = \max\{0, |x| - (T-t)\},$$

$$\nabla_x J^*(t,x) = \max\{0, |x| - (T-t)\} \cdot \mathrm{sgn}(x)$$

Hence,

$$0 = \min_{|u|\le 1} \{1+\mathrm{sgn}(x)\cdot u\}\max\{0, |x|-(T-t)\} \qquad (1) \qquad \text{HJB}$$

Evidently (1) holds and $u=-\mathrm{sgn}(x)$ is a minimizer. Note that when $|x|<(T-t)$ the optimal control is not unique (in fact, any control is optimal then).

Example (HJB)

Scalar system : $\dot{x} = u$ $\qquad u \in \mathbb{R}$ (no constraint)

cost : $J = x(T)^2 + \int_0^T u(t)^2 dt$

$J^*(t,x) = ?$

Guess : optimal $u(\cdot)$ is constant. $u(t) \equiv u$

$$\Rightarrow \; x(\tau) = x(t) + (\tau - t)\cdot u$$

$$\Rightarrow \; J^*(t,x) = \min_u \left[x+(T-t)u\right]^2 + \int_t^T u^2 d\tau$$

$$= \min_u \left[x+(T-t)u\right]^2 + u^2[T-t]$$

To find the optimal control $u^*$ let us differentiate the cost wrt $u$

$$0 = \frac{d}{du}\left\{ [x+(T-t)u]^2 + u^2[T-t]\right\}$$

$$= 2\,[x+(T-t)u](T-t) + 2u(T-t)$$

$$= 2(T-t)\,[x + (T+1-t)u]$$

$$\Rightarrow \; u^* = -\frac{x}{T+1-t}$$ (under the assumption that opt. $u$ is constant)

Hence

$$J^*(t,x) = \left[x + (T-t)\left\{-\frac{x}{T+1-t}\right\}\right]^2 + \frac{x^2}{(T+1-t)^2}(T-t)$$

$$= x^2\left\{\left[1 - \frac{T-t}{T+1-t}\right]^2 + \frac{T-t}{(T+1-t)^2}\right\}$$

$$= x^2\left\{\frac{1}{(T+1-t)^2} + \frac{T-t}{(T+1-t)^2}\right\}$$

$$= \frac{x^2}{T+1-t}$$

Is this really $J^*(t,x)$? Check HJB eqn. (How about the bound. cond.?)

$$\nabla_t J^*(t,x) = \frac{x^2}{(T+1-t)^2} \quad \& \quad \nabla_x J^*(t,x) = \frac{2x}{T+1-t}$$

We have to have

$$0 \overset{?}{=} \nabla_t J^*(t,x) + \min_u \left\{\underbrace{g(x,u)}_{u^2} + \nabla_x J^*(t,x)\cdot \underbrace{f(x,u)}_{u}\right\}$$

$$= \frac{x^2}{(T+1-t)^2} + \min_u \left\{u^2 + \frac{2xu}{T+1-t}\right\}$$

$$= \min_u \left\{u^2 + \frac{2xu}{T+1-t} + \frac{x^2}{(T+1-t)^2}\right\}$$

$$= \min_u \left(u + \frac{x}{T+1-t}\right)^2$$

Clearly HJB eqn. is satisfied & $u^* = -\frac{x}{T+1-t}$ (as expected)

Note that this is sufficient for optimality. Hence our guess (constant $u$) is indeed correct.

Example (LQR)

LTI system: $\dot{x} = Ax + Bu$ $\qquad x \in \mathbb{R}^n, \; A \in \mathbb{R}^{n \times n}, \; B \in \mathbb{R}^{n \times m}$

cost : $x(T)^T H \, x(T) + \int_0^T \left( x(t)^T Q x(t) + u(t)^T R u(t) \right) dt$

where $H, Q$ are symmetric positive semidefinite and $R$ is symmetric pos. definite.

The HJB equation reads

$$0 = \nabla_t V(t,x) + \min_{u \in \mathbb{R}^m} \left\{ x^T Q x + u^T R u + \nabla_x V(t,x)^T (Ax + Bu) \right\}$$

with the bound. cond. $V(T,x) = x^T H x$

Guess: $V(t,x) = x^T P(t) x$ , where $P(t) \in \mathbb{R}^{n \times n}$ is symmetric

Let us see if we can solve HJB eqn. with our guess

$$0 = x^T \dot{P}(t) x + \min_u \{ \underbrace{x^T Q x + u^T R u + 2x^T P(t)(Ax + Bu)}_{\square} \}$$

To find the minimum solve $\nabla_u \square = 0$

$$\nabla_u \square = 2Ru + 2B^T P(t) x \quad \Rightarrow \quad u = -R^{-1} B^T P(t) x$$

Hence, we proceed as

$$0 = x^T \dot{P} x + \left\{ x^T Q x + x^T P B R^{-1} B^T P x + 2x^T P A x - 2x^T P B R^{-1} B^T P x \right\}$$

$$= x^T \left\{ \dot{P} + Q - P B R^{-1} B^T P + A^T P + P A \right\} x \qquad (1)$$

Since (1) should hold for all $x$ we can write

$$\boxed{\dot{P}(t) = -Q + P(t)BR^{-1}B^T P(t) - A^T P(t) - P(t)A} \quad \text{Riccati eqn.}$$

This is a matrix differential equation, the celebrated "Continuous-time Riccati Equation", which should satisfy the terminal cond.

$$\boxed{P(T) = H.}$$

Thanks to sufficiency of HJB eqn. if we can find a matrix $P(t)$ that satisfies the Riccati eqn. & $P(T) = H$ then the optimal cost reads $J^*(t,x) = x^T P(t) x$. Then the optimal control reads

$$u^*(t,x) = -R^{-1}B^T P(t) x.$$

$\boxtimes$

Remark It turns out that if $Q$ is pos. det. and the system is controllable (milder conditions do exist) then the solution to Riccati equation for $T=\infty$ is a constant matrix ($\dot{P}=0$). This matrix can be shown to be positive def. For this case let us answer the following question.

Question The behaviour of the closed loop solutions of $\dot{x} = Ax + Bu \big|_{u=-R^{-1}B^T P x}$ ?

Closed-loop dynamics : $\dot{x} = [A - BR^{-1}B^T P]x =: A_{cl}x$

Riccati eqn. becomes: $0 = -Q + PBR^{-1}B^T P - A^T P - PA$

$$= -Q - PBR^{-1}B^T P - [A - BR^{-1}B^T P]^T P - P[A - BR^{-1}B^T P]$$

That is,

$$A_{cl}^T P + PA_{cl} = -Q - PBR^{-1}B^T P \qquad (*)$$

Meaning of (*) ?

Define $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ as $V(x) = x^T P x$ (Lyapunov function)

Behavior of $V(x(t))$ along the solutions of $\dot{x} = A_{cl}x$ ?

$\frac{d}{dt} V(x(t)) = ?$

$$\frac{d}{dt} V(x(t)) = \langle \nabla_x V(x(t)), \dot{x}(t) \rangle$$

$$= 2x(t)^T P A_{cl} x(t)$$

$$= x(t)^T \{ A_{cl}^T P + P A_{cl} \} x(t)$$

$$\qquad (*)$$

$$= -x(t)^T \{ \underbrace{Q + PBR^{-1}B^T P}_{\text{pos. def}} \} x(t)$$

Therefore $\dot{V}(x(t)) < 0$ (unless $x(t) = 0$)

Hence $V(x(t)) \to 0$ (why?)

$\Rightarrow$ $x(t) \to 0$ (why?)

Result: The closed-loop system $\dot{x} = Ax + Bu \Big|_{u = -R^{-1}B^T P x}$

is asymptotically (exponentially) stable. Equivalently, all the eigenvalues of $A_{cl} = A - BR^{-1}B^T P$ satisfy $\mathrm{Re}\{\lambda\} < 0$.

[ Eqn. (*) is sometimes called the Lyapunov Eqn. ]

## Calculus of Variations

→ Functional vs. Function

<u>function</u> $f : \mathbb{R}^n \to \mathbb{R}$

<u>ex</u> $f : \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\in \mathbb{R}^2} \longmapsto \underbrace{\sqrt{q_1^2 + q_2^2}}_{\in \mathbb{R}}$

<u>functional</u> $J : \underbrace{\{ x : [0,\infty) \to \mathbb{R}^n \}}_{\mathcal{X}} \to \mathbb{R}$

<u>ex</u> $J : \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\in \mathcal{X}} \longmapsto \int_0^1 \sqrt{x_1(t)^2 + x_2(t)^2}\, dt$

Note that domain of functions is finite dimensional whereas domain of functionals is infinite dimensional. Roughly speaking, functional means "function of functions."

→ Linearity

Recall that a function is linear if it satisfies

1) $f(\alpha q) = \alpha f(q)$ $\quad \alpha \in \mathbb{R}$
2) $f(q + p) = f(q) + f(p)$

Same is true for a linear functional. Namely, a linear functional satisfies

1) $J(\alpha x) = \alpha J(x)$
2) $J(x + y) = J(x) + J(y)$

Example For the space $V = \{x : [0,1] \to \mathbb{R} ;\ x \text{ differentiable}\}$ consider the functional $J : V \to \mathbb{R}$ defined as

$$J(x) = \int_0^1 x(t)\,dt$$

Note that $J$ is linear since

1) $J(\alpha x) = \int_0^1 \alpha x(t)\,dt = \alpha \int_0^1 x(t)\,dt = \alpha J(x)$

2) $J(x+y) = \int_0^1 (x(t) + y(t))\,dt = \int_0^1 x(t)\,dt + \int_0^1 y(t)\,dt = J(x) + J(y)$

Question How about $J(x) = \int_0^1 \sqrt{1 + \dot{x}(t)^2}\,dt$ ? $\left(\dot{x} = \frac{d}{dt}x(t)\right)$

— o —

→ Function norm

Let $V = \{x : [t_0, t_f] \to \mathbb{R}^n\}$ (function space)

Definition A function $\|\cdot\| : V \to \mathbb{R}$ is said to be a norm (on $V$) if it satisfies (the norm properties):

1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x(t) \equiv 0$

2) $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all $\alpha \in \mathbb{R}$

3) $\|x + y\| \leq \|x\| + \|y\|$

Remark An important use of the norm is to characterize "closeness" of two functions $x, y$. Namely, the smaller the $\|x - y\|$ the "closer" the functions.

Exercise Show that the below function $\|\cdot\|$ satisfies the properties of norm.

$$\underbrace{\|x\|}_{\text{function norm}} = \max_{t_0 \leq t \leq t_f} \underbrace{\|x(t)\|}_{\text{vector norm}}$$

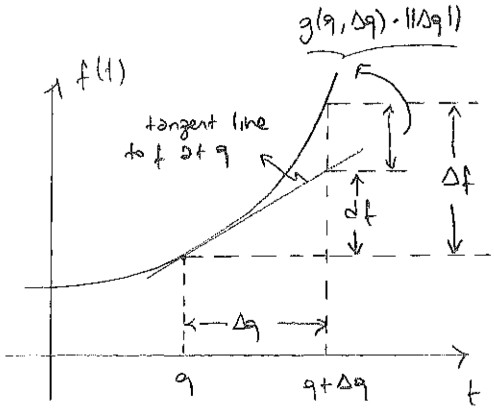
→ Increment & differential

Let us remember these concepts for functions $f : \mathbb{R}^n \to \mathbb{R}$.

Given $q, \Delta q \in \mathbb{R}^n$ the <u>increment</u> of $f$, denoted $\Delta f$, is defined as

$$\Delta f = f(q + \Delta q) - f(q)$$

Note that $\Delta f$ depends both on $q$ & $\Delta q$.




For a given $q \in \mathbb{R}^n$, if we can write (for all $\Delta q$)

$$\Delta f(q, \Delta q) = df(q, \Delta q) + g(q, \Delta q) \cdot \|\Delta q\|$$

where $df$ is linear w.r.t. $\Delta q$ and $g$ satisfies

$$\lim_{\|\Delta q\| \to 0} g(q, \Delta q) = 0$$

then $f$ is said to be <u>differentiable</u> at $q$ and $df$ is the <u>differential</u> of $f$ at $q$.

<u>Example</u> $f(q) = q_1^2 + 2 q_1 q_2$ $\left( q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \right)$

$$\Delta f = f(q + \Delta q) - f(q)$$

$$= (q_1 + \Delta q_1)^2 + 2(q_1 + \Delta q_1)(q_2 + \Delta q_2) - q_1^2 - 2 q_1 q_2$$

$$= 2 q_1 \Delta q_1 + (\Delta q_1)^2 + 2 q_2 \Delta q_1 + 2 q_1 \Delta q_2 + 2 \Delta q_1 \Delta q_2$$

$$= \underbrace{\underbrace{[\,2q_1 + 2q_2 \quad 2q_1\,]}_{\nabla f^T} \underbrace{\begin{bmatrix} \Delta q_1 \\ \Delta q_2 \end{bmatrix}}_{\Delta q}}_{df(q, \Delta q)} + \underbrace{\frac{(\Delta q_1)^2 + 2 \Delta q_1 \Delta q_2}{\sqrt{\Delta q_1^2 + \Delta q_2^2}}}_{g(q, \Delta q)} \underbrace{\sqrt{\Delta q_1^2 + \Delta q_2^2}}_{\|\Delta q\|}$$

$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial q_1} \\ \dfrac{\partial f}{\partial q_2} \end{bmatrix}$, the gradient of $f$.

These concepts can naturally be extended to functionals.

Definition If the increment $\Delta J(x, \delta x) = J(x+\delta x) - J(x)$ of the functional satisfies

$$\Delta J(x,\delta x) = \delta J(x,\delta x) + g(x,\delta x)\cdot \|\delta x\|$$

where $\delta J$ is linear in $\delta x$ and $\lim_{\|\delta x\|\to 0} g(x,\delta x) = 0$ then $J$ is said to be differentiable at $x$ and $\delta J$ is called the variation of $J$ evaluated for the function $x$.

Example Let $x:[0,1]\to\mathbb{R}$ denote a continuous function. Find the variation of the functional

$$J = \int_0^1 [x(t)^2 + 2x(t)]\,dt.$$

Sol'n
$$\begin{aligned}\Delta J(x,\delta x) &= J(x+\delta x) - J(x)\\ &= \int_0^1 [(x(t)+\delta x(t))^2 + 2(x(t)+\delta x(t))]dt - \int_0^1 [x(t)^2 + 2x(t)]dt\\ &= \int_0^1 \{[2x(t)+2]\delta x(t) + [\delta x(t)]^2\}dt\\ &= \underbrace{\int_0^1 (2x(t)+2)\,\delta x(t)\,dt}_{\delta J(x,\delta x)} + \underbrace{\left(\int_0^1 \frac{[\delta x(t)]^2}{\|\delta x\|}dt\right)}_{g(x,\delta x)}\|\delta x\|\end{aligned}$$

Is $\delta J$ linear? YES

$\rightarrow \delta J(x, \alpha\delta x) = \alpha\,\delta J(x,\delta x)$ ✓

$\rightarrow \delta J(x, \delta x_1 + \delta x_2) = \delta J(x,\delta x_1) + \delta J(x,\delta x_2)$ ✓

$\lim_{\|\delta x\|\to 0} g(x,\delta x) \stackrel{?}{=} 0$ YES

because $g(x,\delta x) = \frac{1}{\|\delta x\|}\int_0^1 [\delta x(t)]^2 dt \le \frac{1}{\|\delta x\|}\int_0^1 \|\delta x\|^2 dt = \|\delta x\|$ (for $\|\delta x\| := \max_{t\in[0,1]} |\delta x(t)|$)

→ Extremum points

A function $f : D \to \mathbb{R}$ $(D \subset \mathbb{R}^n)$ has a (relative) extremum at the point $q^*$ if there is an $\varepsilon > 0$ such that for all $q \in D$ satisfying $\|q - q^*\| < \varepsilon$ the increment $\Delta f = f(q) - f(q^*)$ has the same sign.

If $\Delta f \geq 0$ then $f(q^*)$ is called a minimum;

If $\Delta f \leq 0$ then $f(q^*)$ is called a maximum.

A minimum / maximum is global if we can let $\varepsilon = \infty$.

Recall that if $f$ is differentiable and $q^*$ is an interior point of $D$ then that $f(q^*)$ is an extremum implies that the differential at $q^*$ is zero. In particular we have $\nabla f(q^*) = 0$.

Similarly, for functionals:

Definition A functional $J : \Omega \to \mathbb{R}$ has a (relative) extremum at $x^*$ if there exists $\varepsilon > 0$ such that for all $x \in \Omega$ satisfying $\|x - x^*\| < \varepsilon$ the increment $\Delta J = J(x) - J(x^*)$ has the same sign.

If $\Delta J \geq 0$ then $J(x^*)$ is a minimum;

If $\Delta J \leq 0$ then $J(x^*)$ is a maximum.

If $\varepsilon = \infty$ we have global minimum / maximum. $x^*$ is called a extremal and $J(x^*)$ an extremum.

— o —

Assumption: For $\Omega \subset \{ x : [0, \infty) \to \mathbb{R}^n \}$ let $J : \Omega \to \mathbb{R}$ be a differentiable functional. The functions in $\Omega$ are not constrained by any boundaries.

<u>Theorem</u> (The fundamental thm of the calculus of variations) Suppose the above assumption holds. If $x^*$ is an extremal, then the variation of $J$ must vanish on $x^*$. That is,

$$\delta J(x^*, \delta x) = 0 \quad \text{for all admissable } \delta x \text{ (i.e. } x^* + \delta x \in \Omega)$$

<u>proof</u> Suppose not. Then $\delta J(x^*, \delta x) = c \neq 0$ (for $\delta x$ fixed). Without loss of generality let $c > 0$. Recall

$$\Delta J = \delta J(x^*, \delta x) + g(x^*, \delta x) \cdot \|\delta x\| \quad \& \quad \lim_{\|\delta x\| \to 0} g(x^*, \delta x) = 0$$

Hence there exists $\varepsilon > 0$ such that

$$|g(x^*, \alpha\delta x)| \cdot \|\delta x\| < \tfrac{1}{2}c \quad \text{for } |\alpha| < \varepsilon. \qquad (1)$$

Then we can write for all $\alpha$

$$\begin{aligned}\Delta J(x^*, \alpha\delta x) &= \delta J(x^*, \alpha\delta x) + g(x^*, \alpha\delta x) \cdot \|\alpha\delta x\| \\ &= \alpha\left\{\delta J(x^*, \delta x) + \frac{|\alpha|}{\alpha} \cdot g(x^*, \alpha\delta x) \cdot \|\delta x\|\right\} \\ &= \alpha\left\{c + \frac{|\alpha|}{\alpha} g(x^*, \alpha\delta x) \cdot \|\delta x\|\right\} \qquad (2)\end{aligned}$$

Now, (1) & (2) implies

$$\Delta J(x^*, \alpha\delta x) \geq \alpha\frac{c}{2} > 0 \quad \text{for } \alpha \in (0, \varepsilon) \qquad (3)$$

&

$$\Delta J(x^*, \alpha\delta x) \leq \alpha\frac{c}{2} < 0 \quad \text{for } \alpha \in (-\varepsilon, 0) \qquad (4)$$

(3) & (4) implies that $x^*$ cannot be extremal. ☒

## Euler – Lagrange Equation

Problem: Find the extremal of the functional (assume J differentiable)

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\,dt$$

given : $t_0, t_f, x_0 = x(t_0)$, & $x_f = x(t_f)$

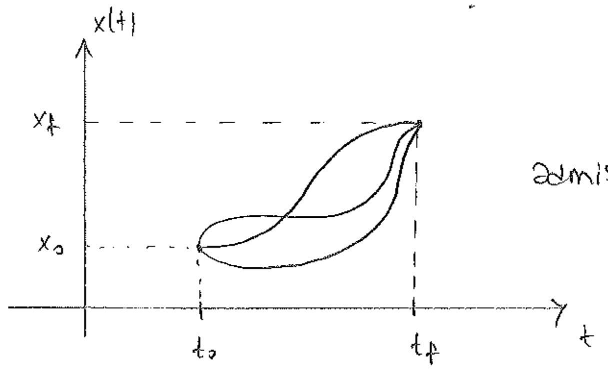
Sol'n $\Delta J = ?$

$$\Delta J(x, \delta x) = \int_{t_0}^{t_f} g(x+\delta x, \dot{x}+\delta\dot{x}, t)\,dt - \int_{t_0}^{t_f} g(x, \dot{x}, t)\,dt$$

$$g(x+\delta x, \dot{x}+\delta\dot{x}, t) = g(x,\dot{x},t) + \underbrace{\frac{\partial g}{\partial x}(x,\dot{x},t)}_{\text{row}} \cdot \underbrace{\delta x}_{\text{column}} + \frac{\partial g}{\partial \dot{x}}(x,\dot{x},t)\cdot\delta\dot{x} + \underbrace{o(\delta x, \delta\dot{x})}_{\text{higher order terms}}$$

$$\Rightarrow \Delta J(x, \delta x) = \int_{t_0}^{t_f} \left\{ \frac{\partial g}{\partial x}\cdot\delta x + \frac{\partial g}{\partial \dot{x}}\delta\dot{x} \right\} dt + o(\delta x, \delta\dot{x}) \qquad (1)$$

Note that $\displaystyle\int_{t_0}^{t_f} \frac{\partial g}{\partial \dot{x}}\delta\dot{x}\,dt = \underbrace{\frac{\partial g}{\partial \dot{x}}\delta x \Big|_{t_0}^{t_f}}_{?} - \int_{t_0}^{t_f} \left\{ \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} \right\} \delta x\,dt$ (2) (integration by parts)




admissible curves

Since endpoints are fixed we have $\delta x(t_0) = \delta x(t_f) = 0$

Hence $\displaystyle\frac{\partial g}{\partial \dot{x}}\delta x \Big|_{t_0}^{t_f} = 0 - 0 = 0$ (3)

$\left( \text{And we have } \displaystyle\int_{t_0}^{t_f} \frac{\partial g}{\partial \dot{x}}\delta\dot{x}\,dt = -\int_{t_0}^{t_f} \left\{ \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} \right\} \delta x\,dt \right)$

Combining (1), (2), & (3) gives

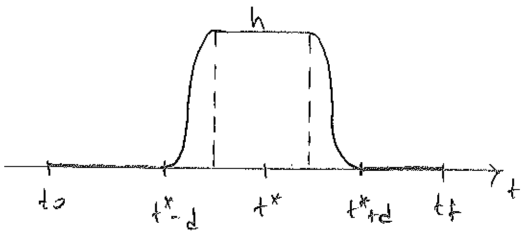
$$\Delta J(x, \delta x) = \underbrace{\int_{t_0}^{t_f} \left\{ \frac{\partial g}{\partial x} - \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} \right\} \delta x \, dt}_{\delta J(x, \delta x)} + o(\delta x, \delta \dot{x}) \qquad (4)$$

By the fund. thm. of calculus of variations we have to have

$$\delta J(x, \delta x) = 0 \quad \forall \delta x \text{ at an extremal } x = x^*. \qquad (5)$$

Lemma (Fund. lemma of the calculus of variations) If a continuous function $f : [t_0, t_f] \to \mathbb{R}^n$ satisfies $\int_{t_0}^{t_f} f(t)^T h(t)\, dt = 0$ for any continuous function $h : [t_0, t_f] \to \mathbb{R}^n$ with $h(t_0) = h(t_f) = 0$ then $f(t) \equiv 0$.

