First name:________________

Last name: Sol'n

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Signature:________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1. 30%

For the scalar system $\dot{x} = -x + u$ starting from the initial condition $x(0) = 1$ the below cost is to be minimized.

$$J = x(1)^2 + \int_0^1 u^2 dt \,.$$

Find $x^*(1)$, the final value of the optimal trajectory.

Hamiltonian : $H = u^2 + p(-x+u)$

state eqn : $\dot{x} = -x + u$ (1)

costate eqn : $\dot{p} = -\frac{\partial H}{\partial x} = p$ (2)

opt. con : $0 = \frac{\partial H}{\partial u} = 2u + p \Rightarrow u = -\frac{1}{2}p$ (3)

bnd. cond. : $\underbrace{\left[\frac{\partial h}{\partial x} - p\right]}_{0} \delta x_f + \left[H + \frac{\partial h}{\partial t}\right] \underbrace{\delta t_f}_{0} = 0$

$\Rightarrow p(1) = 2x(1)$ (4)

(2) & (4) $\Rightarrow$ $p(t) = 2x(1)e^{t-1}$ (5)

(1),(3),(5) $\Rightarrow$ $\dot{x} = -x - \frac{1}{2}\left[2x(1)e^{t-1}\right]$

Hence, $\dot{x} = -x - \frac{x(1)}{e} e^{t}$ with $x(0) = 1$ (6)

(6) implies $x(t) = c_1 e^{-t} + c_2 e^{t}$

$c_1, c_2 = ?$

(6) $\Rightarrow$ $\cancel{-c_1 e^{-t}} + c_2 e^t = -[\cancel{c_1 e^{-t}} + c_2 e^t] - \frac{x(1)}{e} e^t$

$\Rightarrow 2c_2 = -\frac{x(1)}{e} \Rightarrow c_2 = -\frac{x(1)}{2e}$

Also, $c_1 e^{-t} + c_2 e^{t}\Big|_{t=0} = 1 \Rightarrow c_1 = 1 - c_2 = 1 + \frac{x(1)}{2e}$

Therefore,

$x(t) = \left[1 + \frac{x(1)}{2e}\right] e^{-t} - \frac{x(1)}{2e} e^{t}$ (7)

Evaluating (7) at $t=1$ we have

$x(1) = \left[1 + \frac{x(1)}{2e}\right] \frac{1}{e} - \frac{x(1)}{2e} e$

yielding $\boxed{x(1) = \frac{2e}{3e^2 - 1}}$

Q2. 20%

Given t_0, t_f, $x(t_0)$, $x(t_f)$, the cost

$$J = \int_{t_0}^{t_f} x\sqrt{1+u^2}\,dt$$

is to be minimized for the scalar system $\dot{x} = u$. Assuming the boundary conditions are satisfied, determine whether each of the below candidates could be an optimal trajectory. *Hint:* $\dot{H}^* = ?$

(a) $x^*(t) = \cos t + \sin t$.

(b) $x^*(t) = e^{t/2} + e^{-t/2}$.

$$H = x\sqrt{1+u^2} + pu = \text{const.} \quad (\dot{H} = 0) \quad (1)$$

$$0 = \frac{\partial H}{\partial u} = \frac{xu}{\sqrt{1+u^2}} + p \Rightarrow p = -\frac{xu}{\sqrt{1+u^2}} \quad (2)$$

$$(1)\ \&\ (2) \Rightarrow \quad x\sqrt{1+u^2} - \frac{xu^2}{\sqrt{1+u^2}} = \text{const.}$$

$$\Rightarrow \quad \text{Const.} = \frac{x(1+u^2) - xu^2}{\sqrt{1+u^2}} = \frac{x}{\sqrt{1+u^2}}$$

Hence, any optimal trajectory should satisfy

$$\frac{x^2}{1+\dot{x}^2} = \text{constant} \quad (*)$$

a) $$\frac{x^2}{1+\dot{x}^2} = \frac{(\cos t + \sin t)^2}{1 + (-\sin t + \cos t)^2}$$

$$= \frac{1 + 2\cos t \sin t}{2 - 2\cos t \sin t} \neq \text{const.}$$

Hence $x(t) = \cos t + \sin t$ CANNOT be optimal.

b) $$\frac{x^2}{1+\dot{x}^2} = \frac{(e^{t/2} + e^{-t/2})^2}{1 + (\frac{1}{2}e^{t/2} - \frac{1}{2}e^{-t/2})^2}$$

$$= \frac{e^t + e^{-t} + 2}{1 + \frac{1}{4}(e^t + e^{-t} - 2)} = 4 \text{ (const.)}$$

Hence $x(t) = e^{t/2} + e^{-t/2}$ CAN be optimal.

