First name:____________________________

Last name: KEY

Student ID:____________________________

Signature:____________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1. 20%

The state of the scalar system $\dot{x} = x + u$ starting from the initial condition $x(0) = 30$ is to be driven to the origin in $\ln 8$ seconds, i.e., $x(\ln 8) = 0$, while minimizing the cost

$$J = \int_0^{\ln 8} \frac{1}{4} u^4 dt .$$

Find the resulting optimal trajectory $x^*(t)$.

Hamiltonian $\quad H = \frac{1}{4} u^4 + p(x+u)$

$\dot{p} = -\frac{\partial H}{\partial x} = -p \quad \Rightarrow \quad p(t) = c e^{-t}$

$0 = \frac{\partial H}{\partial u} = u^3 + p \quad \Rightarrow \quad u = -p^{1/3} = -c^{1/3} e^{-t/3}$

$\dot{x} = x + u = x - c^{1/3} e^{-t/3}$

$\Rightarrow x(t) = c_1 e^t + c_2 e^{-t/3}$

$x(0) = 30 \quad \Rightarrow \quad c_1 + c_2 = 30 \quad (1)$

$x(\ln 8) = 0 \quad \Rightarrow \quad 8c_1 + \frac{1}{2} c_2 = 0 \quad (2)$

$(1) \,\&\, (2) \quad \Rightarrow \quad c_1 = -2 \;\&\; c_2 = 32$

$\Rightarrow \quad \boxed{x^*(t) = -2e^t + 32e^{-t/3}}$

Q2. 30%

The trajectory $(x_1^*(t),\, x_2^*(t))$ of the following system with bounded input $u \in [-1,\, 1]$

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + u\end{aligned}$$

evolves throughout the interval $t \in [0,\, \pi]$ under the control $u^*(t)$ that minimizes the cost

$$J = \frac{1}{2}\left[x_1(\pi)^2 + x_2(\pi)^2\right].$$

It is measured that $x_1^*(\pi) = 0$ and $x_2^*(\pi) = -1$. Find the initial conditions $x_1^*(0)$ and $x_2^*(0)$.

$H = p_1 x_2 + p_2(-x_1 + u)$

$\left.\begin{aligned}\dot{p}_1 &= -\frac{\partial H}{\partial x_1} = p_2\\ \dot{p}_2 &= -\frac{\partial H}{\partial x_2} = -p_1\end{aligned}\right\} \quad \ddot{p}_2 = -p_2 \Rightarrow p_2(t) = c_1\cos t + c_2 \sin t \quad (1)$

$u^* = \arg\min_u H \quad \Rightarrow \quad u^* = \begin{cases} -1 & p_2 > 0 \\ +1 & p_2 < 0\end{cases}$

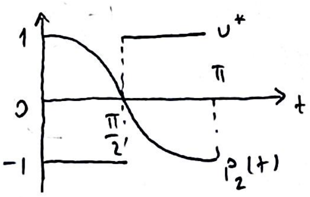
$h = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ & $\left.\frac{\partial h}{\partial x} - p^T\right|_{t=\pi} = 0$

$\Rightarrow p_1(\pi) = x_1(\pi) = 0 \quad \Rightarrow \dot{p}_2(\pi) = -p_1(\pi) = 0 \quad (2)$

& $p_2(\pi) = x_2(\pi) = -1 \quad (3)$

$(1), (2), (3) \Rightarrow \quad p_2(t) = \cos t$

Hence,



i.e. $u^* = \begin{cases} +1 & t \in [\frac{\pi}{2}, \pi] \\ -1 & t \in [0, \frac{\pi}{2}]\end{cases}$

$\ddot{x}_1 = \dot{x}_2 = -x_1 + u$

<u>$t \in [\frac{\pi}{2}, \pi]$</u> $\quad [u^* = 1]$

$\ddot{x}_1 = -x_1 + 1 \quad \Rightarrow x_1(t) = c_3 \cos t + c_4 \sin t + 1 \quad (4)$

$x_1(\pi) = 0 \quad (5) \quad$ & $\dot{x}_1(\pi) = x_2(\pi) = -1 \quad (6)$

$(4), (5), (6) \Rightarrow \quad x_1(t) = \cos t + \sin t + 1$

$\Rightarrow x_1(\frac{\pi}{2}) = 2 \quad (7) \quad$ & $\dot{x}_1(\frac{\pi}{2}) = -1 \quad (8)$

<u>$t \in [0, \frac{\pi}{2}]$</u> $\quad [u^* = -1]$

$\ddot{x}_1 = -x_1 - 1 \quad \Rightarrow x_1(t) = c_5 \cos t + c_6 \sin t - 1 \quad (9)$

$(7), (8), (9) \Rightarrow \quad x_1(t) = \cos t + 3\sin t - 1$

$\Rightarrow \boxed{x_1(0) = 0}$

$x_2(t) = \dot{x}_1(t) = -\sin t + 3\cos t$

$\Rightarrow \boxed{x_2(0) = 3}$

Q3. 30%

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where the triple (C, A, B) is both observable and controllable.