Proof (For the scalar $(n=1)$ case.) Let $f(t^*) > 0$ for some $t^*$, $t_0 < t^* < t_f$. Since $f$ is continuous, $f(t) > c > 0$ in some neighborhood $D$ of the point $t^*$: $t_0 < t^* - d < t < t^* + d < t_f$. Let $h(t)$ be such that $h(t) = 0$ outside $D$, $h(t) \geq 0$ in $D$ and $h(t) = 1$ for $t \in [t^* - \frac{d}{2}, t^* + \frac{d}{2}]$. Then, clearly, $\int_{t_0}^{t_f} f(t) h(t) \geq dc > 0$. This contradiction shows that $f(t^*) = 0$ for all $t^*$, $t_0 < t^* < t_f$.



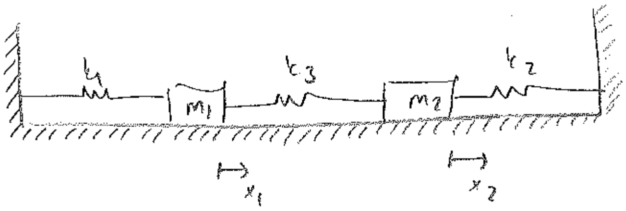

∎

Eq. (4), (5), and the previous lemma tell us that an extremal $x^*$ must satisfy

$$\boxed{\frac{\partial g}{\partial x}(x^*, \dot{x}^*, t) - \frac{d}{dt}\frac{\partial g}{\partial \dot{x}}(x^*, \dot{x}^*, t) = 0}$$

This is the celebrated Euler-Lagrange eqn.

<u>Example</u> Mass - spring system (No friction)




$$KE = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$PE = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_3 (x_1 - x_2)^2$$

let $g = KE - PE$

Given $x(t_0)$ & $x(t_f)$ find the extremal of

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt \qquad \text{where} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

<u>Sol'n</u> $\frac{\partial g}{\partial x} = \left[ \frac{\partial g}{\partial x_1} \quad \frac{\partial g}{\partial x_2} \right]$

$$\frac{\partial g}{\partial x_1} = - \frac{\partial}{\partial x_1} (PE) = - \left[ k_1 x_1 + k_3 (x_1 - x_2) \right]$$

$$\frac{\partial g}{\partial x_2} = - \frac{\partial}{\partial x_2} (PE) = - \left[ k_2 x_2 + k_3 (x_2 - x_1) \right]$$

$$\frac{\partial g}{\partial \dot{x}} = \left[ \frac{\partial g}{\partial \dot{x}_1} \quad \frac{\partial g}{\partial \dot{x}_2} \right] = \left[ m_1 \dot{x}_1 \quad m_2 \dot{x}_2 \right] \Rightarrow \frac{d}{dt} \frac{\partial g}{\partial \dot{x}} = \left[ m_1 \ddot{x}_1 \quad m_2 \ddot{x}_2 \right]$$

Euler Lagrange Eqn : $\frac{\partial g}{\partial x} - \frac{d}{dt} \frac{\partial g}{\partial \dot{x}} = 0$

$$\Rightarrow - \begin{bmatrix} k_1 x_1 + k_3 (x_1 - x_2) \\ k_2 x_2 + k_3 (x_2 - x_1) \end{bmatrix} - \begin{bmatrix} m_1 \ddot{x}_1 \\ m_2 \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} m_1 \ddot{x}_1 + k_1 x_1 + k_3 (x_1 - x_2) = 0 \\ m_2 \ddot{x}_2 + k_2 x_2 + k_3 (x_2 - x_1) = 0 \end{cases}$$

Hence, we've obtained the actual dynamics of the system. That is, the actual trajectories of the mass-spring system is an extremal (in fact minimizer) of the functional $J = \int [KE - PE]$.

<u>Problem</u> Find the extremal of the functional

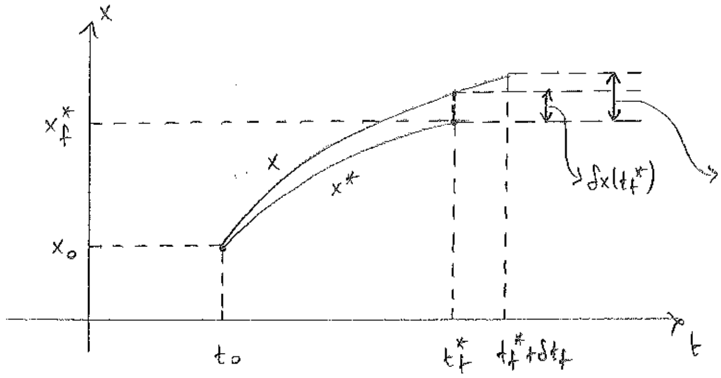
$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt$$

given: $t_0$, $x_0 = x(t_0)$;

this time we let $t_f$ & $x_f = x(t_f)$ be free.

<u>Sol'n</u> $\Delta J = J(x) - J(x^*) = ?$

$$\Delta J = \int_{t_0}^{t_f^*+\delta t_f} g(x,\dot{x},t)dt - \int_{t_0}^{t_f^*} g(x^*,\dot{x}^*,t)dt$$




$x(t) = x^*(t) + \delta x(t)$ for $t \in [t_0, t_f^*]$

(Here we implicitly assume $\delta t_f > 0$, it's easy to see that the end results are the same for $\delta t_f < 0$.)

$$\Rightarrow \Delta J = \underbrace{\int_{t_0}^{t_f^*} \left\{ \frac{\partial}{\partial x} g(x^*,\dot{x}^*,t)\,\delta x + \frac{\partial}{\partial \dot{x}} g(x^*,\dot{x}^*,t)\,\delta\dot{x} \right\} dt}_{F \text{ (first term)}} + \underbrace{\int_{t_f^*}^{t_f^*+\delta t_f} g(x,\dot{x},t)\,dt}_{S \text{ (second term)}} + o(\delta x, \delta\dot{x}) \quad (1)$$

As before, using integration by parts, we can write

$$F = \frac{\partial}{\partial \dot{x}} g(x^*,\dot{x}^*,t)\,\delta x \Big|_{t_0}^{t_f^*} + \int_{t_0}^{t_f^*} \left\{ \frac{\partial}{\partial x} g(x^*,\dot{x}^*,t) - \frac{d}{dt}\frac{\partial}{\partial \dot{x}} g(x^*,\dot{x}^*,t) \right\} \delta x\, dt$$

$\delta x(t_0) = 0$

$$= \frac{\partial}{\partial \dot{x}} g(x^*(t_f^*), \dot{x}^*(t_f^*), t_f^*)\,\delta x(t_f^*) + \int_{t_0}^{t_f^*} \left\{ \frac{\partial}{\partial x} g(x^*,\dot{x}^*,t) - \frac{d}{dt}\frac{\partial}{\partial \dot{x}} g(x^*,\dot{x}^*,t) \right\} \delta x\, dt \quad (2)$$

Also,

$$S = g(x(t_f^*), \dot{x}(t_f^*), t_f^*)\,\delta t_f + o(\delta t_f)$$

$x(t_f^*) = x^*(t_f^*) + \delta x(t_f^*)$

$$= g(x^*(t_f^*), \dot{x}^*(t_f^*), t_f^*)\,\delta t_f + o(\cdot) \quad (3)$$

Combining (1), (2), & (3) yields

$$\Delta J = \frac{\partial}{\partial \dot{x}} g(x^*(t_f^*), \dot{x}^*(t_f^*), t_f^*)\, \delta x(t_f^*) + g(x^*(t_f^*), \dot{x}^*(t_f^*), t_f^*)\, \delta t_f$$

$$+ \int_{t_0}^{t_f^*} \left\{ \frac{\partial}{\partial x} g(x^*, \dot{x}^*, t) - \frac{d}{dt}\frac{\partial}{\partial \dot{x}} g(x^*, \dot{x}^*, t) \right\} \delta x \, dt + o(\cdot) \qquad (4)$$

Note that $\delta x_f = \delta x(t_f^*) + \dot{x}^*(t_f^*)\,\delta t_f + o(\cdot)$ (5)

Using (4) & (5) we can write

$$\left.\begin{aligned} \Delta J = {} & \frac{\partial}{\partial \dot{x}} g(x^*(t_f^*), \dot{x}^*(t_f^*), t_f^*)\, \delta x_f \\ & + \left\{ g(x^*(t_f^*), \dot{x}^*(t_f^*), t_f^*) - \frac{\partial}{\partial \dot{x}} g(x^*(t_f^*), \dot{x}^*(t_f^*), t_f^*)\, \dot{x}^*(t_f^*) \right\} \delta t_f \\ & + \int_{t_0}^{t_f^*} \left\{ \frac{\partial}{\partial x} g(x^*, \dot{x}^*, t) - \frac{d}{dt}\frac{\partial}{\partial \dot{x}} g(x^*, \dot{x}^*, t) \right\} \delta x \, dt \end{aligned}\right\} \delta J$$

$$+ o(\cdot)$$

$$= \delta J(x^*, \delta x, \delta x_f, \delta t_f) + o(\cdot)$$

Now, for $x^*$ to be an extremal we need that the variation vanish, i.e.,

$$\delta J(x^*, \delta x, \delta x_f, \delta t_f) = 0$$

<u>Implications of $\delta J = 0$ for different cases</u>

For all cases the Euler-Lagrange eqn.

$$\boxed{\frac{\partial}{\partial x} g(x^*, \dot{x}^*, t) - \frac{d}{dt}\frac{\partial}{\partial \dot{x}} g(x^*, \dot{x}^*, t) = 0}$$

must be satisfied since the same $x^*$ is clearly an extremal also of the fixed end point problem $\hat{P}$ for $\hat{x}_f = x^*(t_f^*)$ and $\hat{t}_f = t_f^*$

Case 1 $t_f$ and $x_f$ fixed. (We already studied this case) Fixed final time and final state mean $\delta t_f = 0$ & $\delta x_f = 0$. Hence $\delta J = 0$ implies only the Euler-Lagrange equation.

Case 2 $t_f$ fixed, $x_f$ free. Fixed final time means $\delta t_f = 0$. Then $\delta J = 0$ implies the coefficient of $\delta x_f$ must be zero. Hence

$$\boxed{\frac{\partial}{\partial \dot{x}} g(x^*(t_f), \dot{x}^*(t_f), t_f) = 0} \qquad t_f \text{ fixed, } x_f \text{ free}$$

Case 3 $t_f$ free, $x_f$ fixed. This time $\delta x_f = 0$. Then $\delta J = 0$ implies the coefficient of $\delta t_f$ must be zero. Hence

$$\boxed{g(x^*(t_f^*), \dot{x}^*(t_f^*), t_f^*) - \frac{\partial}{\partial \dot{x}} g(x^*(t_f^*), \dot{x}^*(t_f^*), t_f^*)\, \dot{x}^*(t_f^*) = 0} \qquad t_f \text{ free, } x_f \text{ fixed}$$

Case 4 Both $t_f$ and $x_f$ free, unrelated. In this case $\delta t_f$ and $\delta x_f$ are independent of each other and arbitrary. Hence $\delta J = 0$ implies both of their coefficients must be zero. Therefore

$$\boxed{\begin{aligned} &\frac{\partial}{\partial \dot{x}} g(x^*(t_f^*), \dot{x}^*(t_f^*), t_f^*) = 0 \quad \& \\ &g(x^*(t_f^*), \dot{x}^*(t_f^*), t_f^*) = 0 \end{aligned}} \qquad t_f, x_f \text{ free \& independent}$$

Case 5 $t_f$ & $x_f$ are related by a given map $\theta : \mathbb{R} \to \mathbb{R}^n$, i.e., $x(t_f) = \theta(t_f)$: We can write (why?)

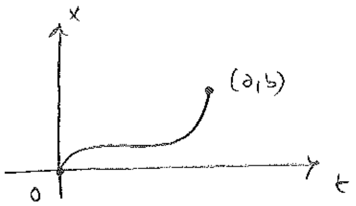
$$\delta x_f = \frac{d}{dt}\theta(t_f^*)\, \delta t_f + o(\cdot)$$

This, together with $\delta J = 0$ imply

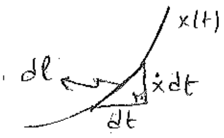
$$\boxed{\begin{aligned} &g(x^*(t_f^*), \dot{x}^*(t_f), t_f^*) + \frac{\partial}{\partial \dot{x}} g(x^*(t_f^*), \dot{x}^*(t_f^*), t_f^*)\left[\dot{\theta}(t_f^*) - \dot{x}(t_f^*)\right] = 0 \quad \& \\ &x^*(t_f^*) = \theta(t_f^*) \end{aligned}} \qquad t_f, x_f \text{ free \& related}$$

<u>Example</u> Find the shortest curve that connects the points

$(t_0, x(t_0)) = (0,0)$ & $(t_f, x(t_f)) = (a,b)$ $\quad a,b>0$




length of a curve ?




$(dl)^2 = (dt)^2 + (\dot{x}\,dt)^2$

$\Rightarrow \; dl = \sqrt{1+\dot{x}^2}\; dt$

$J = \int_{t_0}^{t_f} \sqrt{1+\dot{x}(t)^2}\, dt \quad \Rightarrow \quad g(x,\dot{x},t) = \sqrt{1+\dot{x}^2}$

Euler-Lagrange Eqn. $\quad \underbrace{\dfrac{\partial g}{\partial x}}_{0} - \dfrac{d}{dt}\dfrac{\partial g}{\partial \dot{x}} = 0$

$\Rightarrow \; 0 = \dfrac{d}{dt}\dfrac{\partial g}{\partial \dot{x}} = \dfrac{d}{dt}\dfrac{\dot{x}}{\sqrt{1+\dot{x}^2}} \quad \Rightarrow \quad \dot{x} = \text{constant}$

$\Rightarrow \; x(t) = c_0 + c_1 t$ $\quad$ (the form of an extremal)

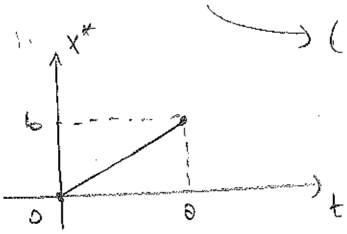
<u>$c_0, c_1 = ?$</u>

$x(0) = 0 \quad \Rightarrow \quad c_0 = 0$

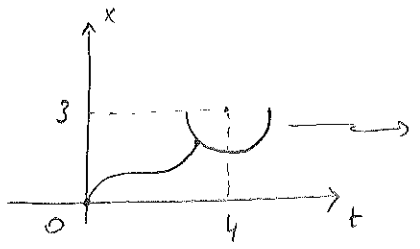
$x(a) = b \quad \Rightarrow \quad c_1 = \dfrac{b}{a}$

Hence the extremal path is a straight line

→ (for our case it is the shortest)



<u>Example</u> Find the shortest curve that connects the point $(t_0, x(t_0)) = (0,0)$ to the surface $(x-3)^2 + (t-4)^2 = 1$



$$x = 3 - \left[1-(t-4)^2\right]^{1/2} =: \theta(t)$$

As before, Euler-Lagrange Eqn. implies $x^*$ is a straight line

$$x^*(t) = c_0 + c_1 t \quad \Rightarrow \quad x^*(t) = c_1 t \quad (\text{since } x^*(0) = 0)$$

Also, we have to have

$$g(x^*(t_f^*), \dot{x}^*(t_f), t_f^*) + \frac{\partial}{\partial \dot{x}} g(x^*(t_f^*), \dot{x}^*(t_f^*), t_f^*)\left[\dot{\theta}(t_f^*) - \dot{x}^*(t_f^*)\right] = 0 \qquad (1)$$

Recall that $g = (1+\dot{x}^2)^{1/2}$. Hence (1) becomes

$$\left[1+\dot{x}^*(t_f^*)^2\right]^{1/2} + \frac{\dot{x}^*(t_f^*)}{\left[1+\dot{x}^*(t_f^*)^2\right]^{1/2}}\left[\frac{t_f^*-4}{\left[1-(t_f^*-4)^2\right]^{1/2}} - \dot{x}^*(t_f^*)\right] = 0$$

$$\Rightarrow \left[1 + \cancel{\dot{x}^*(t_f^*)^2}\right] + \dot{x}^*(t_f^*)\left[\frac{t_f^*-4}{\left[1-[t_f^*-4]^2\right]^{1/2}} - \cancel{\dot{x}^*(t_f^*)}\right] = 0 \qquad \dot{x}^* = c_1$$

$$\Rightarrow \quad 1 + c_1 \cdot \frac{t_f^*-4}{\left[1-[t_f^*-4]^2\right]^{1/2}} = 0 \qquad (2)$$

Also, $x^*(t_f^*) = \theta(t_f^*)$ implies

$$c_1 t_f^* = 3 - \left[1-(t_f^*-4)^2\right]^{1/2} \qquad (3)$$

(2) & (3) $\Rightarrow$ $c_1 = \frac{3}{4}$ & $t_f^* = \frac{16}{5}$

Hence $x^*(t) = \frac{3}{4} t$.

Example Find the extremal for

$$J(x) = \int_0^{\pi/2} \{ \dot{x}_1^2 + \dot{x}_2^2 + 2x_1x_2 \} dt \qquad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; x(\tfrac{\pi}{2}) = \begin{bmatrix} \text{free} \\ 1 \end{bmatrix}$$

$$\delta J = \underbrace{\frac{\partial}{\partial \dot{x}} g(x^*(\tfrac{\pi}{2}), \dot{x}^*(\tfrac{\pi}{2}), \tfrac{\pi}{2}) \delta x_f}_{} + \int_0^{\frac{\pi}{2}} \underbrace{\left\{ \frac{\partial}{\partial x} g(x^*, \dot{x}^*, t) - \frac{d}{dt}\frac{\partial}{\partial \dot{x}} g(x^*, \dot{x}^*, t) \right\}}_{=0} \delta x \, dt = 0$$

$$\begin{bmatrix} \frac{\partial g}{\partial \dot{x}_1} & \frac{\partial g}{\partial \dot{x}_2} \end{bmatrix} \begin{bmatrix} \delta x_{1f} \\ \delta x_{2f} \end{bmatrix} \qquad \delta x_{2f} = 0 \text{ because } x_2(t_f) \text{ is fixed.}$$

$$\Rightarrow \frac{\partial}{\partial \dot{x}_1} g(x^*(\tfrac{\pi}{2}), \dot{x}^*(\tfrac{\pi}{2}), \tfrac{\pi}{2}) = 0$$

$$\Rightarrow 2\dot{x}_1(\tfrac{\pi}{2}) = 0 \Rightarrow \dot{x}_1(\tfrac{\pi}{2}) = 0$$

Also, the Euler-Lagrange Eqn. yields

$$\begin{bmatrix} 2x_2 \\ 2x_1 \end{bmatrix} - \frac{d}{dt}\begin{bmatrix} 2\dot{x}_1 \\ 2\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left. \begin{aligned} \ddot{x}_1 &= x_2 \\ \ddot{x}_2 &= x_1 \end{aligned} \right\} \overset{(4)}{x_1} = x_1 \Rightarrow (D^4 - 1)x_1 = 0$$

$\Rightarrow$ char. poly. $d(s) = s^4 - 1 = (s-1)(s+1)(s-j)(s+j)$

$\Rightarrow x_1(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$ (we need 4 eqn. to determine $c_1, c_2, c_3, c_4$)

$x_1(0) = 0 \Rightarrow c_1 + c_2 + c_3 = 0$ (1)

$x_2(0) = 0 \Rightarrow \ddot{x}_1(0) = 0 \Rightarrow c_1 + c_2 - c_3 = 0$ (2)

$x_2(\frac{\pi}{2}) = 1 \Rightarrow \ddot{x}_1(\frac{\pi}{2}) = 1 \Rightarrow c_1 e^{\frac{\pi}{2}} + c_2 e^{-\frac{\pi}{2}} - c_4 = 1$ (3)

$\dot{x}_1(\frac{\pi}{2}) = 0 \Rightarrow c_1 e^{\frac{\pi}{2}} - c_2 e^{-\frac{\pi}{2}} - c_3 = 0$ (4)

(1), (2), (3), (4) $\Rightarrow c_4 = -1$ & $c_1 = c_2 = c_3 = 0$

$\Rightarrow x_1^*(t) = -\sin t$ & $x_2 = \ddot{x}_1 \Rightarrow x_2^*(t) = \sin t$

$$\Rightarrow \boxed{x^*(t) = \begin{bmatrix} -\sin t \\ \sin t \end{bmatrix}}$$

Example Find the extremal for

$$J(x) = \int_0^1 \{x\dot{x} + \ddot{x}^2\}dt \qquad \begin{array}{l} x(0)=0, \quad \dot{x}(0)=1 \\ x(1)=2, \quad \dot{x}(1)=4 \end{array}$$

How to deal with $\ddot{x}$ ?