Q3. 30%

Consider the collection of systems

$$\begin{aligned}\dot{x} &= Ax + B_i u\\ y &= C_j x\end{aligned}$$

where

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad C_1 = [1 \ \ 1], \quad C_2 = [1 \ \ 0].$$

The cost to be minimized is

$$J = \int_0^\infty (y^2 + u^2)\, dt\,.$$

(a) For which of the four systems (C_1, A, B_1), (C_1, A, B_2), (C_2, A, B_1), (C_2, A, B_2) can you claim that the optimal stabilizing Riccati solution P exists? How about its positive definiteness?

(b) Solve the optimal control problem for the system (C_1, A, B_1). That is, compute the optimal control and minimum value of the cost. *Hint:* $\dot{y} = ?$

$|sI-A| = \begin{vmatrix} s+1 & -1 \\ -1 & s+1 \end{vmatrix} = (s+1)^2 - 1 = s(s+2)$

$\Rightarrow$ eigenvalues of A : $\lambda_1 = 0$, $\lambda_2 = -2$

Corresponding eigenvectors (of both A & A^T)

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} v_1 = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} v_2 = 0 \Rightarrow v_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

<u>(A, B_1)</u>

$\left.\begin{array}{l} B_1^T v_1 \neq 0 \\ B_1^T v_2 \neq 0 \end{array}\right\}$ (A, B_1) controllable

<u>(A, B_2)</u>

$B_2^T v_1 = 0 \Rightarrow (A, B_2)$ NOT stabilizable

<u>(C_1, A)</u>

$\left.\begin{array}{l} C_1 v_1 \neq 0 \\ C_1 v_2 = 0 \end{array}\right\}$ (C_1, A) detectable, but NOT observable

<u>(C_2, A)</u>

$\left.\begin{array}{l} C_2 v_1 \neq 0 \\ C_2 v_2 \neq 0 \end{array}\right\}$ (C_2, A) observable

a) For stabilizing solution P we need stabilizability & detectability. For pos. def. we need (in addition) observability.

Hence,

$\boxed{\begin{array}{l} P \text{ exists for } (C_1, A, B_1) \ \& \ (C_2, A, B_1) \\ P \text{ pos. def. for } (C_2, A, B_1) \end{array}}$

b) $\dot{y} = C_1 \dot{x} = C_1 (Ax + B_1 u)$

$= [1 \ \ 1] \left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \right)$

$= u$

$\Rightarrow J = \int_0^\infty (y^2 + u^2)\, dt$ & $\dot{y} = u$

ARE : $ap + pa + q - \dfrac{pbbp}{r} = 0$ $(a=0, b=1, q=r=1)$

$\Rightarrow p^2 = 1 \Rightarrow p = 1$

$\Rightarrow u^*(t) = -\dfrac{bp}{r} y(t) \Rightarrow \boxed{u^*(t) = -y(t)}$

$J_{min} = p\, y(0)^2 \Rightarrow \boxed{J_{min} = y(0)^2}$

Q4. 20%

Consider the linear system

$$\dot{x} = Ax + Bu$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and the pair (A, B) is stabilizable. Let the symmetric matrix $P \in \mathbb{R}^{n \times n}$ satisfy

$$\min_{u(\cdot)} \int_0^\infty (x^T x + u^T u)\, dt = x(0)^T P x(0)$$

for all initial conditions $x(0) \in \mathbb{R}^n$.

(a) Write the Riccati equation satisfied by P.

(b) Show that the feedback law $u = -\frac{1}{2} B^T P x$ exponentially stabilizes the system.

a) $A^T P + PA + Q - PBR^{-1}B^T P = 0$ $\quad Q = I,\ R = I$

$\Rightarrow \boxed{A^T P + PA + I - PBB^T P = 0}$

b) Closed-loop system : $\dot{x} = \underbrace{[A - \frac{1}{2}BB^T P]}_{H} x$

$H^T P + PH = [A - \frac{1}{2}BR^T P]^T P + P[A - \frac{1}{2}BB^T P]$

$= A^T P + PA - PBB^T P$

$= -I$

$\Rightarrow H^T P + PH = -I \qquad (*)$

Since the pair (Q, A) is observable P is pos. def. Hence $(*)$ is a Lyapunov equation. That is, H is Hurwitz. ∎