First name:______________________________

Last name:______________________________

Student ID:______________________________

Signature:______________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1. 30%

For the scalar system $\dot{x} = -x + u$ starting from the initial condition $x(0) = 1$ the below cost is to be minimized.

$$J = x(1)^2 + \int_0^1 u^2 dt \,.$$

Find $x^*(1)$, the final value of the optimal trajectory.

Q2. 20%

Given t_0, t_f, $x(t_0)$, $x(t_f)$, the cost

$$J = \int_{t_0}^{t_f} x\sqrt{1+u^2}\,dt$$

is to be minimized for the scalar system $\dot{x} = u$. Assuming the boundary conditions are satisfied, determine whether each of the below candidates could be an optimal trajectory. *Hint:* $\dot{H}^* = ?$

(a) $x^*(t) = \cos t + \sin t$.

(b) $x^*(t) = e^{t/2} + e^{-t/2}$.

Q3. 30%

Consider the collection of systems

$$\begin{aligned} \dot{x} &= Ax + B_i u \\ y &= C_j x \end{aligned}$$

where

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad C_1 = [1 \ \ 1], \quad C_2 = [1 \ \ 0].$$

The cost to be minimized is

$$J = \int_0^\infty (y^2 + u^2)\, dt\,.$$

(a) For which of the four systems (C_1, A, B_1), (C_1, A, B_2), (C_2, A, B_1), (C_2, A, B_2) can you claim that the optimal stabilizing Riccati solution P exists? How about its positive definiteness?

(b) Solve the optimal control problem for the system (C_1, A, B_1). That is, compute the optimal control and minimum value of the cost. *Hint:* $\dot{y} = ?$

Q4. 20%

Consider the linear system

$$\dot{x} = Ax + Bu$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and the pair (A, B) is stabilizable. Let the symmetric matrix $P \in \mathbb{R}^{n\times n}$ satisfy

$$\min_{u(\cdot)} \int_0^\infty (x^T x + u^T u)\, dt = x(0)^T P x(0)$$

for all initial conditions $x(0) \in \mathbb{R}^n$.

(a) Write the Riccati equation satisfied by P.

(b) Show that the feedback law $u = -\frac{1}{2} B^T P x$ exponentially stabilizes the system.