Define $z_1 = x$, $z_2 = \dot{x}$ and let $p \in \mathbb{R}$

Let $z = \begin{bmatrix} z_1 \\ z_2 \\ p \end{bmatrix}$. Note that $z_1, z_2$ are not independent, but $\dot{z}_1 = z_2$

Consider the functional

$$\hat{J}(z) = \int_0^1 \{z_1 z_2 + \dot{z}_2^2 + p(\dot{z}_1 - z_2)\}dt \qquad \begin{array}{l} z_1(0)=0, \quad z_2(0)=1 \\ z_1(1)=2, \quad z_2(1)=4 \end{array}$$

Claim The extremal $x^*$ of $J$ equals the extremal $z_1^*$ of $\hat{J}$.

proof Exercise.

Euler-Lagrange Eqn. $\dfrac{\partial \hat{g}}{\partial z} - \dfrac{d}{dt}\dfrac{\partial \hat{g}}{\partial \dot{z}} = 0$

$$\frac{\partial \hat{g}}{\partial z} = [z_2 \quad z_1 - p \quad \dot{z}_1 - z_2] \qquad \frac{\partial \hat{g}}{\partial \dot{z}} = [p \quad 2\dot{z}_2 \quad 0]$$

$$\Rightarrow \quad \begin{aligned} \dot{p} &= z_2 \\ \ddot{z}_2 &= \frac{1}{2}(z_1 - p) \\ 0 &= \dot{z}_1 - z_2 \end{aligned}$$

$\Rightarrow$ since the extremal $z^*$ should satisfy this ...
we have $J(x^*) = \hat{J}(z^*)$

$$\Rightarrow \dddot{z}_2 = \frac{1}{2}(\underbrace{\dot{z}_1}_{z_2} - \underbrace{\dot{p}}_{z_2}) = 0$$

Since $z_2 = \dot{x}$ we have $x^{(4)} = 0 \Rightarrow x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$

$x(0) = 0 \Rightarrow c_0 = 0$

$\dot{x}(0) = 1 \Rightarrow c_1 = 1$

$x(1) = 2 \Rightarrow 1 + c_2 + c_3 = 2$

$\dot{x}(1) = 4 \Rightarrow 1 + 2c_2 + 3c_3 = 4$

$\Rightarrow c_2 = 0, \; c_3 = 1$

$$\boxed{x^*(t) = t^3 + t}$$

<u>Example</u> Find the extremal for

$$J(x) = \int_0^1 (\dot{x}^2 + t^2)\,dt \quad \text{subj. to} \quad \begin{array}{l} x(0) = 0 \\ x(1) = 0 \end{array} \quad \& \quad \int_0^1 x^2\,dt = 2$$

Note that $J(x) = \int_0^1 \dot{x}^2\,dt + \underbrace{\int_0^1 t^2\,dt}_{\text{constant}}$. Hence we can let $g = \dot{x}^2$

Let $z_1 = x$ & $\dot{z}_2 = z_1^2$ & $z_2(0) = 0$

Then $2 = \int_0^1 x^2\,dt = \int_0^1 \dot{z}_2\,dt = z_2(1) - z_2(0) \Rightarrow z_2(1) = 2$

let $z = \begin{bmatrix} z_1 \\ z_2 \\ p \end{bmatrix}$ and consider the functional

$$\hat{J}(z) = \int_0^1 \{\dot{z}_1^2 + p(\dot{z}_2 - z_1^2)\}\,dt \quad \text{subj. to} \quad \begin{array}{ll} z_1(0) = 0, & z_2(0) = 0 \\ z_1(1) = 0, & z_2(1) = 2 \end{array}$$

Euler-Lagrange eqn. $\dfrac{\partial \hat{g}}{\partial z} - \dfrac{d}{dt}\dfrac{\partial \hat{g}}{\partial \dot{z}} = 0$

$[-2z_1 p \quad 0 \quad \dot{z}_2 - z_1^2]$ $\qquad [2\dot{z}_1 \quad p \quad 0]$

$$\Rightarrow \quad \begin{aligned} \ddot{z}_1 &= -pz_1 \\ \dot{p} &= 0 \\ 0 &= \dot{z}_2 - z_1^2 \end{aligned}$$

$p$ is constant & $\ddot{z}_1 = -pz_1$

$p > 0$ otherwise ($p < 0$) the constraints cannot be met.

$z_1(t) = c_1 \cos \omega t + c_2 \sin \omega t$

$z_1(0) = 0 \Rightarrow c_1 = 0$

$z_1(1) = 0 \Rightarrow \omega = n\pi \quad n = 1, 2, 3, \ldots$

$\Rightarrow z_1(t) = c_2 \sin(n\pi t)$

Also, $\int_0^1 z_1^2\,dt = 2 \Rightarrow \dfrac{c_2^2}{2} = 2 \Rightarrow c_2 = \mp 2$

Hence, $\boxed{x^*(t) = \mp 2 \sin(n\pi t)} \quad n = 1, 2, \ldots$

## Optimal control

**Problem:** minimize the cost (by choosing "optimal control" $u^*: [t_0, t_f] \to U$)

$$J = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t)\, dt \qquad (x \in \mathbb{R}^n,\ u \in \mathbb{R}^m)$$

subject to $\dot{x} = f(x, u, t)$

given: $t_0$, $x_0 = x(t_0)$.

Solution approach: We will use the variational techniques we developed earlier.

To this end we first construct an augmented cost function $J_a$ that

- → yields the same optimal control $u^*$
- → does not have a terminal cost $h(x(t_f), t_f)$
- → is not (explicitly) subject to the constraint $\dot{x} = f(x, u, t)$

[Initially we will suppose $U = \mathbb{R}^m$, i.e., we have no constraint on the input $u$]

— o —

We can write

$$h(x(t_f), t_f) = \int_{t_0}^{t_f} \left\{ \frac{d}{dt} h(x, t) \right\} dt + h(x(t_0), t_0)$$

$$\Rightarrow \quad J = \int_{t_0}^{t_f} \left\{ g(x, u, t) + \frac{d}{dt} h(x, t) \right\} dt + h(x(t_0), t_0)$$

this term is fixed hence we can ignore it

Now, we can reformulate the problem as

$$\text{minimize } J = \int_{t_0}^{t_f} \left\{ g(x, u, t) + \frac{\partial}{\partial x} h(x, t)\, \dot{x} + \frac{\partial}{\partial t} h(x, t) \right\} dt$$

subject to $\dot{x} = f(x, u, t)$

Define the augmented integrand

$$g_a(x, \dot{x}, u, p, t) = g(x, u, t) + \frac{\partial}{\partial x} h(x,t)\,\dot{x} + \frac{\partial}{\partial t} h(x,t) + p^T\left(f(x,u,t) - \dot{x}\right)$$

where $p \in \mathbb{R}^n$ is called the Lagrange multiplier. Lagrange multiplier allows us to remove the constraint $\dot{x} = f(x,u,t)$. Now, consider the cost

$$J_a = \int_{t_0}^{t_f} g_a(x, \dot{x}, u, p, t)\,dt .$$

The original cost $J$ and the augmented cost $J_a$ share the same minimizer $u^*$.
To see this let us study the variation $\delta J_a$ (using our earlier analyses)

$$\delta J_a = \int_{t_0}^{t_f^*} \left\{ \frac{\partial g_a}{\partial x}\delta x + \frac{\partial g_a}{\partial \dot{x}}\delta \dot{x} + \frac{\partial g_a}{\partial u}\delta u + \frac{\partial g_a}{\partial p}\delta p \right\} dt + \int_{t_f^*}^{t_f^* + \delta t_f} g_a(x, \dot{x}, u, p, t)\,dt \qquad (1)$$

these partial derivatives are evaluated at optimal trajectories

Recall that

$$\int_{t_0}^{t_f^*} \frac{\partial g_a}{\partial \dot{x}}\delta \dot{x}\,dt = \frac{\partial g_a}{\partial \dot{x}}\delta x \Big|_{t_0}^{t_f^*} - \int_{t_0}^{t_f^*} \frac{d}{dt}\left\{\frac{\partial g_a}{\partial \dot{x}}\right\}\delta x\,dt \qquad (2)$$

$$\frac{\partial g_a}{\partial \dot{x}}\delta x \Big|_{t_0}^{t_f^*} = \frac{\partial}{\partial \dot{x}} g_a \underbrace{\left(x^*(t_f^*), \dot{x}^*(t_f^*), u^*(t_f^*), p^*(t_f^*), t_f^*\right)}_{w^*(t_f^*)} \delta x(t_f^*) \qquad (3)$$

$$\int_{t_f^*}^{t_f^* + \delta t_f} g_a(x, \dot{x}, u, p, t)\,dt = g_a\left(w^*(t_f^*)\right)\delta t_f^* + o(\cdot) \qquad (4)$$

$$\delta x_f = \delta x(t_f^*) + \dot{x}^*(t_f^*)\,\delta t_f + o(\cdot) \qquad (5)$$

Combining (1), (2), (3), (4), (5) we have

$$\delta J_a = \frac{\partial}{\partial \dot{x}} g_a(w^*(t_f^*))\delta x_f + \left\{ g_a(w^*(t_f^*)) - \frac{\partial}{\partial \dot{x}} g_a(w^*(t_f^*))\, \dot{x}^*(t_f^*) \right\} \delta t_f$$

$$+ \int_{t_0}^{t_f^*} \left\{ \left( \frac{\partial g_a}{\partial x} - \frac{d}{dt}\frac{\partial g_a}{\partial \dot{x}} \right)\delta x + \frac{\partial g_a}{\partial u}\delta u + \frac{\partial g_a}{\partial p}\delta p \right\} dt$$

evaluated at $w^*(t) = (x^*(t), \dot{x}^*(t), u^*(t), p^*(t), t)$

Optimality requires $\delta J_a = 0$, which gives us four equations

First $\quad \frac{\partial}{\partial \dot{x}} g_a(w^*(t_f^*))\,\delta x_f + \left\{ g_a(w^*(t_f^*)) - \frac{\partial}{\partial \dot{x}} g_a(w^*(t_f^*))\, \dot{x}^*(t_f^*) \right\} \delta t_f = 0$

$$\Rightarrow \left[ \frac{\partial}{\partial x} h(x^*(t_f^*), t_f^*) - p^*(t_f^*)^T \right] \delta x_f$$

$$+ \left[ g(x^*(t_f^*), u^*(t_f^*), t_f^*) + \frac{\partial}{\partial t} h(x^*(t_f^*), t_f^*) + p^*(t_f^*)^T f(x^*(t_f^*), u^*(t_f^*), t_f^*) \right] \delta t_f = 0$$

Second $\quad \frac{\partial g_a}{\partial x} - \frac{d}{dt}\frac{\partial g_a}{\partial \dot{x}} = 0$

$$\Rightarrow \frac{\partial g}{\partial x} + \cancel{\frac{\partial}{\partial x}\left[\frac{d}{dt} h\right]} + p^T \frac{\partial f}{\partial x} - \frac{d}{dt}\left\{ \cancel{\frac{\partial h}{\partial x}} - p^T \right\} = 0$$

$$\Rightarrow \dot{p}^*(t)^T = -\frac{\partial}{\partial x} g(x^*(t), u^*(t), t) - p^*(t)^T \frac{\partial}{\partial x} f(x^*(t), u^*(t), t)$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x} \\ \vdots \\ \frac{\partial f_n}{\partial x} \end{bmatrix}_{n \times n}$$

Third $\quad \frac{\partial g_a}{\partial u} = 0$

$$\Rightarrow \frac{\partial}{\partial u} g(x^*(t), u^*(t), t) + p^*(t)^T \frac{\partial}{\partial u} f(x^*(t), u^*(t), t) = 0$$

Fourth $\quad \frac{\partial g_a}{\partial p} = 0 \quad \Rightarrow \dot{x}^*(t) = f(x^*(t), u^*(t), t).$

These four equations can be simplified notationwise by introducing the "Hamiltonian"

$$\boxed{H(x, u, p, t) = g(x, u, t) + p^T f(x, u, t)}$$

Let us summarize our results (four equations) in terms of Hamiltonian.

<u>Problem</u> minimize $J = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t)\, dt$

subject to $\dot{x} = f(x, u, t)$

Optimal control $u^*(t)$ and the resulting trajectory $x^*(t)$ satisfy

$$\boxed{\left.\begin{aligned} \dot{x} &= \nabla_p H \\ \dot{p} &= -\nabla_x H \\ 0 &= \nabla_u H \end{aligned}\right\} \text{ for all } t \in [t_0, t_f^*]}$$

Moreover, we have

$$\boxed{\left[\frac{\partial h}{\partial x} - p^T\right]\delta x_f + \left[H + \frac{\partial h}{\partial t}\right]\delta t_f = 0 \qquad \text{for } t = t_f^*}$$

<u>Claim</u> Suppose $g$ & $f$ are time invariant. That is, $g = g(x, u)$ & $f = f(x, u)$. Then $H = H(x, u, p)$ is constant along optimal trajectories.

proof
$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial p}\dot{p} + \cancelto{0}{\frac{\partial H}{\partial u}}\dot{u}$$

$$= \langle \nabla_x H, \nabla_p H \rangle + \langle \nabla_p H, -\nabla_x H \rangle$$

$$= 0$$

▨

Notes

1) The auxiliary variable $p \in \mathbb{R}^n$ is called the <u>costate</u>. (Recall that we call $x$ the <u>state</u>.) Hence $\dot{x} = \nabla_p H$ is called the state equation and $\dot{p} = -\nabla_x H$ is called the costate equation.

2) When we have control constraints $u \in U$ (as opposed to $u \in \mathbb{R}^m$) the equation $0 = \nabla_u H$ is generalized into

$$u^*(t) = \arg\min_{u \in U} H(x^*(t), u, p^*(t), t)$$

(See the text for the derivation). In this more general form the set of conditions

$$\boxed{\begin{array}{ll} \left.\begin{array}{l} \dot{x} = \nabla_p H \\ \dot{p} = -\nabla_x H \\ u^*(t) = \arg\min_{u \in U} H(x^*(t), u, p^*(t), t) \end{array}\right\} & t \in [t_0, t_f^*] \\ \left[\frac{\partial h}{\partial x} - p^T\right]\delta x_f + \left[H + \frac{\partial h}{\partial t}\right]\delta t_f = 0 & t = t_f^* \end{array}}$$

are called <u>Pontryagin's Minimum Principle</u>.

3) These conditions are only necessary. That is, there may exist non-optimal solutions satisfying Pontryagin's Min. Principle.

<u>Example</u> (Revisited) Find the shortest curve that connects the points

$(t_0, x(t_0)) = (0, 0)$ & $(t_f, x(t_f)) = (a, b)$ $\quad a, b > 0$

Recall that the problem is

$$\text{minimize} \quad J = \int_{t_0}^{t_f} \sqrt{1 + \dot{x}(t)^2}\, dt$$

<u>Q</u>: How to formulate this problem as an optimal control problem?

A: Consider the following problem

$$\underset{u\in\mathbb{R}}{\text{minimize}} \quad J(u) = \int_{t_0}^{t_f} \sqrt{1+u(t)^2}\, dt \qquad \text{subject to} \qquad \dot{x} = u$$

Hamiltonian

$$H(x,u,p) = g(x,u,t) + p^T f(x,u,t) \qquad ( g = \sqrt{1+u^2}\ ,\ f = u )$$

$$= \sqrt{1+u^2} + pu$$

State eqn

$$\dot{x} = \nabla_p H = \frac{\partial}{\partial p}\left\{\sqrt{1+u^2} + pu\right\} = u$$

$$\Rightarrow \dot{x} = u$$

Costate eqn

$$\dot{p} = -\nabla_x H = -\frac{\partial}{\partial x}\left\{\sqrt{1+u^2} + pu\right\} = 0$$

$$\Rightarrow \dot{p} = 0 \quad \Rightarrow \quad p(t) \equiv p(0) \ (\text{constant})$$

optimal u

$$0 = \nabla_u H = \frac{\partial}{\partial u}\left\{\sqrt{1+u^2} + p(0)\cdot u\right\} = p(0) + \frac{u}{\sqrt{1+u^2}}$$

$$\Rightarrow u(t) \equiv c_1 \ (\text{constant})$$

boundary condition

Since $t_f$ & $x(t_f)$ are fixed we have $\delta x_f = 0$ & $\delta t_f = 0$. Hence

$$\left[\frac{\partial h}{\partial x} - p^T\right]\underbrace{\delta x_f}_{0} + \left[H + \frac{\partial h}{\partial t}\right]\underbrace{\delta t_f}_{0} = 0 \qquad (h=0)$$

$\Rightarrow \quad 0 = 0$ (i.e., boundary condition gives us nothing for this problem)

<u>Solution</u> Since optimal control $u^*(t) = c_1$ (constant) we have

$\dot{x} = c_1 \Rightarrow x(t) = c_1 t + c_0$

$$\left.\begin{array}{l} x(0) = 0 \Rightarrow c_0 = 0 \\ x(a) = b \Rightarrow c_1 = \frac{b}{a} \end{array}\right\} \quad \boxed{x^*(t) = \frac{b}{a} t} \quad \text{(as expected)}$$

<u>Example</u> Consider the previous example, this time under the criterion

$(t_0, x(t_0)) = (0, 0)$ & $(t_f, x(t_f)) = (a, \text{free})$

$$\min \quad J(u) = \int_{t_0}^{t_f} \sqrt{1 + u(t)^2}\, dt \quad \text{subject to} \quad \dot{x} = u$$

$H = \sqrt{1+u^2} + pu$ (as before)

$\dot{x} = u$ (as before)

$\dot{p} = 0$ (as before)

$0 = p(0) + \frac{u}{\sqrt{1+u^2}}$ (1) (as before)

<u>but</u> boundary condition implies

$$\Big[\frac{\partial h}{\partial x} - p\Big] \underbrace{\delta x_f}_{\text{free}} + \Big[H + \frac{\partial h}{\partial t}\Big] \underbrace{\delta t_f}_{0} = 0$$

$\Rightarrow \left. \frac{\partial h}{\partial x} - p \right|_{t = t_f} = 0$ since we have no terminal cost ($h \equiv 0$)

this means $p(t_f) = 0$.

Now, $\dot{p} = 0 \Rightarrow p(0) = p(t_f) = 0$

Then, (1) $\Rightarrow u^*(t) \equiv 0 \Rightarrow \boxed{x^*(t) = x(0)}$ (optimal trajectory is horizontal)

Example (Resource Allocation [Bertsekas])

A producer with production rate $x(t)$ at time $t$ may allocate a portion $u(t)$ of her production rate to reinvestment and $1-u(t)$ to production of a storable good. Thus $x(t)$ evolves according to

$$\dot{x} = \gamma u x \qquad (\gamma > 0 \text{ is a given constant})$$

The producer wants to maximize the total amount of product stored

$$\int_0^T (1-u(t))\, x(t)\, dt$$

subject to $0 \le u(t) \le 1$ for all $t \in [0,T]$. Assume $x(0) > 0$ is given.

Problem ?

$$\min_{u \in U} \; J = -\int_0^T (1-u)\, x\, dt = \int_0^T (u-1)\, x\, dt$$

subject to $\dot{x} = \gamma u x$

$$(t_0, x(t_0)) = (0, x(0)) \quad \& \quad (t_f, x(t_f)) = (T, \text{free})$$

Hamiltonian $\quad H = (u-1)x + \gamma p u x$

state eqn. $\quad \dot{x} = \nabla_p H = \gamma u x$

costate eqn. $\quad \dot{p} = -\nabla_x H = 1 - (1+\gamma p) u$

boundary cond. $\quad \frac{\partial h}{\partial x} - p = 0 \Rightarrow p^*(T) = 0$

optimal control $\quad u^*(t) = \arg \min_{u \in [0,1]} \; -x^*(t) + (1+\gamma p^*(t))\, u$

Therefore $\quad u^*(t) = \begin{cases} 1 & 1+\gamma p^*(t) < 0 \quad (p^*(t) < -\frac{1}{\gamma}) \\ 0 & 1+\gamma p^*(t) > 0 \quad (p^*(t) > -\frac{1}{\gamma}) \end{cases}$

Note that the evolution of $(p^*(t), u^*(t))$ does not depend on the state $x^*(t)$ !