(a) Show that the triple $(C, [A + \alpha I], B)$ is both detectable and stabilizable for all $\alpha \in \mathbb{R}$.

(b) Let $\alpha > 0$ and R be symmetric positive definite. Show that the below cost

$$J = \int_0^\infty e^{2\alpha t}(y^T y + u^T Ru)dt$$

admits a minimizer in the form $u_{\min}(t) = -Kx(t)$ where K is a constant feedback gain. *Hint: Employ the change of variables $z := e^{\alpha t}x$ and $v := e^{\alpha t}u$.*

(c) What can be said about the eigenvalue locations of the closed-loop system matrix $A - BK$?

a) Let $\tilde{A} = A + \alpha I$. Let (λ, v) be an eigenvalue-eigenvector pair for A: $Av = \lambda v$. Then

$\tilde{A}v = [A + \alpha I]v = Av + \alpha v = (\lambda + \alpha)v$. That is, A & $\tilde{A}$ share the same eigenvectors. By eigenvector tests for obs. & cont. we have

(C, A, B) obs. & cont. $\Rightarrow$ $(C, \tilde{A}, B)$ obs. & cont.

Since obs. $\Rightarrow$ det. & cont. $\Rightarrow$ stab. we have $(C, \tilde{A}, B)$ both det. & stab. ∎

b) $\dot{z} = \frac{d}{dt} e^{\alpha t}x = \alpha e^{\alpha t}x + e^{\alpha t}\dot{x}$

$= \alpha e^{\alpha t}x + Ae^{\alpha t}x + Be^{\alpha t}u$

$= [A + \alpha I]e^{\alpha t}x + Be^{\alpha t}u$

$= \tilde{A}z + Bv$

Then

$J = \int_0^\infty e^{2\alpha t}(y^T y + u^T R u)dt$

$= \int_0^\infty [z^T C^T C z + v^T R v]\, dt$

Since $(C, \tilde{A}, B)$ both det. & obs. LQR has a sol'n and the optimal control v^* reads

$v^*(t) = -Kz(t)$ $(K = R^{-1}B^T P)$

$\Rightarrow \cancel{e^{\alpha t}} u^*(t) = -K\cancel{e^{\alpha t}} x(t) \Rightarrow u^*(t) = -Kx(t)$ ∎

c) We know that $\tilde{A} - BK$ is Hurwitz. Then:

$0 > \mathrm{Re}\, \lambda_i [\tilde{A} - BK]$

$= \mathrm{Re}\, \lambda_i [\alpha I + [A - BK]]$

$= \alpha + \mathrm{Re}\, \lambda_i [A - BK]$

$\Rightarrow \boxed{\mathrm{Re}\, \lambda_i [A - BK] < -\alpha}$

Q4. 20%

Consider the linearization of the (normalized) pendulum system without friction

$$\ddot{y} + y = u$$

where y is the (angular) position of the pendulum and u is the torque applied to the system. Cost to be minimized is

$$J = \int_0^\infty (3y^2 + 3\dot{y}^2 + u^2)dt\,.$$

(a) Find $J_{\min}$ in terms of $y(0)$ and $\dot{y}(0)$.

(b) Let the optimal control law be $u_{\min} = -k_1 y - k_2 \dot{y}$. Find k_1 and k_2.

a) let $x_1 = y$ & $x_2 = \dot{y}$ & $x = [x_1\ x_2]^T$. Then

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$J = \int_0^\infty \left(x^T \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}}_{Q} x + u^T \underbrace{[1]}_{R} u \right) dt$$

ARE : $A^T P + PA + Q - PBR^{-1}B^T P = 0$.

Let $P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Then (ARE) implies

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$- \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -2b & a-c \\ a-c & 2b \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} b^2 & bc \\ bc & c^2 \end{bmatrix} = 0$$

$\Rightarrow \quad 0 = b^2 + 2b - 3 = (b+3)(b-1) \Rightarrow b = -3$ or $+1$

& $\quad c^2 = 2b + 3 \quad \Rightarrow$ b must be 1 & $c = \sqrt{5}$

& $\quad a - c = bc \quad \Rightarrow a = (b+1)c = 2\sqrt{5}$

Hence $P = \begin{bmatrix} 2\sqrt{5} & 1 \\ 1 & \sqrt{5} \end{bmatrix}$

$$J = x(0)^T P x(0) = \boxed{2\sqrt{5}\, y(0)^2 + 2y(0)\dot{y}(0) + \sqrt{5}\, \dot{y}(0)^2}$$

b) $[k_1\ k_2] = R^{-1}B^T P = [1\ \sqrt{5}]$

$\Rightarrow \boxed{k_1 = 1 \ \& \ k_2 = \sqrt{5}}$