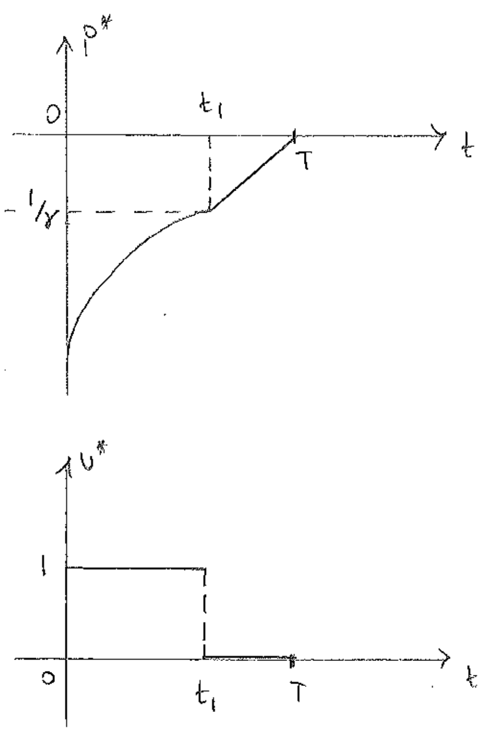
Integrating backward in time (since we only know $p^*(T)$) we obtain

$$\dot{p}^*(t) = \begin{cases} t - T & t \in [T - \frac{1}{\gamma}, T] & (\dot{p} = 1, \ u = 0) \\ -\frac{1}{\gamma} e^{-\gamma(t - t_1)} & t \in [0, T - \frac{1}{\gamma}) & (\dot{p} = -\gamma p, \ u = 1) \end{cases} \qquad \left( t_1 = T - \frac{1}{\gamma} \right.$$

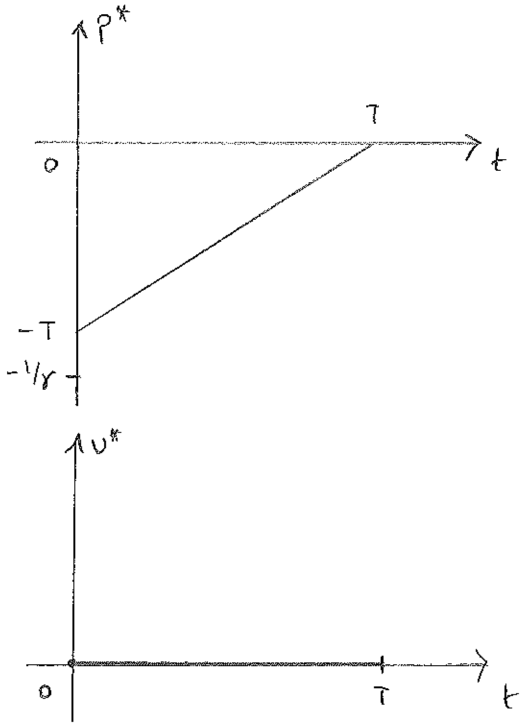
and

$$u^*(t) = \begin{cases} 0 & t \in [T - \frac{1}{\gamma}, T] \\ 1 & t \in [0, T - \frac{1}{\gamma}) \end{cases}$$

<u>Case 1</u> $T > \frac{1}{\gamma}$

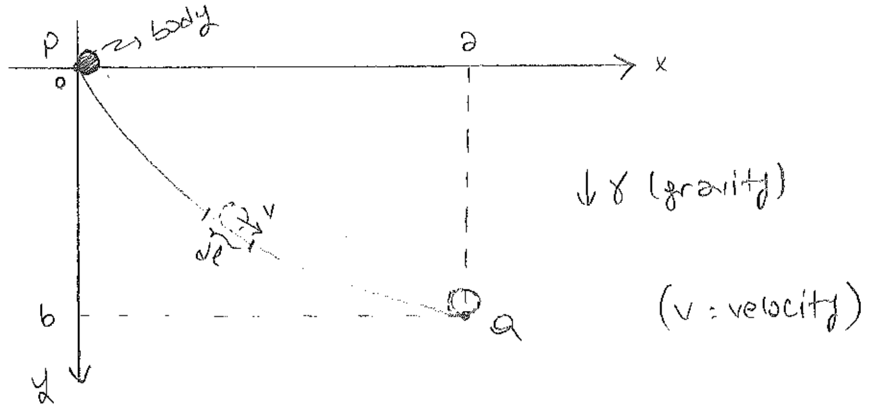



<u>Case 2</u> $T < \frac{1}{\gamma}$




<u>Conclusion</u> As the second case indicates, if the horizon (T) is not long enough, it does not pay to reinvest.

<u>Example</u> (Brachistochrone Problem) In 1696 the great mathematician Bernoulli asked the following question. Given two points $P=(0,0)$ & $Q=(a,b)$ $(a,b>0)$ on plane, what is the optimal shape of the curve $y(x)$ that connects P & Q such that a body with zero initial velocity left at point P reaches point Q (under the influence of gravity) in shortest time?




<u>Problem formulation</u>

$$v(y) = ? \qquad \underbrace{\tfrac{1}{2}mv^2}_{K.E.} = \underbrace{m\gamma y}_{P.E.} \quad \Rightarrow \quad v = \sqrt{2\gamma y}$$

Let
- $dt$ : time elapsed while covering the distance $dl$
- $dl$ : the (local) length of the curve

Then $\dfrac{dl}{dt} = v$

Note that $dl = \sqrt{dx^2 + dy^2}$

Then $$dt = \frac{\sqrt{dx^2+dy^2}}{\sqrt{2\gamma y}} = \frac{\sqrt{1+\dot{y}^2}\,dx}{\sqrt{2\gamma y}} \qquad \text{where } \dot{y} = \frac{dy}{dx}$$

Hence the problem can be stated as

$$\min \int_P^Q dt = \min \int_0^a \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2\gamma y}}\,dx$$

Hence we've obtained the following optimal control problem :

$$\min_{u} \int_0^a \frac{\sqrt{1+u^2}}{\sqrt{2\gamma y}}\, dx \qquad \text{subject to } \dot{y} = u \quad (1) \qquad \left(\dot{y} = \frac{dy}{dx}\right)$$

Hamiltonian $\quad H = \dfrac{\sqrt{1+u^2}}{\sqrt{2\gamma y}} + pu$

costate eqn. $\quad \dot{p} = -\nabla_y H = \dfrac{\gamma\sqrt{1+u^2}}{(2\gamma y)^{3/2}} \qquad (2)$

optimal control $\quad 0 = \nabla_u H = \dfrac{u}{\sqrt{1+u^2}\sqrt{2\gamma y}} + p \qquad (3)$

Note that

$$\dot{H} = \nabla_y H \dot{y} + \nabla_p H \dot{p} + \cancel{\nabla_u H}^{\,0} \dot{u}$$
$$= (\nabla_y H)(\nabla_p H) + (\nabla_p H)(-\nabla_y H)$$
$$= 0$$

Hence $H$ = constant . That is,

$$\frac{\sqrt{1+u^2}}{\sqrt{2\gamma y}} + pu = C \qquad (4)$$

$$(3),(4) \Rightarrow \frac{\sqrt{1+u^2}}{\sqrt{2\gamma y}} - \frac{u^2}{\sqrt{1+u^2}\sqrt{2\gamma y}} = C$$

$$\Rightarrow \frac{1}{\sqrt{1+u^2}\sqrt{2\gamma y}} = C \qquad (5)$$

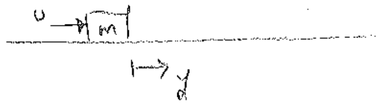
$$(1),(5) \Rightarrow \sqrt{1+\dot{y}^2}\sqrt{2\gamma y} = \frac{1}{C}$$

"cycloid" eqn.

$$\Rightarrow 1+\dot{y}^2 = \frac{1}{C(2\gamma y)} \Rightarrow \boxed{\dot{y}^*(x) = \sqrt{\frac{1}{2\gamma C\, y^*(x)} - 1}}$$

the unknown $C$ & the solutions are found using constraints $y(0)=0$ & $y(a)=b$ (cycloid)

<u>Example</u> (Minimum time problem)




Bring the mass m to rest at the origin as quickly as possible. ($t_f$ : free)

$m = 1$

$u$ : force

$y$ : position

$y(0), \dot{y}(0)$ : given

$u \in [-1, 1]$

<u>Problem formulation</u> Let $x_1 = y$ & $x_2 = \dot{y}$ , $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow$ system $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = \begin{bmatrix} x_2 \\ u \end{bmatrix} =: f(x, u)$

problem : given $x(0)$ = something & $x(t_f) = 0$

minimize $t_f = \int_0^{t_f} 1 \cdot dt$ $(g \equiv 1)$

subject to $\dot{x} = f(x, u)$

$H = g + p^T f = 1 + [p_1 \ p_2] \begin{bmatrix} x_2 \\ u \end{bmatrix} = 1 + p_1 x_2 + p_2 u$

costate eqn. $\dot{p} = -\nabla_x H \Rightarrow \dot{p}_1 = -\frac{\partial H}{\partial x_1} = 0$

$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = -p_1$

optimal control : $u^* = \arg\min_{-1 \le u \le 1} H = \arg\min_u 1 + p_1 x_2 + p_2 u$

$\Rightarrow u^*(t) = \begin{cases} 1 & \text{if } p_2^*(t) < 0 \\ -1 & \text{if } p_2^*(t) > 0 \end{cases}$

$\Rightarrow u^*$ is piecewise constant.

Note that $u^*(t)$ can take any (admissible) value during an interval $t \in [t_1, t_2]$ $(t_2 > t_1)$ if $p_2^*(t) = 0$ during $t \in [t_1, t_2]$. However, such case is not possible. Why?

Because: $\left.\begin{array}{l} \dot{p}_1 = 0 \\ \dot{p}_2 = -p_1 \end{array}\right\} \Rightarrow \begin{array}{l} p_1(t) = c_1 \\ p_2(t) = c_2 - c_1 t \end{array}$

Hence $p_2^*(t) = 0$ for $t \in [t_1, t_2]$ $\Rightarrow$ $c_1, c_2 = 0$ $\Rightarrow$ $p_1^*(t) = 0$, $p_2^*(t) = 0$ at all times

$\Rightarrow$ $H = 1 + p_1^* x_2^* + p_2^* u^* = 1$ $\Rightarrow$ $H = 1$ at all times (1)

Now, consider the boundary condition

$$\underbrace{\left[\frac{\partial h}{\partial x} - p^T\right]\delta x_f}_{0 \ (x_f \text{ fixed})} + \underbrace{\left[H + \frac{\partial h}{\partial t}\right]\delta t_f}_{\text{free}} = 0 \quad \text{at } t = t_f^*$$

$\Rightarrow$ $H + \underbrace{\frac{\partial h}{\partial t}}_{0 \ (h=0)} = 0$ $\Rightarrow$ $H = 0$ at $t = t_f^*$ (2)

(1) & (2) $\Rightarrow$ contradiction.

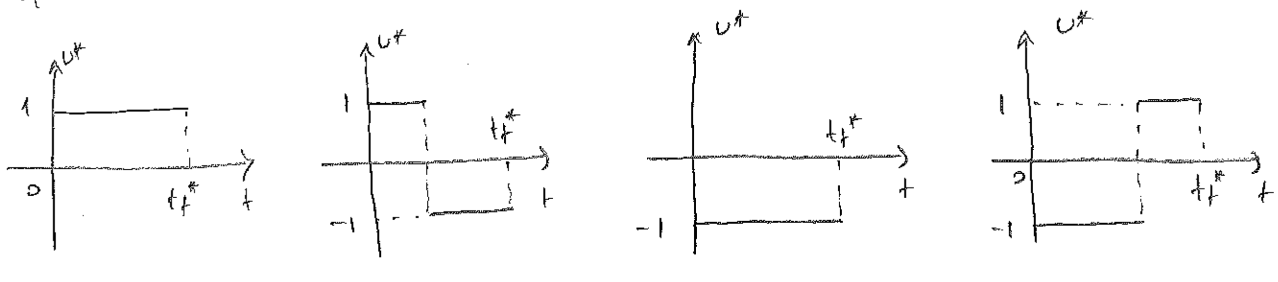
— o —

We've obtained:

$$u^*(t) = -\operatorname{sgn}(p_2^*(t))$$

& $p_2^*(t) = c_2 - c_1 t$ (and not both $c_1$ & $c_2$ can be zero)

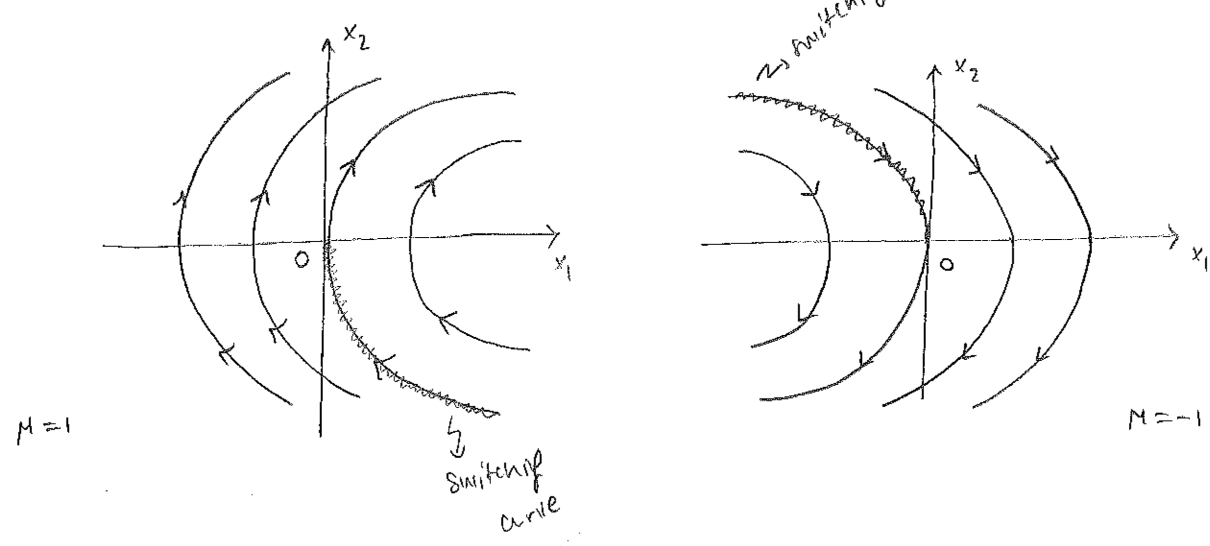
Hence $p_2^*(t)$ can change sign <u>at most once</u>. Then there are four possible cases for optimal control.



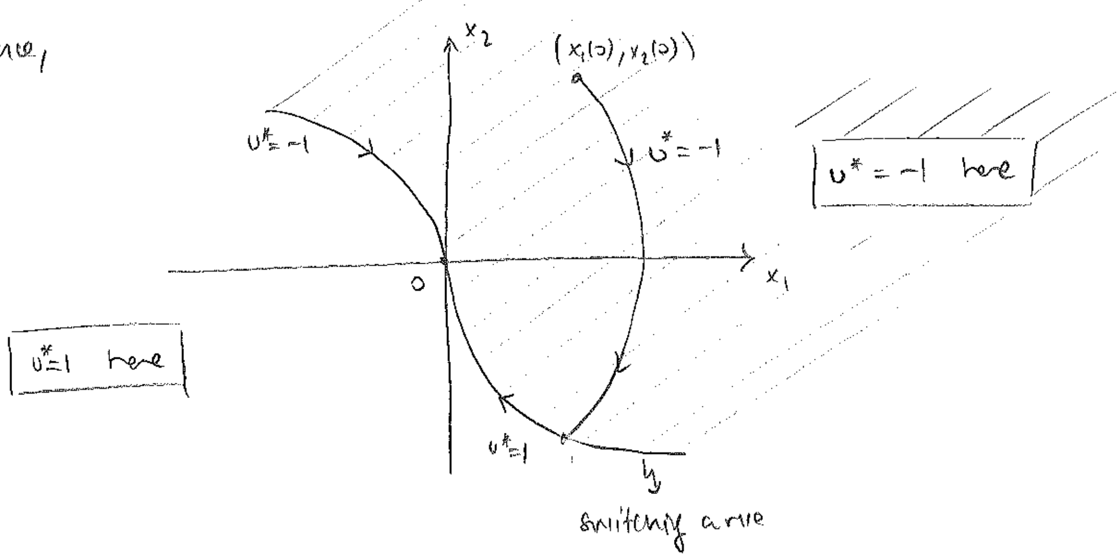
## When to switch?

$$\dot{x}_1 = x_2 \qquad\qquad x_2(t) = x_2(0) + \mu t$$
$$\dot{x}_2 = \mu \quad (\mu = 1 \text{ or } -1) \qquad x_1(t) = x_1(0) + x_2(0)t + \tfrac{1}{2}\mu t^2$$

$$\Rightarrow \quad x_1(t) - \frac{1}{2\mu} x_2(t)^2 = x_1(0) - \frac{1}{2\mu} x_2(0)^2 \qquad (x_1 = \alpha x_2^2 + \beta)$$




Hence,




## Minimum-time strategy

→ Apply the control $u^* = \mp 1$ that transfers the state to the switching curve

→ Once on the switching curve, change sign and apply $u^* = \pm 1$.

Remark: If $(x_1(0), x_2(0))$ is on the switching curve, then the origin is reached without any switching.

Example (LQR revisited)

$$\min_{u(\cdot)} \frac{1}{2}\left[ x(t_f)^T N\, x(t_f) + \int_0^{t_f} \{ x(t)^T Q\, x(t) + u(t)^T R\, u(t) \} dt \right]$$

subject to $\dot{x} = Ax + Bu$ (1) $\qquad$ ($t_f$ : fixed)

where $N, Q \in \mathbb{R}^{n \times n}$ are symmetric pos. semidef.

& $R \in \mathbb{R}^{m \times m}$ is symmetric pos. def.

Hamiltonian $\quad H = \frac{1}{2}[x^T Q x + u^T R u] + p^T (Ax + Bu)$

costate eqn. $\quad \dot{p} = -\nabla_x H = -Qx - A^T p \quad (2)$

boundary cond. $\quad \underbrace{\left[ \frac{\partial h}{\partial x} - p^T \right]}_{=0} \underbrace{\delta x_f}_{\text{free}} + \left[ H + \frac{\partial h}{\partial t} \right] \underbrace{\delta t_f}_{0} = 0 \qquad$ (at $t = t_f$)

$$\Rightarrow \quad p(t_f) = \nabla_x \left\{ \frac{1}{2} x^T N x \right\} \Big|_{t = t_f} = N x(t_f) \quad (3)$$

optimal control : $\quad 0 = \nabla_u H = Ru + B^T p$

$$\Rightarrow \quad u = -R^{-1} B^T p \quad (4)$$

Combining (1), (2), (4)

$$\dot{x} = Ax - BR^{-1}B^T p$$

$$\dot{p} = -Qx - A^T p$$

Guess : $\quad p = K(t)\, x \qquad$ for some $K(t) \in \mathbb{R}^{n \times n}$ (note : $K$ does not depend on $x$)

$$\Rightarrow \quad 0 = \frac{d}{dt} \{ Kx - p \}$$

$$\Rightarrow \quad 0 = \dot{K}x + K\dot{x} - \dot{p}$$

$$= \dot{K}x + K\{Ax - BR^{-1}B^T Kx\} + Qx + A^T Kx$$

$$= \{\dot{K} + KA + A^T K + Q - KBR^{-1}B^T K\} x$$

$$\Rightarrow \quad \dot{K} = -KA - A^T K - Q + KBR^{-1}B^T K \qquad \text{(Riccati Eqn.)}$$

with boundary cond. $K(t_f) = N$ (due to (3))

That is, the optimal control $u^*(t)$ is obtained by

→ first solving the Riccati eqn. (backward in time)

→ then setting $u^*(t) = -R^{-1}B^T K(t) x(t)$.

Remark: $K$ is a symmetric matrix (why?) Hence Riccati eqn. is of order $\frac{n(n+1)}{2}$.

Example (Linear tracking)

Given: reference signal $r : [0, \infty) \to \mathbb{R}^n$

Want: the state $x(t)$ follow $r(t)$ as closely as possible ($\dot{x} = Ax + Bu$)

Formulation:

$$\min_u \frac{1}{2}\left[ \|x(t_f) - r(t_f)\|_N^2 + \int_0^{t_f} \{\|x(t) - r(t)\|_Q^2 + \|u(t)\|_R^2\} dt \right]$$

subject to $\dot{x} = Ax + Bu$ ($t_f$ : fixed)

where $N, Q \in \mathbb{R}^{n \times n}$ are symmetric pos. semidef.

& $R \in \mathbb{R}^{m \times m}$ is symmetric pos. def.

& $\|\eta\|_Q^2 := \eta^T Q \eta$

Hamiltonian $\quad H = \frac{1}{2}\left[(x - r(t))^T Q (x - r(t)) + u^T R u\right] + p^T (Ax + Bu)$

costate eqn. $\dot{p} = -\nabla_x H = -Q(x-r) - A^T p$

boundary cond. $\underbrace{\left[\frac{\partial h}{\partial x} - p^T\right]}_{=0} \underbrace{\delta x_f}_{\text{free}} + \left[H + \frac{\partial h}{\partial t}\right]\underbrace{\delta t_f}_{0} = 0$ (at $t = t_f$)

$\Rightarrow \quad p(t_f) = N(x(t_f) - r(t_f))$ (1)

optimal control: $0 = \nabla_u H = Ru + B^T p$

$\Rightarrow \quad u = -R^{-1}B^T p$

Hence,
$$\dot{x} = Ax - BR^{-1}B^T p$$
$$\dot{p} = -Qx + Qr - A^T p$$

Guess: $p(t) = K(t)x(t) + s(t)$ for some $K \in \mathbb{R}^{n \times n}$ ( $K$ does not depend on $x$ or $r$)

$$\Rightarrow \quad 0 = \frac{d}{dt}\{Kx + s - p\}$$
$$= \dot{K}x + K\dot{x} + \dot{s} - \dot{p}$$
$$= \dot{K}x + K\{Ax - BR^{-1}B^T Kx - BR^{-1}B^T s\} + \dot{s} + Qx - Qr + A^T Kx + A^T s$$
$$= \{\underbrace{\dot{K} + KA + A^T K + Q - KBR^{-1}B^T K}_{=0}\}x + \{\underbrace{\dot{s} + (A^T - KBR^{-1}B^T)s - Qr}_{=0}\}$$

$\Rightarrow \quad \dot{K} = -KA - A^T K - Q + KBR^{-1}B^T K$ (Riccati Eqn.)

& $\dot{s} = -(A^T - KBR^{-1}B^T)s + Qr$

} diff. eqn.

How about boundary conditions for these diff. equations?

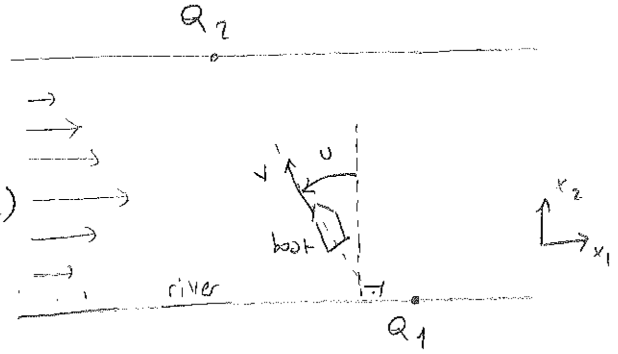
(1) $\Rightarrow \quad Kx + s = Nx - Nr$ at $t = t_f$

Hence $K(t_f) = N$

& $s(t_f) = -Nr(t_f)$

} boundary conditions.

Example




v : velocity of the boat (constant)

u : steering angle

$s(x_2)$ : velocity of current in $x_1$ direction

a) obtain the model

b) Obtain the conditions for the boat to move from point $Q_1$ to point $Q_2$ in minimum time.

c) let $s(x_2) = \bar{s}$ (constant) when does minimum time solution exist?

What is the optimal u?

Sol'n a)

$$\dot{x}_1 = -v\sin u + s(x_2)$$
$$\dot{x}_2 = v\cos u$$

b) $$\min_u \int_0^{t_f} 1\, dt \quad \text{given} \quad x(0) = Q_1 \ \& \ x(t_f) = Q_2$$

Hamiltonian $$H = 1 + p^T f = 1 + p_1(-v\sin u + s(x_2)) + p_2 v\cos u$$

costate eqn. $\dot{p} = -\nabla_x H$

$$\Rightarrow \quad \dot{p}_1 = -\frac{\partial}{\partial x_1} H = 0$$

$$\dot{p}_2 = -\frac{\partial}{\partial x_2} H = -p_1 \frac{\partial s}{\partial x_2}$$

boundary cond.

$$\underbrace{\left[\frac{\partial h}{\partial x} - p^T\right]\delta x_f}_{0} + \left[H + \frac{\partial h}{\partial t}\right]\underbrace{\delta t_f}_{\text{free}} = 0 \quad \Rightarrow \quad H\Big|_{t=t_f^*} = 0$$

Since $\dot{H}=0$ (at all times) we have $H=0$ at all times.

$\Rightarrow \quad 1+p_1(-v\sin u + s(x_2)) + p_2 v\cos u = 0$

optimal control : $0 = \nabla_u H \Rightarrow -p_1 v\cos u - p_2 v \sin u = 0$

$\Rightarrow \quad p_1\cos u + p_2 \sin u = 0$

In summary, the minimum time solution satisfies :

$$\left.\begin{aligned} \dot{x}_1 &= -v\sin u + s(x_2) \\ \dot{x}_2 &= v\cos u \end{aligned}\right\} \text{ state eqn.}$$

$$\left.\begin{aligned} \dot{p}_1 &= 0 \\ \dot{p}_2 &= -p_1 \frac{\partial s}{\partial x_2} \end{aligned}\right\} \text{ costate eqn.}$$

$$1 - p_1 v\sin u + p_1 s + p_2 v\cos u = 0 \quad \} \text{ hamiltonian identically zero}$$

$$p_1\cos u + p_2\sin u = 0 \quad \} \text{ optimal control}$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = a_1 \quad \& \quad \begin{pmatrix} x_1(t_f) \\ x_2(t_f) \end{pmatrix} = a_2 \quad \} \text{ boundary cond. (given)}$$

c) If $s(x_2)=\bar{s} > 0$ is constant we need in general $v > \bar{s}$ for the existence of solution. Then:

$\dot{p}=0 \Rightarrow p(t) = \text{constant}$

$\Rightarrow p_1\cos u + p_2\sin u = 0 \Rightarrow u^*(t) = \bar{u}$ (constant)

How to find $\bar{u}$ ?

$$\left.\begin{aligned} x_1(t) &= x_1(0) + (\bar{s} - v\sin\bar{u})t \\ x_2(t) &= x_2(0) + (v\cos\bar{u})t \end{aligned}\right\} \Rightarrow \text{solve } \underbrace{\frac{x_1(t_f)-x_1(0)}{x_2(t_f)-x_2(0)}}_{\text{known}} = \frac{\bar{s} - v\sin\bar{u}}{v\cos\bar{u}}$$

$\blacksquare$

Example

$$\text{system}\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = -x_1 + [1-x_1^2]x_2 + u\end{cases}$$

The system is to be transferred from the origin to the plane

$$15x_1(t) + 20x_2(t) + 12t = 60$$

while minimizing the cost

$$J = \frac{1}{2}\int_0^{t_f} u^2(t)\,dt \qquad (t_f \text{ is free})$$

a) Determine the costate eqn.

b) Find the optimal control for

i) $u$ not bounded

ii) $u \in [-1, 2]$

c) Determine the boundary conditions at $t = t_f^*$.

Sol'n $\quad H = \frac{1}{2}u^2 + p^T f(x,u) = \frac{1}{2}u^2 + p_1 x_2 + p_2\{-x_1 + [1-x_1^2]x_2 + u\}$

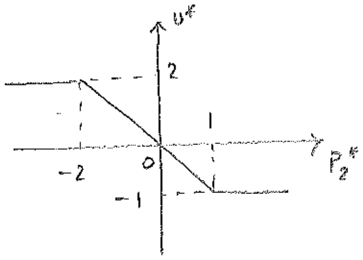
a)
$$\left.\begin{aligned}\dot{p}_1 &= -\frac{\partial H}{\partial x_1} = -\{-p_2 - 2x_1x_2p_2\}\\ \dot{p}_2 &= -\frac{\partial H}{\partial x_2} = -\{p_1 + p_2[1-x_1^2]\}\end{aligned}\right\} \Rightarrow \begin{aligned}\dot{p}_1 &= [1+2x_1x_2]p_2\\ \dot{p}_2 &= -p_2 + [x_1^2-1]p_1\end{aligned}$$

b) $u$ not bounded

$$0 = \frac{\partial H}{\partial u} = u + p_2 \quad \Rightarrow \quad u^*(t) = -p_2^*(t)$$

$u \in [-1, 2]$

$$u^* = \underset{u\in[-1,2]}{\arg\min}\ H \quad \Rightarrow$$




c) Firstly, we should have

$$15x_1^*(t_f^*) + 20x_2^*(t_f^*) + 12t_f^* = 60$$

Also,

$$\left[\frac{\partial h}{\partial x} - p^T\right]\delta x_f + \left[H + \frac{\partial h}{\partial t}\right]\delta t_f = 0 \quad \Rightarrow \quad -p^T\delta x_f + H\delta t_f = 0 \quad \text{at } t = t_f^* \qquad (1)$$

Now, although $x_f$ & $t_f$ are free we cannot take the coefficients of $\delta x_f$ & $\delta t_f$ zero because $\delta x_f$ & $\delta t_f$ are related :

$$60 = 15x_1(t_f) + 20x_2(t_f) + 12t_f = 15[x_1^*(t_f^*) + \delta x_{1f}] + 20[x_2^*(t_f^*) + \delta x_{2f}] + 12[t_f^* + \delta t_f]$$

$$= \underbrace{15x_1^*(t_f^*) + 20x_2^*(t_f^*) + 12t_f^*}_{60} + \underbrace{15\delta x_{1f} + 20\delta x_{2f} + 12\delta t_f}_{0}$$

$$\Rightarrow \quad \delta t_f = -\frac{5}{4}\delta x_{1f} - \frac{5}{3}\delta x_{2f} \qquad (2)$$

$$(1) \,\&\, (2) \Rightarrow \quad -p^T\delta x_f + H\left[-\frac{5}{4} \;\; -\frac{5}{3}\right]\delta x_f = 0$$

$$\Rightarrow \underbrace{\left\{p^T + H\left[\frac{5}{4} \;\; \frac{5}{3}\right]\right\}}_{=0 \;\; (\text{at } t = t_f^*)} \delta x_f = 0 \qquad \text{free}$$

Hence, the boundary conditions are :

$$\begin{cases} 15x_1^*(t_f^*) + 20x_2^*(t_f^*) + 12t_f^* = 60 \\ p_1^*(t_f^*) + \frac{5}{4}\left(H\big|_{t=t_f^*}\right) = 0 \\ p_2^*(t_f^*) + \frac{5}{3}\left(H\big|_{t=t_f^*}\right) = 0 \end{cases}$$

<u>Example</u> For the system

$$\dot{x} = x + u \qquad (1)$$

a) Find the unconstrained control in feedback form ($u = K_T(t)x$) which minimizes

$$J = \int_0^T \left[\frac{3}{2}x^2 + \frac{1}{2}u^2\right]dt \quad ; \quad T \text{ fixed}, \; x(T) \text{ free}$$

b) Show that $\lim_{T\to\infty} K_T(t) = K$ (constant). Find $K$.

<u>Sol'n</u> $\quad H = \frac{3}{2}x^2 + \frac{1}{2}u^2 + p(x+u)$

costate eqn. $\quad \dot{p} = -\frac{\partial H}{\partial x} = -3x - p \qquad (2)$

optimal cont. $\quad 0 = \frac{\partial H}{\partial u} = u + p \Rightarrow u = -p \qquad (3)$

combining (1), (2), (3)

$$\left.\begin{aligned}\dot{x} &= x - p\\ \dot{p} &= -3x - p\end{aligned}\right\} \quad \dot{\begin{bmatrix} x \\ p \end{bmatrix}} = \underbrace{\begin{bmatrix} 1 & -1 \\ -3 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ p \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x(T) \\ p(T) \end{bmatrix} = e^{A(T-t)} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \qquad (4)$$

Note that $p(T) = 0$ (why?) Therefore (4) implies

$$[0 \quad 1]\, e^{A(T-t)} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = 0 \qquad (5)$$

$$e^{At} = \mathcal{L}^{-1}\{[sI - A]^{-1}\} = \mathcal{L}^{-1}\left\{ \begin{bmatrix} s-1 & 1 \\ 3 & s+1 \end{bmatrix}^{-1} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{(s-2)(s+2)} \begin{bmatrix} s+1 & -1 \\ -3 & s-1 \end{bmatrix} \right\}$$

Now,
$$-\frac{3}{(s-2)(s+2)} = \frac{3/4}{s+2} - \frac{3/4}{s-2} \xrightarrow{\mathcal{L}^{-1}} \frac{3}{4}e^{-2t} - \frac{3}{4}e^{2t} =: a_{21}(t)$$

&
$$\frac{s-1}{(s-2)(s+2)} = \frac{3/4}{s+2} + \frac{1/4}{s-2} \xrightarrow{\mathcal{L}^{-1}} \frac{3}{4}e^{-2t} + \frac{1}{4}e^{2t} =: a_{22}(t)$$

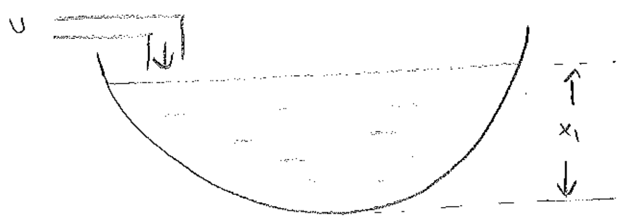
$$(5) \Rightarrow \quad a_{21}(T-t)\,x(t) + a_{22}(T-t)\,p(t) = 0$$

$$\Rightarrow \quad p(t) = -\frac{a_{21}(T-t)}{a_{22}(T-t)}\,x(t) \overset{(3)}{=} -u(t)$$

$$\Rightarrow \quad \boxed{K_T(t) = \frac{3\left(e^{-2(T-t)} - e^{2(T-t)}\right)}{3e^{-2(T-t)} + e^{2(T-t)}}}$$

b) $\lim_{T\to\infty} K_T(t) = \boxed{-3 = K}$

Example (5-13) Consider the leaky reservoir




$$\dot{x}_1 = -\frac{1}{10}x_1 + u$$

$u \in [0, M]$ (bounded control)

Find the optimal control law that minimizes

a) $J = \int_0^{100} -x_1(t)\,dt$

b) $J = \int_0^{100} -x_1(t)\,dt$ subj. to $\int_0^{100} u(t)\,dt = k$ (a known constant)

c) $J = -x_1(100)$ subj. to $\int_0^{100} u(t)\,dt = k$

Sol'n a) $H = -x_1 + p_1\{-\frac{1}{10}x_1 + u\}$

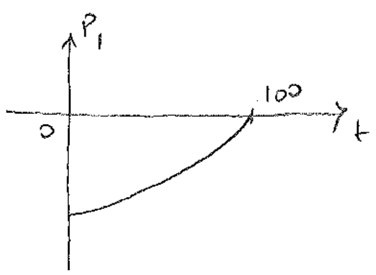
costate eqn. $\dot{p}_1 = -\frac{\partial H}{\partial x_1} = 1 + \frac{1}{10}p_1$ (1)

optimal control. $u^* = \arg\min_{u \in [0,M]} H \Rightarrow u^* = \begin{cases} 0 & \text{for } p_1 > 0 \\ M & \text{for } p_1 < 0 \end{cases}$

boundary cond. $\underbrace{[\frac{\partial h}{\partial x} - p^T]}_{0}\underbrace{\delta x_f}_{\text{free}} + [H + \frac{\partial h}{\partial t}]\underbrace{\delta t_f}_{0 \ (t_f = 100,\ \text{fixed})} = 0$ at $t = t_f$

$\Rightarrow p_1(100) = 0$ (2)

(1) & (2) $\Rightarrow p_1(t) = 10\{e^{(t-100)/10} - 1\}$




$\Rightarrow p_1(t) < 0$ for all $t$ $\Rightarrow$ $\boxed{u^*(t) \equiv M}$

b) To deal with the constraint $\int_0^{100} u(t)\,dt = k$, introduce a new state variable, $x_2$ with $\dot{x}_2 = u$. Then we can express the constraint as

$$k = \int_0^{100} u(t)\,dt = \int_0^{100} \dot{x}_2(t)\,dt = x_2(100) \quad (\text{letting } x_2(0) = 0).$$

Here is the reformulation of the problem:

$$\text{system} \begin{cases} \dot{x}_1 = -\frac{1}{10}x_1 + u \\ \dot{x}_2 = u \end{cases}$$

$$J = \int_0^{100} -x_1(t)\,dt \quad , \quad \& \quad \begin{cases} x_1(100) \text{ free} \\ x_2(100) = k \quad (\text{fixed}) \end{cases}$$

$$H = -x_1 + p_1\left\{-\frac{1}{10}x_1 + u\right\} + p_2 u$$

$$\text{costate eqn.} \begin{cases} \dot{p}_1 = 1 + \frac{1}{10}p_1 \\ \dot{p}_2 = 0 \quad \Rightarrow \quad p_2 = \text{constant} \end{cases}$$

$$\text{optimal control} \quad u^* = \arg\min_{u \in [0,M]} H = \arg\min_{u \in [0,M]} (p_1 + p_2)u$$

$$\Rightarrow u^* = \begin{cases} 0 & \text{for } p_1 + p_2 > 0 \\ M & \text{for } p_1 + p_2 < 0 \end{cases}$$

$$\text{boundary cond.} \quad \left[\frac{\partial h}{\partial x} - p^T\right] \begin{bmatrix} \delta x_{1f} \\ \delta x_{2f} \end{bmatrix} + \left[H + \frac{\partial h}{\partial t}\right]\underbrace{\delta t_f}_{0} = 0$$

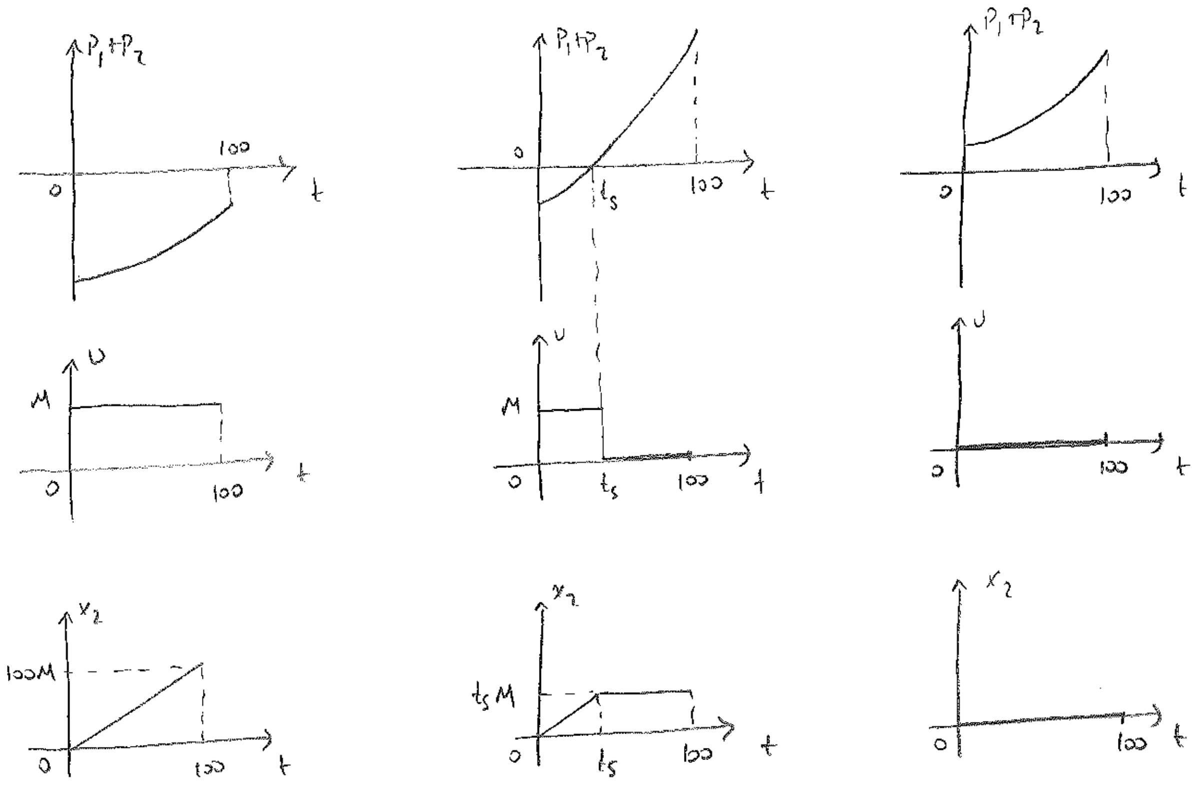
$\delta x_{1f}$: free

$\delta x_{2f}$: $= 0$ ($x_2(t_f)$ fixed)

$$\Rightarrow \quad p_1 \delta x_{1f} + p_2 \delta x_{2f} = 0$$

$$\Rightarrow \quad p_1(100) = 0 \quad \Rightarrow \quad p_1(t) = 10\left\{e^{(t-100)/10} - 1\right\} \quad (\text{as before})$$

Note that since $p_1$ is monotone increasing & $p_2$ is constant $p_1+p_2$ is monotone inc.
Hence there can be at most one switching (in control)




Now, $k = x_2(100) = t_s M \Rightarrow t_s = \frac{k}{M}$ (switching time)

Therefore $$\boxed{u^*(t) = \begin{cases} M & \text{for } t \in [0, t_s] \\ 0 & \text{for } t \in (t_s, 100] \end{cases}}$$

(Note that if $\frac{k}{M} \notin [0,100]$ then the problem is not feasible.)

c) Again we study the augmented system

sys. $\begin{cases} \dot{x}_1 = -\frac{1}{10}x_1 + u \\ \dot{x}_2 = u \end{cases}$

$J = h(x(100))$ (terminal cost $h(x) = -x_1$)

$x_1(100)$ : free

$x_2(100) = k$ , $x_2(0) = 0$

$H = P_1\{-\frac{1}{10}x_1 + u\} + P_2 u$

costate eqn. $\begin{cases} \dot{P}_1 = \frac{1}{10}P_1 \\ \dot{P}_2 = 0 \quad (P_2 = \text{constant}) \end{cases}$

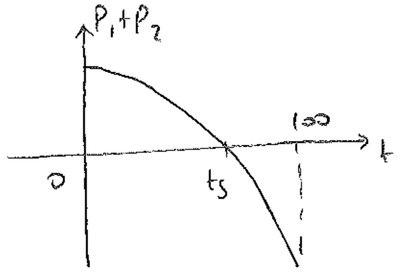
optimal control $u^* = \begin{cases} 0 & \text{for } P_1+P_2 > 0 \\ M & \text{for } P_1+P_2 < 0 \end{cases}$

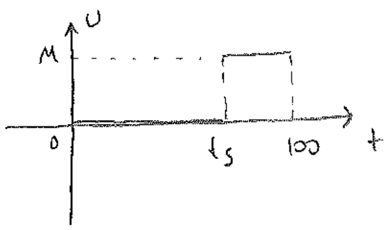
boundary cond. $\left[\frac{\partial h}{\partial x} - p^T\right]\begin{bmatrix}\delta x_{1f} \\ \partial x_{2f}\end{bmatrix} = 0 \qquad (\delta x_{2f} = 0)$

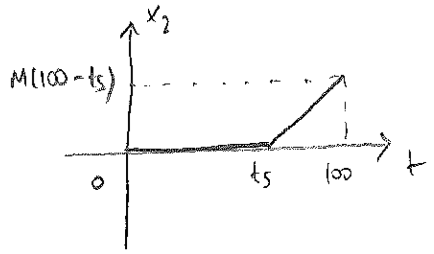
$\Rightarrow \left.\frac{\partial h}{\partial x_1} - P_1 = 0\right|_{t=100} \Rightarrow P_1(100) = -1$

Now, $\dot{P}_1 = \frac{1}{10}P_1$ & $P_1(100) = -1$ imply $P_1(t) = -e^{(t-100)/10}$

Since $P_2$ is constant we have



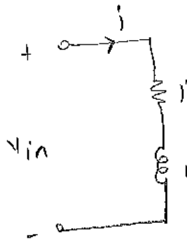



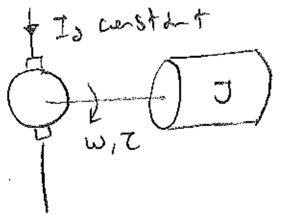

Now, $K = x_2(100) = M(100 - t_s)$

$\Rightarrow t_s = 100 - \frac{K}{M}$

$$\boxed{\Rightarrow u^*(t) = \begin{cases} 0 & \text{for } t \in [0, t_s] \\ M & \text{for } t \in (t_s, 100] \end{cases}}$$

Example [5-15] DC motor (under constant armature current $I_a$) driving a load




torque $\tau = K I_a i$

moment of inertia : $J$

angular velocity : $\omega$

system $\begin{cases} L\frac{d}{dt}i + Ri = v_{in} \\ J\frac{d}{dt}\omega = \tau \end{cases}$

find the optimal control $v_{in}$ minimizes

$$J = \int_0^\infty [\, i^2(t) + \omega^2(t) + \rho v_{in}^2(t) \,]\, dt. \qquad (\rho > 0)$$

Sol'n This is infinite-horizon LQR problem.

$$\dot{x} = Ax + Bu \qquad\qquad (A, B) \text{ stabilizable}$$

$$J = \int_0^\infty (x^T Q x + u^T R u)\, dt \qquad\qquad Q = Q^T \geq 0 \;\; \& \;\; R = R^T > 0$$

optimal control is in feedback form

$$u^* = -R^{-1}B^T P x$$

where the matrix $P = P^T \geq 0$ solves the algebraic Riccati eqn.

$$0 = -PA - A^T P - Q + PBR^{-1}B^T P$$

For our case, let $x_1 = i$, $x_2 = \omega$, and $u = v_{in}$. Then

$$\dot{x} = \underbrace{\begin{bmatrix} -R/L & 0 \\ \frac{KI_a}{J} & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1/L \\ 0 \end{bmatrix}}_{B} u \quad \text{and} \quad J = \int_0^\infty \Big( x^T \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{Q} x + u^T \underbrace{[\rho]}_{R} u \Big) dt$$

Exercise [MATLAB] Let all parameters except $\rho$ be unity. Obtain the root locus plot of the poles of the closed-loop system $\dot{x} = [A - BR^{-1}B^T P]x$ as $\rho : 0 \to \infty$.

Relevant commands : "ss" & "lqr".

# LINEAR QUADRATIC REGULATOR

Consider the LTI system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ $\quad$ $x$: state, $u$: control input

$C \in \mathbb{R}^{k\times n}$ $\quad$ $y$: output

Question 1: When does an optimal control $u^*$ exist that minimizes the infinite-horizon cost

$$J = \int_0^\infty \{ y(t)^T y(t) + u(t)^T R u(t) \} dt$$

$R = R^T > 0$, $\in \mathbb{R}^{m\times m}$

That is, when is $J < \infty$ ?

Question 2: What is $u^*$ ? (If exists)

Question 3: What is the behavior of the trajectories $x(t)$ under $u^*$ ?

Question 4: What do all these have to do with the algebraic Riccati equation? (which we solve for symmetric matrix $P \in \mathbb{R}^{n\times n}$)

$$A^T P + PA + C^T C - PBR^{-1}B^T P = 0 \qquad \text{(ARE)}$$

To be able to answer these interrelated questions we require tools from linear systems theory. In particular, these questions are closely related to the central concepts: stability, controllability (stabilizability), and observability (detectability).

Let us review the fundamentals of LTI system theory first.

Review of LTI system theory

$$\text{system} \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$x \in \mathbb{R}^n$ (state), $u \in \mathbb{R}^m$ (input), $y \in \mathbb{R}^k$ (output)

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{k\times n}$

Given the initial condition $x(0) \in \mathbb{R}^n$ and input signal $u(t) \in \mathbb{R}^m$ the solution $x(t)$ can be written as

$$\boxed{x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}$$

where $e^{At} \in \mathbb{R}^{n\times n}$ is the state transition matrix defined as

$$e^{At} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

Properties of $e^{At}$

P1) The function $e^{At}$ is the unique solution $\Phi(t)$ to

$$\frac{d}{dt}\Phi(t) = A\Phi(t) \qquad \text{with} \quad \Phi(0) = I$$

Remark $\quad$ P1 $\Rightarrow$ $e^{At} = \mathcal{L}^{-1}\{[sI - A]^{-1}\}$

P2) For every $t_1, t_2 \in \mathbb{R}$ we have

$$e^{At_1}e^{At_2} = e^{A(t_1+t_2)}$$

P3) For every $t$ and $A$, $e^{At}$ is nonsingular and

$$[e^{At}]^{-1} = e^{-At}$$

P4) $e^{At}$ commutes with $A$ :

$$e^{At}A = Ae^{At}$$

Definition Matrix $A \in \mathbb{R}^{n\times n}$ is said to be Hurwitz if all eigenvalues $\lambda_i$ of $A$ satisfy $\mathrm{Re}(\lambda_i) < 0$.

Thm $\lim_{t\to\infty} e^{At} = 0_{n\times n}$ if and only if $A$ is Hurwitz.

Internal Stability

Definition System $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ is said to be exponentially stable

if for all initial conditions $x(t) \to 0$ when $u(t) \equiv 0$. this means $u(t)=0 \ \forall t$

Remark Note that exp. stability has nothing to do with B & C matrices. It only depends on the system matrix A. (There is also another type of stability, called bounded input bounded output (BIBO) stability which does depend on the triple $(A, B, C)$)

Thm System $\dot{x} = Ax$ is exp. stable if and only if $A$ is Hurwitz.

Thm Let $P, Q \in \mathbb{R}^{n\times n}$ be symmetric pos. def. and satisfy the Lyapunov eqn.

$$A^T P + PA + Q = 0$$

Then the system $\dot{x} = Ax$ is exp. stable proof Let $\lambda \in \mathbb{C}$ be an arbitrary eigenvalue of $A$ with the corresponding eigenvector $v \in \mathbb{C}^n$. We can write

$$0 = v^*(A^T P + PA + Q)v \qquad (v^* : \text{conjugate transpose of } v)$$

$$= (Av)^* Pv + v^* P(Av) + v^* Qv \qquad \downarrow Av = \lambda v$$

$$= \lambda^* (v^* Pv) + \lambda (v^* Pv) + v^* Qv$$

$$= (\lambda^* + \lambda)(v^* Pv) + v^* Qv$$

$$= 2\mathrm{Re}(\lambda) \cdot \underbrace{v^* Pv}_{>0} + \underbrace{v^* Qv}_{>0}$$

$\Rightarrow \mathrm{Re}(\lambda) < 0 \Rightarrow A$ Hurwitz $\Rightarrow \dot{x} = Ax$ exp. stable. ☒

Observability

Definition System $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ is said to be observable if

$$\left.\begin{array}{l} y(t) \equiv 0 \\ u(t) \equiv 0 \end{array}\right\} \Rightarrow x(t) \equiv 0$$

Remark Observability depends only on $A$ and $C$ matrices. (The matrix $B$ has nothing to do with observability.) If a system is observable then we can compute / construct the solution $x(t)$ from the knowledge of the signals $y(t)$ & $u(t)$.

Thm. System $\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases}$ (or pair $(C, A)$) is observable if and only if null $(C)$ contains no eigenvector of $A$.

<u>Thm</u> Let $\dot{x} = Ax$ be exponentially stable. Then the following are equivalent

1) The pair $(C, A)$ is observable.
2) The below Lyapunov equation has a unique symmetric pos. def solution $P \in \mathbb{R}^{n \times n}$

$$A^T P + PA + C^T C = 0.$$

<u>proof</u> $2 \Rightarrow 1$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A$ with eigenvector $v \in \mathbb{C}^n$.
Since $\dot{x} = Ax$ is exp. stable $Re(\lambda) < 0$. We can write

$$\begin{aligned} 0 &= v^* (A^T P + PA + C^T C) v \\ &= (Av)^* P v + v^* P (Av) + (Cv)^* C v \qquad \} \; Av = \lambda v \\ &= \lambda^* (v^* P v) + \lambda (v^* P v) + \|Cv\|^2 \\ &= \underbrace{2Re(\lambda)}_{<0} \cdot \underbrace{(v^* P v)}_{>0} + \|Cv\|^2 \end{aligned}$$

$\Rightarrow \|Cv\|^2 > 0 \Rightarrow Cv \neq 0 \Rightarrow$ No eigenvector of $A$ belongs to null $C$

$\Rightarrow (C, A)$ is observable. ∎

<u>Detectability</u>

<u>Definition</u> System $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ is said to be detectable if

$$\left.\begin{aligned} y(t) &\equiv 0 \\ u(t) &\equiv 0 \end{aligned}\right\} \Rightarrow x(t) \to 0 \quad (\text{as } t \to \infty)$$

<u>Remark</u> Note that observability $\Rightarrow$ detectability (but not vice versa)
Like observability, detectability depends only on $A$ & $C$ matrices.
If a system is detectable then we can construct a signal $\hat{x}(t)$ from the knowledge of the signals $y(t)$ & $u(t)$ such that $\|\hat{x}(t) - x(t)\| \to 0$ as $t \to \infty$, where $x(t)$ is the actual trajectory.

<u>Thm</u> The pair $(C, A)$ is detectable if and only if null $(C)$ contains no eigenvector $v$ of $A$ $(Av = \lambda v)$ with $Re(\lambda) \geq 0$.

<u>Controllability & Stabilizability</u>

<u>Definition</u> System $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ is said to be controllable if for each initial condition $x(0)$ we can find an input function $u(t)$ that steers the state to origin in finite time, i.e., $x(T) = 0$ for some $T < \infty$.

<u>Definition</u> System $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ is said to be stabilizable if for each initial condition $x(0)$ we can find an input $u(t)$ that yields $x(t) \to 0$ as $t \to \infty$.

<u>Remark</u> Controllability & stabilizability depend only on $(A, B)$. (Matrix $C$ is irrelevant.) Note also that cont. $\Rightarrow$ stab. (but not vice versa).

<u>Remark</u> It turns out that controllability & observability (stabilizability & detectability) are dual concepts. By "dual" we mean:

$$\boxed{\begin{array}{lcl} (A, B) \text{ controllable} & \Longleftrightarrow & (B^T, A^T) \text{ observable} \\ (A, B) \text{ stabilizable} & \Longleftrightarrow & (B^T, A^T) \text{ detectable} \end{array}}$$

or $\begin{cases} (A^T, C^T) \text{ controllable} \Longleftrightarrow (C, A) \text{ observable} \\ (A^T, C^T) \text{ stabilizable} \Longleftrightarrow (C, A) \text{ detectable} \end{cases}$

By duality all the results we listed (and many more we didn't list) on obs. / def. can be dualized into con. / stab. results without any effort. For instance, the eigenvector test for stabilizability reads:

Thm. The pair $(A, B)$ is stabilizable if and only if null $(B^T)$ contains no eigenvector $v$ of $A^T$ $(A^T v = \lambda v)$ with $Re(\lambda) \geq 0$.

Moreover, we have :

Thm The system $\dot{x} = Ax + Bu$ $(x \in \mathbb{R}^n; u \in \mathbb{R}^m)$ is stabilizable if and only if there exists $K \in \mathbb{R}^{m \times n}$ such that the feedback law $u = -Kx$ renders the closed-loop system $(\dot{x} = [A - BK]x)$ exp. stable.

Thm If the pair $(A, B)$ is controllable then for each set of complex numbers $S = \{\mu_1, \mu_2, \cdots, \mu_n\}$ with the condition $\lambda \in S \Rightarrow \lambda^* \in S$ we can find a gain matrix $K \in \mathbb{R}^{m \times n}$ such that the eigenvalues of the closed-loop system matrix $[A - BK]$ are $\mu_1, \mu_2, \cdots, \mu_n$. [In other words, we can 'place' the poles of the closed-loop system as we wish. The associated MATLAB command is "place".]

— END OF LTI SYS. REVIEW —

Back to LQR

System : $\begin{cases} \dot{x} = Ax + Bu & x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ y = Cx & y \in \mathbb{R}^k \end{cases}$

Cost : $J = \int_0^\infty (x^T Q x + u^T R u) dt$

with $Q = C^T C \in \mathbb{R}^{n \times n}$ & $R = R^T > 0$ (i.e., $R \in \mathbb{R}^{m \times m}$ is sym. pos. det.)

(Note that $Q = Q^T \geq 0$, i.e., $Q$ is sym. pos. semidef.)

ARE : $A^T P + PA + Q - PBR^{-1}B^T P = 0$

__Theorem__ Assume that there exists a symmetric solution $P$ to ARE for which $[A - BR^{-1}B^TP]$ is Hurwitz. Then the feedback law

$$u(t) = -Kx(t) \quad \forall t \geq 0 \qquad \text{where} \quad K := R^{-1}B^TP$$

minimizes the cost $J$ and leads to

$$J_{min} = x(0)^T P x(0)$$

__Proof__ Note that $u(t) = -Kx(t)$ yields the closed-loop system $\dot{x} = [A - BR^{-1}B^TP]x$ the solutions of which satisfy $x(t) \to 0$ since $[A - BR^{-1}B^TP]$ is Hurwitz. To prove $u(t) = -Kx(t)$ is optimal suppose otherwise. That is, suppose there exists a "better" control $\tilde{u}(t) = -Kx(t) + v(t)$ that minimizes $J$ while making $x(t) \to 0$. We can write

$$\tilde{J} = \int_0^\infty (x^TQx + \tilde{u}^TR\tilde{u})dt$$

$$= \int_0^\infty \{x^TQx + (v - R^{-1}B^TPx)^T R (v - R^{-1}B^TPx)\}dt \qquad Q = -A^TP - PA + PB^TR^{-1}BP$$

$$= \int_0^\infty \{x^T(-A^TP - PA + PB^TR^{-1}BP)x + (v - R^{-1}B^TPx)^T R (v - R^{-1}B^TPx)\}dt$$

$$= \int_0^\infty \{-x^TP(\underbrace{[A - BR^{-1}B^TP]x + Bv}_{\dot{x}}) - ([A - BR^{-1}B^TP]x + Bv)^T Px + v^TRv\}dt$$

$$= -\int_0^\infty (x^TP\dot{x} + \dot{x}^TPx)dt + \int_0^\infty v^TRv\,dt$$

$$= -\int_0^\infty \left\{\frac{d}{dt} x(t)^TPx(t)\right\}dt + \int_0^\infty v(t)^TRv(t)dt$$

$$= x(0)^TPx(0) - \underbrace{\lim_{t\to\infty} x(t)^TPx(t)}_{=0} + \underbrace{\int_0^\infty v(t)^TRv(t)\,dt}_{\text{always } \geq 0 \text{ \& } = 0 \text{ only when } v(t) \equiv 0}$$

Hence $u(t) = -Kx(t)$ is the minimizer and $J_{min} = x(0)^TPx(0)$.

∎

The previous theorem makes studying the following question worthwhile:

Q: When does a symmetric solution $P$ to ARE for which $[A - BR^{-1}B^TP]$ is Hurwitz exist? (ARE: $A^TP + PA + Q - PBR^{-1}B^TP = 0$)

To answer this question we first define the "Hamiltonian Matrix":

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}_{2n \times 2n}$$

Note that ARE can be expressed as

$$[P \quad -I]\, H \begin{bmatrix} I \\ P \end{bmatrix} = 0$$

<u>Definition</u> A Hamiltonian matrix is said to be in the "domain of the Riccati operator" if there exist square matrices $H_-$, $P \in \mathbb{R}^{n\times n}$ such that $H_-$ is Hurwitz (all its eigenvalues have strictly negative real parts) and we have

$$H \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} H_- .$$

<u>Theorem</u> Suppose a Hamiltonian matrix $H$ be in the domain of Riccati operator with $H_-$, $P$ defined as above. Then the following hold

1) $P$ satisfies ARE.
2) $A - BR^{-1}B^TP$ is Hurwitz.
3) $P$ is symmetric.

<u>Remark</u>: Note that by the following thm we can reformulate our question we asked above as:

Q': When does $H$ belong to the domain of Riccati operator?

proof of Thm

1) $$[P \ -I] H \begin{bmatrix} I \\ P \end{bmatrix} = \underbrace{[P \ -I] \begin{bmatrix} I \\ P \end{bmatrix}}_{0} H_- = 0$$

2) $$A - BR^{-1}B^T P = [I \ 0] \underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}}_{H} \begin{bmatrix} I \\ P \end{bmatrix} = \underbrace{[I \ 0] \begin{bmatrix} I \\ P \end{bmatrix}}_{I} H_- = \underbrace{H_-}_{\text{Hurwitz}}$$

3) $$(P - P^T) H_- = [-P^T \ I] \begin{bmatrix} I \\ P \end{bmatrix} H_-$$

$$= [-P^T \ I] \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix}$$

$$= \underbrace{-P^T A - A^T P - Q + P^T B R^{-1} B^T P}_{\text{symmetric}}$$

$\Rightarrow$ $(P - P^T) H_-$ is symmetric

Hence, we can write

$$(P - P^T) H_- + H_-^T (P - P^T) = (P - P^T) H_- - [(P - P^T) H_-]^T = 0$$

Then

$$0 = e^{H_-^T t} \{ (P - P^T) H_- + H_-^T (P - P^T) \} e^{H_- t}$$

$$= e^{H_-^T t} (P - P^T) H_- e^{H_- t} + e^{H_-^T t} H_-^T (P - P^T) e^{H_- t}$$

$$= \frac{d}{dt} \{ e^{H_-^T t} (P - P^T) e^{H_- t} \}$$

$$\Rightarrow \quad e^{H_-^T t} (P - P^T) e^{H_- t} = \text{constant}$$

$$\Rightarrow \quad P - P^T \overset{!}{=} e^{H_-^T t} (P - P^T) e^{H_- t} \Big|_{t=0} = \lim_{t \to \infty} e^{H_-^T t} (P - P^T) e^{H_- t} = 0$$

(because $e^{H_- t} \to 0$, $H_-$ Hurwitz)

$\square$

<u>Lemma</u> $H$ is similar to $-H$. I.e., there exists a change of coordinate matrix $T \in \mathbb{R}^{2n \times 2n}$ such that $T^{-1} H T = -H$.

<u>Proof</u> Let $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$. Note that $J^{-1} = -J$ and $J^{-1}[-H]J = JHJ = H^T$

Therefore $-H \sim H^T$. Since $H^T \sim H$, we have $-H \sim H$. ▨

<u>Remark</u> Note that $-H \sim H$ implies that if $\lambda$ is an eigenvalue of $H$, so is $-\lambda$. Moreover, the characteristic polynomial of $H$ is written as

$$d(s) = \Delta_-(s)\, \Delta_0(s)\, \Delta_+(s)$$

where the roots of:

- $\rightarrow$ $\Delta_-(s)$ are the eigenvalues with negative real part
- $\rightarrow$ $\Delta_0(s)$ " " " " zero " "
- $\rightarrow$ $\Delta_+(s)$ " " " " positive " "

Then $\deg \Delta_-(s) = \deg \Delta_+(s)$. (Equivalently, $\dim \mathcal{N}\Delta_-(H) = \dim \mathcal{N}\Delta_+(H)$.)

<u>Lemma</u> Suppose $(A, B)$ is stabilizable and $(Q, A)$ is detectable. Then $H$ has no eigenvalue on the imaginary axis. (I.e. $\deg \Delta_0(s) = 0$.)

<u>Proof</u> Suppose not. Then $H$ has an eigenvalue $\lambda = j\omega$ ($\omega \in \mathbb{R}$) with corresponding eigenvector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq 0$. Now, $Hv = j\omega v$ implies

$$\begin{bmatrix} j\omega v_1 \\ j\omega v_2 \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} Av_1 - BR^{-1}B^T v_2 \\ -Qv_1 - A^T v_2 \end{bmatrix}$$

$$\Rightarrow \quad [A - j\omega I]v_1 = BR^{-1}B^T v_2 \qquad (1)$$

$$\& \quad [A - j\omega I]^* v_2 = -Qv_1 \qquad (2)$$

Using (1) & (2) we can write

$$v_2^* B R^{-1} B^T v_2 + v_1^* Q v_1 = v_2^* [A - j\omega I] v_1 - v_1^* [A - j\omega I]^* v_2$$

real since $Q \geq 0$, $R \geq 0$

$$= j \, 2 \, \mathrm{Im} \{ v_2^* [A - j\omega I] v_1 \}$$

$$= 0$$

Hence we have

$$v_2^* B R^{-1} B^T v_2 = 0 \implies B^T v_2 = 0 \quad (3)$$

& $$v_1^* Q v_1 = 0 \implies Q v_1 = 0 \quad (4)$$

Combining (1), (3), (4) and detectability of $(Q, A)$ we have

$$\begin{bmatrix} A - j\omega I \\ Q \end{bmatrix} v_1 = 0 \implies v_1 = 0 .$$

Likewise, combining (2), (3), (4) and stabilizability of $(A, B)$ we have

$$\begin{bmatrix} A^T + j\omega I \\ B^T \end{bmatrix} v_2 = 0 \implies v_2 = 0 .$$

Therefore $v = 0$, which contradicts that $v$ is an eigenvector. $\boxtimes$

<u>Theorem</u> Suppose $(A, B)$ is stabilizable & $(Q, A)$ is detectable. Then the Hamiltonian $H$ is in the domain of Riccati operator.

<u>Proof</u> By the previous lemma $H_{2n \times 2n}$ has no eigenvalues on the imaginary axis. Therefore, since $H$ & $-H$ are similar, there exist matrices $V_1, V_2 \in \mathbb{R}^{n \times n}$ and a Hurwitz matrix $F \in \mathbb{R}^{n \times n}$ such that

$$H \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} F \quad \& \quad \mathrm{rank} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = n$$

If $V_1$ were nonsingular then we could write

$$H\begin{bmatrix} I \\ V_2V_1^{-1}\end{bmatrix} = H\begin{bmatrix} V_1 \\ V_2\end{bmatrix}V_1^{-1} = \begin{bmatrix} V_1 \\ V_2\end{bmatrix}F V_1^{-1} = \begin{bmatrix} I \\ V_2V_1^{-1}\end{bmatrix}[V_1 F V_1^{-1}]$$

and the result would follow by definition once we set $P = V_2V_1^{-1}$ & $H_- = V_1 F V_1^{-1}$ since $H_-$ would inherit Hurwitzness from $F$. To prove our result all we need therefore is that $V_1$ is nonsingular.

<u>Step1</u> Show that $V_1^T V_2$ is symmetric. Let $J = \begin{bmatrix} 0 & -I \\ I & 0\end{bmatrix}$. Observe that $JH$ is symmetric. Therefore

$$[V_2^T V_1 - V_1^T V_2]F = \begin{bmatrix} V_1 \\ V_2\end{bmatrix}^T J \begin{bmatrix} V_1 \\ V_2\end{bmatrix} F = \underbrace{\begin{bmatrix} V_1 \\ V_2\end{bmatrix}^T JH \begin{bmatrix} V_1 \\ V_2\end{bmatrix}}_{\text{symmetric}}$$

Hence

$$[V_2^TV_1 - V_1^TV_2]F + F^T[V_2^TV_1 - V_1^TV_2] = [V_2^TV_1 - V_1^TV_2]F - ([V_2^TV_1 - V_1^TV_2]F)^T$$

$$= 0 \qquad (1)$$

Now, (1) implies

$$\frac{d}{dt}\left\{ e^{F^Tt}[V_2^TV_1 - V_1^TV_2]e^{Ft}\right\} = 0 \quad \Rightarrow \quad e^{F^Tt}[V_2^TV_1 - V_1^TV_2]e^{Ft} = \text{constant}$$

$$\Rightarrow V_2^TV_1 - V_1^TV_2 = e^{F^Tt}[V_2^TV_1 - V_1^TV_2]e^{Ft}\Big|_{t=0} = \lim_{t\to\infty} e^{F^Tt}[V_2^TV_1 - V_1^TV_2]e^{Ft} = 0$$

(since $e^{Ft}\to 0$, $F$ Hurwitz)

$\Rightarrow V_1^TV_2 = V_2^TV_1 = (V_1^TV_2)^T$. Symmetry is established.

<u>Step2</u> Show that $\text{null}\, V_1 = \{0\}$. First we will show $\text{null}\, V_1$ is invariant under $F$. (That is, $V_1x = 0 \Rightarrow V_1Fx = 0$ for all $x \in \mathbb{C}^n$)

Choose an arbitrary vector $x \in \text{null } V_1$. We can write

$$-x^* V_2^T B R^{-1} B^T V_2 x = x^* V_2^T [A V_1 - B R^{-1} B^T V_2] x$$

$$= x^* V_2^T [I \;\; 0] H \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} x$$

$$= x^* V_2^T [I \;\; 0] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} F x$$

$$= x^* V_2^T V_1 F x \qquad V_1^T V_2 = V_2^T V_1 \quad \text{(from step 1)}$$

$$= x^* V_1^T V_2 F x$$

$$= (V_1 x)^* V_2 F x \qquad V_1 x = 0$$

$$= 0$$

Since $R > 0$, this means $B^T V_2 x = 0$. Hence we can proceed as

$$V_1 F x = [I \;\; 0] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} F x$$

$$= [I \;\; 0] H \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} x$$

$$= [A \;\; -B R^{-1} B^T] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} x = \underbrace{A V_1 x}_{0} - \underbrace{B R^{-1} B^T V_2 x}_{0}$$

$= 0$ which was to be shown.

Using the invariance of null $V_1$ under $F$ we now show null $V_1 = \{0\}$. Suppose null $V_1 \neq \{0\}$. Then, due to invariance, null $V_1$ must contain an eigenvector of $F$. Let $x \in \text{null } V_1$ be this eigenvector, i.e.,

$$F x = \lambda x \qquad (\lambda \in \mathbb{C}, \; x \neq 0)$$

Since $F$ is Hurwitz, we have $\text{Re}\{\lambda\} < 0$.

Now, we can write

$$[A^T + \lambda I] V_2 x = A^T V_2 x + V_2(\lambda x) \quad \Big\downarrow\ Fx = \lambda x$$

$$= A^T V_2 x + V_2 F x \quad \Big\downarrow\ V_1 x = 0$$

$$= Q V_1 x + A^T V_2 x + V_2 F x$$

$$= [Q \quad A^T] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} x + V_2 F x$$

$$= [0 \ -I] H \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} x + V_2 F x \quad \Big\downarrow\ H \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} F$$

$$= [0 \ -I] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} F x + V_2 F x$$

$$= -V_2 F x + V_2 F x$$

$$= 0 \qquad (1)$$

Also, we have previously obtained

$$V_1 x = 0 \Rightarrow B^T V_2 x = 0 \qquad (2)$$

(1) & (2) imply

$$\begin{bmatrix} A^T + \lambda I \\ B^T \end{bmatrix} V_2 x = 0$$

Since $(A, B)$ is stabilizable & $\mathrm{Re}\{\lambda\} < 0$ we have to have $V_2 x = 0$.

Then we can write

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} x = 0$$

But this means (since $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ is full column rank) $x = 0$, which contradicts that $x$ is an eigenvector. ☒

## Summary of LQR

system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$

cost: $J = \int_0^\infty (x^T Q x + u^T R u)\,dt$ $\qquad (Q = C^T C)$

$\rightarrow Q = Q^T \geq 0$ pos. semidef

$\rightarrow R = R^T > 0$ pos. def.

ARE $\quad A^T P + PA + Q - PBR^{-1}B^T P = 0$

If the system is both stabilizable & detectable then there exists a (stabilizing) solution $P = P^T$ to ARE for which $[A - BR^{-1}B^T P]$ is Hurwitz. Moreover, $u(t) = -R^{-1}B^T P x(t)$ is the optimal control that minimizes the cost.

How to compute $P$?

Construct Hamiltonian $\quad H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$

Let $V_1, V_2 \in \mathbb{R}^{n \times n}$ be such that

$$H \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} H_- \quad \text{where } H_- \in \mathbb{R}^{n \times n} \text{ is Hurwitz.}$$

Then $\boxed{P = V_2 V_1^{-1}}$

Recall that we've earlier obtained

$$\min_{u(\cdot)} \int_0^\infty [x(t)^T Q x(t) + u(t)^T R u(t)]\,dt = x(0)^T P x(0)$$

Since the cost is never negative and $x(0)$ is arbitrary, this means

$P \geq 0$, i.e., $P$ is pos. semidef.

Now, we ask: " when is $P$ positive definite ? "

<u>Theorem</u> (In addition to standard assumptions) if the system (i.e. the pair $(C, A)$ or, equivalently, the pair $(Q, A)$) is observable then $P$ is pos. def.

<u>Proof</u> Let $x \in \mathbb{C}^n$. Then $Px = 0 \Rightarrow Cx = 0 \Rightarrow PAx = 0$. To see this

<u>Step 1</u> Pre & post multiply ARE with $x^T$ & $x$.

$$0 = x^T \{ A^T P + PA + C^T C - PBR^{-1}B^T P \} x \quad \overset{Px = 0}{\downarrow} \qquad (C^T C = Q)$$

$$= x^T C^T C x$$

$$= \|Cx\|^2$$

Hence $Cx = 0$

<u>Step 2</u> Post-multiply ARE with $x$

$$0 = \{ A^T P + PA + C^T C - PBR^{-1}B^T P \} x \quad \overset{Cx = 0,\ Px = 0}{\downarrow}$$

$$= PAx$$

Suppose now $P$ is not pos. def. Then we can find $x \neq 0$ such that $Px = 0$. Using the chain $Px = 0 \Rightarrow Cx = 0 \Rightarrow PAx = 0$ we deduce $Cx = 0, CAx = 0, CA^2x = 0, \ldots$

Hence,

$$\underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{\text{observability matrix}} x = 0$$

Since $x \neq 0$ this means the obs. matrix is not full rank. This contradicts our assumption that the system is observable. Hence $P$ is pos. def. $\square$

# Optimization-Based Estimation of LTI Systems

Consider the following DT LTI dynamics

system : $x_{k+1} = Ax_k$

tracker : $\hat{x}_{k+1} = A\hat{x}_k + Bu_k$

where $x, \hat{x} \in \mathbb{R}^n$ , $u \in \mathbb{R}$ (scalar)

$x_k$ denotes the value of the state at time $k$ $(k=0,1,2,\ldots)$

Shorthand notation :

$x^+ = Ax$

$\hat{x}^+ = A\hat{x} + Bu$

<u>Want</u> : The tracker tracks the system, i.e., $\|\hat{x}_k - x_k\| \to 0$ as $k \to \infty$.

<u>Assume</u> $\rightarrow$ rank $[B \; AB \cdots A^{n-1}B] = n$ (controllability)

$\rightarrow$ full state information $x_k ; \hat{x}_k$ is available to the tracker.

Here is a solution to our problem:

<u>Algorithm</u> 1. At each time $k$, set the control of the tracker as $u_k = u_0^{[k]}$ where $u_0^{[k]}$ is the first entry of the sequence $(u_0^{[k]}, u_1^{[k]}, \ldots, u_{n-1}^{[k]})$ that solves

$$A^n\hat{x}_k + A^{n-1}Bu_0^{[k]} + A^{n-2}Bu_1^{[k]} + \cdots + Bu_{n-1}^{[k]} = A^n x_k . \quad (1)$$

<u>Remark</u> Note that the control suggested by the algorithm can be expressed in feedback form $u_k = K(x_k - \hat{x}_k)$. Since

$$(1) \Rightarrow \underbrace{[B \; AB \cdots A^{n-1}B]}_{F_{n\times n}} \begin{bmatrix} u_{n-1}^{[k]} \\ \vdots \\ u_0^{[k]} \end{bmatrix}_{n\times 1} = A^n(x_k - \hat{x}_k)$$

$$\Rightarrow \begin{bmatrix} u_{n-1}^{[k]} \\ \vdots \\ u_0^{[k]} \end{bmatrix} = F^{-1} A^n (x_k - \hat{x}_k)$$

$$\Rightarrow \quad u_0^{[k]} = e_n^T F^{-1} A^n (x_k - \hat{x}_k) \quad \text{where} \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n$$

Hence, $\boxed{K = e_n^T F^{-1} A^n}$ $\quad (K \in \mathbb{R}^{1 \times n})$

Theorem [Min. time tracking] Consider the system-tracker pair

$$\begin{cases} x^+ = Ax \\ \hat{x}^+ = A\hat{x} + Bu \end{cases}$$

where the tracker is run under Algorithm 1. Then $\hat{x}_k = x_k$ for all $k \geq n$.

Proof Multiplying both sides of (1) with $A$ we have

$$A^{n+1}\hat{x}_k + A^n B u_0^{[k]} + \cdots + AB u_{n-1}^{[k]} = A^{n+1} x_k$$

$$\Rightarrow A^n \big( \underbrace{A\hat{x}_k + Bu_0^{[k]}}_{\hat{x}_{k+1}} \big) + A^{n-1} B u_1^{[k]} + \cdots + AB u_{n-1}^{[k]} = A^n \big( \underbrace{Ax_k}_{x_{k+1}} \big)$$

$$\Rightarrow A^{n-1} B u_1^{[k]} + A^{n-2} B u_2^{[k]} + \cdots + AB u_{n-1}^{[k]} = A^n ( x_{k+1} - \hat{x}_{k+1} ) \qquad (2)$$

Rewriting (1) for $k+1$ we have

$$A^n \hat{x}_{k+1} + A^{n-1} B u_0^{[k+1]} + \cdots + B u_{n-1}^{[k+1]} = A^n x_{k+1}$$

$$\Rightarrow A^{n-1} B u_0^{[k+1]} + A^{n-2} B u_1^{[k+1]} + \cdots + B u_{n-1}^{[k+1]} = A^n (x_{k+1} - \hat{x}_{k+1}) \qquad (3)$$

Combining (2) & (3) we obtain

$$A^{n-1} B u_0^{[k+1]} + A^{n-2} B u_1^{[k+1]} + \cdots + B u_{n-1}^{[k+1]} = A^{n-1} B u_1^{[k]} + A^{n-2} B u_2^{[k]} + \cdots + AB u_{n-1}^{[k]}$$

$$\Rightarrow A^{n-1} B (u_0^{[k+1]} - u_1^{[k]}) + A^{n-2} B (u_1^{[k+1]} - u_2^{[k]}) + \cdots + AB (u_{n-2}^{[k+1]} - u_{n-1}^{[k]}) + B u_{n-1}^{[k+1]} = 0 \qquad (4)$$

Since the vectors $A^{n-1}B, A^{n-2}B, \ldots, B$ are lin. ind. we have to have

$$u_0^{[k+1]} = u_1^{[k]}$$
$$u_1^{[k+1]} = u_2^{[k]}$$
$$\vdots$$
$$u_{n-2}^{[k+1]} = u_{n-1}^{[k]}$$
$$u_{n-1}^{[k+1]} = 0$$

Define the vector $v^{[k]} \in \mathbb{R}^n$ as

$$v^{[k]} = \begin{bmatrix} u_0^{[k]} \\ u_1^{[k]} \\ \vdots \\ u_{n-1}^{[k]} \end{bmatrix}$$

Hence we can write

$$v^{[k+1]} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}}_{J} v^{[k]} \quad \Rightarrow \quad v^{[k]} = J^k v^{[0]}$$

Since $J^n = 0$ we have $v^{[k]} = 0$ for $k \geq n$. This implies two things

1) For $k \geq n$ we have $u_k = 0$. Hence the tracker dynamics becomes $\hat{x}^+ = A\hat{x}$ for $k \geq n$. This allows us to write

$$\hat{x}_{l+n} = A^l \hat{x}_n \quad \text{for} \quad l = 0, 1, 2, \ldots \qquad (1)$$

2) The equation of the Algorithm at time $k = n$ reads

$$A^n \hat{x}_n = A^n x_n \qquad (2)$$

combining (1) & (2) with $l = n$ we have

$$\hat{x}_{2n} = A^n \hat{x}_n = A^n x_n = x_{2n} \qquad (3)$$

Now, recall that we can write $\hat{x}^+ = A\hat{x} + BK(x - \hat{x})$ with $K = e_n^T F^{-1} A^n$.

Defining the error $d = \hat{x} - x$ we can write

$$d^+ = \hat{x}^+ - x^+ = A\hat{x} + BK(x - \hat{x}) - Ax$$

$$= (A - BK)d$$

(Do not confuse $e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ with error $e_k$!)

That is, $d_k = [A-BK]^k d_0$. Eq. (3) implies that $d_{2n} = 0$. Hence

$$[A-BK]^{2n} d_0 = 0.$$

Since $d_0$ is arbitrary we must have $[A-BK]^{2n} = 0$ which implies that all the eigenvalues of $[A-BK]$ must be zero. Then by Cayley-Hamilton Thm we have

$$[A-BK]^n = 0.$$

Therefore $d_k = 0$ for all $k \geq n$. Equivalently, $\hat{x}_k = x_k$ for $k \geq n$. ☒

Remark This type of behaviour, where the convergence $\|\hat{x}_k - x_k\| \to 0$ is achieved exactly $(\hat{x}_k = x_k)$ in finite time is called "deadbeat". Hence our tracker is called deadbeat tracker.

Note that we have established the following result as a byproduct.

Corollary Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$ be such that the matrix

$$F = [B \;\; AB \;\; \cdots \;\; A^{n-1}B]_{n\times n}$$

is nonsingular. Define $K \in \mathbb{R}^{1\times n}$ as $K = e_n^T F^{-1} A^n$. $\left( e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1} \right)$

Then the matrix $[A-BK]$ is nilpotent. That is, its characteristic poly. reads $d(s) = s^n$.

Exercise For various values of matrices $A$ and $B$ verify the above corollary in MATLAB.

## From Deadbeat Tracker to Deadbeat Observer

Consider the following DT LTI dynamics

system: $x^+ = Ax$ , $y = Cx$

observer: $z^+ = A\eta$ , $\hat{x} = A^{N-1}z$ (1)

where $x, z, \eta \in \mathbb{R}^n$ , $y \in \mathbb{R}$ (scalar) is the output (measurement) , $N \geq 1$ (integer)

Want : The observer generates meaningful estimates $\hat{x}$ of the state $x$.
That is, $\|\hat{x}_k - x_k\| \to 0$ as $k \to \infty$.

Constraint : Information available to the observer is $z_k$ & $y_k$.

Assume rank $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$ (observability)

Let us now make some use of our earlier analysis.

Question How should $\eta = \eta(z,y)$ and $N$ be chosen such that the observer becomes a deadbeat observer, i.e., $\hat{x}_k = x_k$ for all $k \geq n$?

Recall the dynamics of the deadbeat tracker

$$\hat{x}^+ = A\hat{x} + BK(x - \hat{x})$$

where the feedback gain was $K = e_n^T [B \ AB \cdots A^{n-1}B]^{-1} A^n$. By duality the dynamics of the observer should read

$$\hat{x}^+ = A\hat{x} + L(y - C\hat{x}) = A\hat{x} + LC(x - \hat{x}) \qquad (2)$$

with the observer gain

$$L = A^n \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1} e_n \qquad (3) \qquad \left( e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1} \right)$$

Exercise : Show that (2)&(3) make a deadbeat observer. Hint: $[A - LC]^n = ?$

Letting $G = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ and using (1), (2), (3) we can proceed as

$$A^N \eta = A^{N-1}(A\eta) = A^{N-1} z^+ = \hat{x}^+$$

$$= A\hat{x} + A^n G^{-1} e_n (y - C\hat{x})$$

$$= A^N z + A^n G^{-1} e_n (y - CA^{N-1} z)$$

If we let $N=n$ we can write

$$A^n \eta = A^n (z + G^{-1} e_n (y - CA^{n-1} z))$$

which suggests that we choose $\eta$ as

$$\eta = z + G^{-1} e_n (y - CA^{n-1} z)$$

which implies

$$G(\eta - z) = e_n (y - CA^{n-1} z)$$

$$\Rightarrow \begin{bmatrix} C(\eta - z) \\ CA(\eta - z) \\ \vdots \\ CA^{n-2}(\eta - z) \\ CA^{n-1}(\eta - z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ y - CA^{n-1} z \end{bmatrix} \Rightarrow \begin{bmatrix} C(\eta - z) \\ \vdots \\ CA^{n-2}(\eta - z) \\ CA^{n-1}\eta - y \end{bmatrix} = 0$$

Hence, we've obtained the dual of Algorithm 1:

<u>Algorithm 2</u> At each time $k$ choose $\eta_k$ that solves

$$z_k, y_k \longrightarrow \boxed{\begin{aligned} & CA^l \eta_k = CA^l z_k \quad \text{for } l = 0, 1, \ldots, n-2 \\ & \& \; CA^{n-1} \eta_k = y_k \end{aligned}} \longrightarrow \eta_k$$

sys. $x^+ = Ax$, $y = Cx$

obs. $z^+ = A\eta$, $\hat{x} = A^{n-1} z$

(we've taken $N=n$)

<u>Remark</u> Algorithm 2, though we obtained it through linear system analysis is valuable also for nonlinear systems. Let's see why.

Consider the (nonlinear) system

$$x^+ = f(x) \quad , \quad y = h(x) \qquad (1)$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ & $h : \mathbb{R}^n \to \mathbb{R}$.

Assume [nonlinear observability] For each $\xi \in \mathbb{R}^n$ the equation

$$\begin{bmatrix} h(\eta) \\ h(f(\eta)) \\ \vdots \\ h(f^{n-1}(\eta)) \end{bmatrix} = \xi \quad \text{has a unique solution } \eta \in \mathbb{R}^n. \qquad \left(f^3(x) = f(f(f(x)))\right)$$

Now, consider the observer

$$z^+ = f(\eta) \quad , \quad \hat{x} = f^{n-1}(z) \qquad (2)$$

where $\eta$ satisfies

$$\left.\begin{aligned} h(f^\ell(\eta)) &= h(f^\ell(z)) \quad \text{for } \ell = 0, 1, \ldots, n-2 \\ h(f^{n-1}(\eta)) &= y . \end{aligned}\right\} \text{ compare this to fig. 2 !}$$

Theorem [Glad] Consider the system (1) & the observer (2). We have $\hat{x}_k = x_k$ for all $k \geq n$.

Proof The result follows trivially for $n = 1$. Suppose $n \geq 2$ and for some $p \in \{1, 2, \ldots, n-1\}$ and some $k \geq 0$ we have

$$h(f^{n-q}(\eta_k)) = y_{k-q+1} \qquad \forall q \in \{1, 2, \ldots, p\} \qquad (3)$$

Then we can write

$$\begin{aligned} h(f^{n-(q+1)}(\eta_{k+1})) &= h(f^{n-(q+1)}(z_{k+1})) \\ &= h(f^{n-(q+1)}(f(\eta_k))) \\ &= h(f^{n-q}(\eta_k)) \\ &= y_{k-q+1} \\ &= y_{(k+1)-(q+1)+1} \end{aligned}$$

Also, $h(f^{n-1}(\eta_{k+1})) = y_{k+1}$ holds by definition. Hence (3) implies

$$h(f^{n-q}(\eta_{k+1})) = y_{(k+1)-q+1} \qquad \forall q \in \{1, 2, \ldots, p+1\}$$

Now, (3) holds at all times $k \geq 0$ with $p=1$. By induction therefore it holds for all $k \geq n-1$ with $p = n$. That is,

$$h(f^{n-q}(\eta_k)) = y_{k-q+1} \qquad \text{for } k \geq n-1 \ \& \ q = 1, 2, \ldots, n.$$

Hence, we can write

$$\begin{bmatrix} h(\eta_k) \\ \vdots \\ h(f^{n-2}(\eta_k)) \\ h(f^{n-1}(\eta_k)) \end{bmatrix} = \begin{bmatrix} y_{k-n+1} \\ \vdots \\ y_{k-1} \\ y_k \end{bmatrix} = \begin{bmatrix} h(x_{k-n+1}) \\ \vdots \\ h(f^{n-2}(x_{k-n+1})) \\ h(f^{n-1}(x_{k-n+1})) \end{bmatrix} \qquad \text{for } k \geq n-1.$$

Then by our observability assumption we have $\eta_k = x_{k-n+1}$ for all $k \geq n-1$. The result then follows since

$$\hat{x}_{k+1} = f^n(\eta_k) = f^n(x_{k-n+1}) = x_{k+1} \qquad \blacksquare$$

Example [Chaotic oscillator]

$$x_1^+ = 1 + x_2 - a x_1^2$$
$$x_2^+ = b x_1 + x_3$$
$$x_3^+ = -b x_1$$

From "Chaos from switched-capacitor circuits: discrete maps" Chua et al. Proceedings of the IEEE, 1987.

For this (nonlinear) system suppose we can measure $y = x_3 =: h(x)$. Design an observer of the form

$$z^+ = f(\eta(z, y))$$
$$\hat{x} = f^2(z)$$

using Glad's method.

Sol'n For $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ we have $x^+ = f(x)$ where

$$f(x) = \begin{bmatrix} 1 + x_2 - a x_1^2 \\ b x_1 + x_3 \\ -b x_1 \end{bmatrix} . \quad \text{Hence } f : \mathbb{R}^3 \to \mathbb{R}^3 \quad (n = 3)$$

We want to solve $\begin{bmatrix} h(\eta) \\ h(f(\eta)) \\ h(f^2(\eta)) \end{bmatrix} = \begin{bmatrix} h(z) \\ h(f(z)) \\ y \end{bmatrix}$ where $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$ & $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

Now, $h(\eta) = h(z) \Rightarrow \eta_3 = z_3$ (1)

$h(f(\eta)) = h(f(z)) \Rightarrow -b\eta_1 = -bz_1 \Rightarrow \eta_1 = z_1$ (2)

$h(f^2(\eta)) = y \Rightarrow -b(1 + \eta_2 - a\eta_1^2) = y$ & $\eta_1 = z_1$

$\Rightarrow \eta_2 = -\frac{1}{b} y + a z_1^2 - 1$ (3)

Hence, combining (1),(2),(3) we obtain

$$\eta(z, y) = \begin{bmatrix} z_1 \\ -\frac{1}{b} y + a z_1^2 - 1 \\ z_3 \end{bmatrix}$$

The observer dynamics are then

$$z^+ = f(\eta) = \begin{bmatrix} -\frac{1}{b} y \\ b z_1 + z_3 \\ -b z_1 \end{bmatrix} , \quad \hat{x} = f^2(z)$$

Exercise Verify experimentally (in MATLAB) that this observer works.

In particular, we expect $\hat{x}(k) = x(k)$ for $k \geq 3$ for all initial conditions $x(0)$ & $z(0)$.

From Deadbeat Observer to Optimal Observer

System : $x^+ = Ax$ , $y = Cx$

Observer : $z^+ = A\eta$ , $\hat{x} = A^{N-1} z$

Assumption : $\operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$ (observability)

Deadbeat observer algorithm (assume $y$ scalar & $N=n$) :

→ At each time $k$, $\eta_k$ is obtained by solving

$$\begin{bmatrix} C \\ \vdots \\ CA^{N-2} \\ CA^{N-1} \end{bmatrix} \eta = \begin{bmatrix} Cz \\ \vdots \\ CA^{N-2}z \\ y \end{bmatrix} \qquad (1)$$

Under our assumptions " $y$ scalar & $N=n$ " the solution to (1) uniquely exists. What if $y$ is not necessarily scalar & $N \geq n$ ? In that case (1) does not always admit a solution.

Q: How to obtain $\eta$ then ?

A: Least squares approximation.

Optimal (least squares) observer algorithm* : At each time $k$, $\eta_k$ is obtained by $\eta = \arg\min_{\xi} J(\xi, z, y)$ where the cost $J$ is

$$J(\xi, z, y) = \|CA^{N-1}\xi - y\|^2 + \sum_{i=0}^{N-2} \|CA^i\xi - CA^i z\|^2 .$$

Remark : Note that for $N=n$ & $y$ scalar, the least squares observer algorithm boils down to the deadbeat observer algorithm.

* : Generalization of DB obs. alg.

Explicit form of the optimal observer

system : $x^+ = Ax$ , $y = Cx$

opt. observer : $z^+ = A\eta$ , $\hat{x} = A^{N-1} z$

$$\eta = \arg\min_{\xi} \|CA^{N-1}\xi - y\|^2 + \sum_{i=0}^{N-2} \|CA^i\xi - CA^i z\|^2$$

Assumptions : $(C,A)$ observable , $N \geq n$

Define $Q = \sum_{i=0}^{N-2} A^{iT} C^T C A^i = [C^TC + A^TC^TCA + \cdots + A^{(N-2)T} C^T C A^{N-2}]$

$H = A^{(N-1)T} C^T C A^{(N-1)}$

Then we can write the cost $J(\xi, z, y) = \|CA^{N-1}\xi - y\|^2 + \sum_{i=0}^{N-2} \|CA^i(\xi - z)\|^2$ as

$$J(\xi, z, y) = \{\xi^T H \xi - 2\xi^T A^{(N-1)T} C^T y + y^T y\} + (\xi - z)^T Q (\xi - z)$$

To obtain $\eta$ we solve $\frac{\partial J}{\partial \xi} = 0$ which yields

$$H\xi - A^{(N-1)T} C^T y + Q(\xi - z) = 0$$

$$\Rightarrow \quad (Q+H)\xi = A^{(N-1)T} C^T y + Qz$$

$$\Rightarrow \quad \eta = (Q+H)^{-1} A^{(N-1)T} C^T y + (Q+H)^{-1} Q z$$ (why $(Q+H)^{-1}$ exists ?)

$$= (Q+H)^{-1} A^{(N-1)T} C^T y + (Q+H)^{-1} \{Q + H - H\} z$$

$$= z + (Q+H)^{-1} \{A^{(N-1)T} C^T y - Hz\}$$

$H = A^{(N-1)T} C^T C A^{(N-1)}$

$$= z + (Q+H)^{-1} A^{(N-1)T} C^T \{y - CA^{N-1} z\}$$

Hence the dynamics can be written as

$$\text{system:} \quad x^+ = Ax, \quad y = Cx$$

$$\text{observer:} \quad z^+ = A\eta, \quad \hat{x} = A^{N-1} z$$

$$\eta = z + (\mathcal{Q}+H)^{-1} A^{(N-1)T} C^T \{ y - CA^{N-1} z \}$$

Can we further simplify? Yes:

$$\begin{aligned}
\hat{x}^+ &= A^{N-1} z^+ \\
&= A^{N-1} A\eta \\
&= A^N \{ z + (\mathcal{Q}+H)^{-1} A^{(N-1)T} C^T \{ y - CA^{N-1} z \} \} \\
&= A(A^{N-1} z) + A^N (\mathcal{Q}+H)^{-1} A^{(N-1)T} C^T \{ y - C(A^{N-1} z) \} \\
&= A\hat{x} + A^N (\mathcal{Q}+H)^{-1} A^{(N-1)T} C^T (y - C\hat{x})
\end{aligned}$$

Therefore, the observer dynamics can be written as

$$\hat{x}^+ = A\hat{x} + L(y - C\hat{x})$$

where the observer gain $L$ reads

$$L = A^N \left[ \sum_{i=0}^{N-1} A^{iT} C^T C A^i \right]^{-1} A^{(N-1)T} C^T$$

Note that we have gotten rid of the $z$ dynamics!

<u>Error dynamics?</u> Define the error $e = \hat{x} - x$. Then

$$\begin{aligned}
e^+ = \hat{x}^+ - x^+ &= A\hat{x} + L(y - C\hat{x}) - Ax = A\hat{x} + L(Cx - C\hat{x}) - Ax \\
&= [A - LC](\hat{x} - x) \\
&= [A - LC] e
\end{aligned}$$

Note that the convergence $\|\hat{x}_k - x_k\| \to 0$ is equivalent to $\|e_k\| \to 0$, which is equivalent to that all the eigenvalues of the matrix $[A - LC]$ satisfy $|\lambda_i| < 1$.

<u>Theorem</u> Consider the optimal observer ( system observable, $N \geq n$)

system: $x^+ = Ax$ , $y = Cx$ $(x \in \mathbb{R}^n)$

observer: $\hat{x}^+ = A\hat{x} + L(y - C\hat{x})$ with $L = A^N \left[ \sum_{i=0}^{N-1} A^{iT} C^T C A^i \right]^{-1} A^{(N-1)T} C^T$

We have $\|\hat{x}_k - x_k\| \to 0$ as $k \to \infty$. In particular the matrix $[A - LC]$ is Schur, i.e., all its eigenvalues satisfy $|\lambda_i| < 1$.

[Proof is lengthy & omitted]

<u>Exercise</u> For various parameter values $A, C,$ & $N$ verify numerically (in MATLAB) the above theorem.

<u>Exercise</u> For $y$ scalar & $N = n$ show that $L$ becomes the deadbeat observer gain. That is,

$$A^n \left[ \sum_{i=0}^{n-1} A^{iT} C^T C A^i \right]^{-1} A^{(n-1)T} C^T = A^n \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1} e_n \qquad \left( e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right.$$

<u>From Optimal Observer to Optimal tracker</u>

Consider the dynamics

system: $x^+ = Ax$ $x \in \mathbb{R}^n$

tracker : $\hat{x}^+ = A\hat{x} + Bu$ $(A, B)$ controllable

Let the tracker input be $u = K(x - \hat{x})$ with tracker gain $K$ is chosen as the dual of the optimal observer gain $L$. That is,

$$K = B^T A^{(N-1)T} \left[ \sum_{i=0}^{N-1} A^i B B^T A^{iT} \right]^{-1} A^N \qquad (N \geq n)$$

Then we at once have the convergence $\|\hat{x}_k - x_k\| \to 0$. At this point an interesting question is : "With respect to what is this $K$ optimal ?"

<u>Optimal (minimum norm) tracker algorithm</u>* At each time $k$, set the control of the tracker as $u_k = u_0^{[k]}$ where $u_0^{[k]}$ is the first entry of the sequence $(u_0^{[k]}, u_1^{[k]}, \ldots, u_{N-1}^{[k]})$ that solves

$$(u_0^{[k]}, \ldots, u_{N-1}^{[k]}) = \arg\min_{(v_0, \ldots, v_{N-1})} \sum_{i=0}^{N-1} \|v_i\|^2$$

$$\text{subject to} \quad A^N \hat{x}_k + A^{N-1} B v_0 + \cdots + A B v_{N-2} + B v_{N-1} = A^N x_k . \qquad (1)$$

Let's compute $u_0^{[k]}$ explicitly.

Let $V = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}$. Then (1) yields

$$\underbrace{[A^{N-1}B \quad A^{N-2}B \quad \cdots}_{W} \quad B]\, V = A^N (x - \hat{x}) \qquad (2)$$

Since $W$ is full row rank eqn (2) has many solutions: the min norm solution reads

$$V_{min} = \underbrace{[W^T (W W^T)^{-1} W]}_{proj_{\mathcal{N}(W)^\perp}} V_{\text{arbitrary solution to (2)}}$$

$W V = A^N (x - \hat{x})$

$$= W^T (W W^T)^{-1} A^N (x - \hat{x})$$

$$\Rightarrow \quad V_{min} = \begin{bmatrix} B^T A^{(N-1)T} (W W^T)^{-1} A^N (x - \hat{x}) \\ \vdots \\ B^T (W W^T)^{-1} A^N (x - \hat{x}) \end{bmatrix} = \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

Hence
$$u_0^{[k]} = B^T A^{(N-1)T} (W W^T)^{-1} A^N (x_k - \hat{x}_k)$$

$$= B^T A^{(N-1)T} \left[ A^{N-1} B B^T A^{(N-1)T} + \cdots + A B B^T A + B B^T \right]^{-1} A^N (x_k - \hat{x}_k).$$

$$= K (x_k - \hat{x}_k)$$

# MODEL PREDICTIVE CONTROL (MPC)

Model predictive control (a.k.a. Receding Horizon Control) is a generalization of the optimal tracker algorithm (we've seen earlier) to nonlinear systems. The main idea is the same: obtain a sequence of control inputs minimizing a finite horizon cost, apply the first entry of the sequence to move the system one step further, then repeat the procedure for the new initial condition. Solving an optimization problem at each time is costly, but has the advantage of rendering the system more responsive to disturbances and more robust to modeling errors/uncertainties.

System: $x^+ = f(x, u)$

constraints: $x \in \mathcal{X}$, $u \in \mathcal{U}$

goal: drive the state to some desired equilibrium, say the origin, $x_k \to 0$ while respecting the constraints $x_k \in \mathcal{X}$ & $u_k \in \mathcal{U}$

MPC algorithm: At each time $k$, set the control of the system as $u_k = u_0^{[k]}$ where $u_0^{[k]}$ is the first entry of the sequence $(u_0^{[k]}, \ldots, u_{N-1}^{[k]})$ that solves

$$(u_0^{[k]}, \ldots, u_{N-1}^{[k]}) = \underset{(v_0, \ldots, v_{N-1})}{\arg\min} \sum_{i=0}^{N-1} g(\xi_i, v_i)$$

$$\text{subject to} \quad \begin{cases} \xi_0 = x_k \\ \xi_{i+1} = f(\xi_i, v_i) \\ \xi_i \in \mathcal{X}, \; v_i \in \mathcal{U} \\ \xi_N = 0 \end{cases}$$

<u>Question</u> under what conditions does MPC algorithm let us achieve our goal?

(Before answering this question let us review the Lyapunov Theory for DT systems.)

Review of Stability for DT dynamics

DT nonlinear autonomous sys:

$x^+ = f(x)$ , $f: X \to X$ $(X \subset \mathbb{R}^n)$

Solution:

$x_k = f^k(x_0) = \underbrace{f(f(\cdots f(x_0)))}_{k \text{ times}}$

Equilibrium point? $x_e$ is an equilibrium point of the system if $x_k = x_e$ for $k = 0, 1, 2, \dots$ is a solution. Equivalently, if

$$\boxed{f(x_e) = x_e}$$

Henceforth, without loss of generality, we will let $x=0$ be an equilibrium point of our system, i.e., $f(0)=0$.

Definition [Stability] Consider the system $x^+ = f(x)$. The equilibrium $x=0$ is

→ stable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x_0\| < \delta \Rightarrow \|x_k\| < \varepsilon \quad \text{for all } k = 0, 1, 2, \dots$$

→ unstable if not stable

→ asy. stable if stable and $\delta$ can be chosen such that

$$\|x_0\| < \delta \Rightarrow \lim_{k \to \infty} \|x_k\| = 0$$

→ globally asy. stable if stable and $\lim_{k \to \infty} \|x_k\| = 0$ for all $x_0$.

→ globally exp. stable if there exist $c > 0$ and $0 < \gamma < 1$ such that all solutions satisfy

$$\|x_k\| \le c \gamma^k \|x_0\| \quad \text{for all } k = 0, 1, 2, \dots$$

Theorem The origin of the linear system $x^+ = Ax$ is stable if and only if all the eigenvalues $\lambda_i$ of $A$ satisfy $|\lambda_i| \le 1$ and whenever $|\lambda_i| = 1$, the corresponding Jordan block is of size $1 \times 1$.

<u>Theorem</u> For $x^+ = Ax$ the following are equivalent

1) The origin is asy. stable.
2) $|\lambda_i| < 1$ for all eigenvalues of $A$.
3) For each $Q = Q^T > 0$ there exists $P = P^T > 0$ that uniquely satisfies

$$A^T P A - P + Q = 0$$

4) The origin is GES.

<u>Theorem</u> [Lyapunov] Consider the system $x^+ = f(x)$ with $f: \mathcal{X} \to \mathcal{X}$ continuous and $f(0) = 0$. Let $V: \mathcal{X} \to \mathbb{R}$ be a continuous pos. def. function satisfying

$$c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2$$

& $$V(f(x)) - V(x) \leq -c_3 \|x\|^2$$

for some positive constants $c_1, c_2, c_3$. Then the origin is globally exp. stable.

<u>Proof</u> Let $x_k = f^k(x_0)$ be an arbitrary solution. We can write

$$\begin{aligned} V(x_{k+1}) &= V(f(x_k)) \\ &\leq V(x_k) - c_3 \|x_k\|^2 \qquad \text{) } V(x) \leq c_2 \|x\|^2 \\ &\leq V(x_k) - \frac{c_3}{c_2} V(x_k) \\ &= \left(1 - \frac{c_3}{c_2}\right) V(x_k) \qquad \text{) } \alpha := 1 - \frac{c_3}{c_2} \\ &= \alpha V(x_k) \qquad (1) \end{aligned}$$

Note that $\alpha < 1$ and cannot be negative (why?). Now, (1) implies

$$V(x_k) \leq \alpha^k V(x_0) \qquad \text{(why?)}$$

Then we have

$$\|x_k\|^2 \leq \frac{1}{c_1} V(x_k) \leq \frac{1}{c_1} \alpha^k V(x_0) \leq \frac{c_2}{c_1} \alpha^k \|x_0\|^2$$

Hence, $\|x_k\| \leq \sqrt{\frac{c_2}{c_1}} \left(\sqrt{\alpha}\right)^k \|x_0\|$. ▨

Stability under MPC

Consider the system

$$x^+ = f(x,u) \quad , \quad f : X \times U \to \mathbb{R}^n \qquad (X \subset \mathbb{R}^n \ , \ U \subset \mathbb{R}^m)$$

Suppose for all $x \in X$ the following optimization problem admits a solution

$$\mathrm{Prob}_N(x) \left\{ \begin{array}{l} V_N(x) = \min\limits_{(v_0, v_1, \dots, v_{N-1})} \sum\limits_{k=0}^{N-1} g(\xi_k, v_k) \\ \\ \text{subject to} \left\{ \begin{array}{ll} \xi_0 = x & \\ \xi_{k+1} = f(\xi_k, v_k) & \forall k \\ \xi_k \in X \ , \ v_k \in U & \forall k \\ \xi_N = 0 & \end{array} \right. \end{array} \right.$$

Let the feedback law $K : X \to U$ be such that for each $x \in X$

$K(x) = v_0^*$ where $(v_0^*, \dots, v_{N-1}^*)$ is a minimizing control sequence for $\mathrm{Prob}_N(x)$.

Theorem [Stability] Assume there exist positive constants $c_1, c_2$ such that

1) $g(x,u) \geq c_1 \|x\|^2$ for all $u \in U$

2) $V_N(x) \leq c_2 \|x\|^2$

Then the origin of the closed-loop system $x^+ = f(x, K(x))$ is exp. stable.

Proof We can write $V_N(x) \geq \min\limits_{u \in U} g(x,u) \geq c_1 \|x\|^2$. Hence

$$c_1 \|x\|^2 \leq V_N(x) \leq c_2 \|x\|^2 \qquad (1).$$

Now we claim the following. There exists $u \in U$ that satisfies

$$g(0,u) = 0 \quad \& \quad f(0,u) = 0. \qquad (2)$$

To see this let $(\xi_0^\circ, \xi_1^\circ, \dots, \xi_N^\circ)$ and $(v_0^\circ, v_1^\circ, \dots, v_{N-1}^\circ)$ be optimal sequences for $\mathrm{Prob}_N(0)$. Note that $\xi_0^\circ = 0$ by def. and $V_N(0) = 0$ by (1).

we can write

$$0 = V_N(0) = \sum_{k=0}^{N-1} g(\xi_k^\circ, v_k^\circ) \geq \sum_{k=0}^{N-1} c_1 \|\xi_k^\circ\|^2 \geq 0$$

Hence $g(\xi_k^\circ, v_k^\circ) = 0$ for all $k$. For $k=0$ we have

$$0 = g(\xi_0^\circ, v_0^\circ) = g(0, v_0^\circ)$$

For $k=1$ we can write

$$0 = g(\xi_1^\circ, v_1^\circ) = g(f(\xi_0^\circ, v_0^\circ), v_1^\circ) = g(f(0, v_0^\circ), v_1^\circ) \geq c_1 \| f(0, v_0^\circ)\|^2$$

Hence we have $g(0, v_0^\circ) = 0$ & $f(0, v_0^\circ) = 0$, as was to be shown.

Let $x \in X$ be given. Then let $(\xi_0^*, \xi_1^*, \ldots, \xi_{N-1}^*)$ and $(v_0^*, v_1^*, \ldots, v_{N-2}^*)$ be optimal sequences for $\text{Prob}_{N-1}(f(x, K(x)))$. Note that $\xi_{N-1}^* = 0$. Choose now some $\bar{v} \in U$ that satisfies (2). Then construct the sequences

$$(\hat{\xi}_0, \hat{\xi}_1, \ldots, \hat{\xi}_{N-1}, \hat{\xi}_N) := (\xi_0^*, \xi_1^*, \ldots, \xi_{N-1}^*, 0)$$

$$(\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_{N-2}, \hat{v}_{N-1}) := (v_0^*, v_1^*, \ldots, v_{N-2}^*, \bar{v})$$

Note that these new sequences are feasible for $\text{Prob}_N(f(x, K(x)))$. Therefore by optimality we can write

$$V_N(f(x, K(x))) \leq \sum_{k=0}^{N-1} g(\hat{\xi}_k, \hat{v}_k) = \sum_{k=0}^{N-2} g(\xi_k^*, v_k^*) + g(0, \bar{v})$$

$$= V_{N-1}(f(x, K(x))) \qquad (3)$$

By principle of optimality we have

$$V_N(x) = g(x, K(x)) + V_{N-1}(f(x, K(x))) \qquad (4)$$

Combining (3) & (4) we can write

$$\begin{aligned} V_N(f(x, K(x))) - V_N(x) &\leq V_{N-1}(f(x, K(x))) - V_N(x) \\ &= -g(x, K(x)) \\ &\leq -c_1 \|x\|^2 \qquad (5) \end{aligned}$$

Finally, (1) & (5) yield exp. stability. ▨