First name:________________

Last name: KEY

Student ID:________________

Signature:________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1. 30%

(a) Write the variation δJ of the functional

$$J(x_1, x_2) = \int_{t_0}^{t_f} (x_1^2 + x_1 x_2 + x_2^2 + 2\dot{x}_1\dot{x}_2)\, dt$$

where the endpoints (initial time, initial state, final time, final state) are given.

(b) Find the extremal $x^*(t)$ of the following functional

$$J(x) = \int_0^7 (x^2 + 2x\dot{x} - \dot{x}^2)\, dt \quad \text{subject to} \quad x(0) = 1 \text{ and } x(7) = 0\,.$$

(c) Find the extremal $x^*(t)$ of the following functional

$$J(x) = \int_0^1 \left(\frac{1}{2}\dot{x}^2 + x\dot{x} + \dot{x} + x\right) dt \quad \text{subject to} \quad x(0) = \frac{1}{2} \text{ and } x(1) \text{ free}\,.$$

a) $\delta J = \int_{t_0}^{t_f} \left[\frac{\partial g}{\partial x} - \frac{d}{dt}\frac{\partial g}{\partial \dot{x}}\right]\delta x \, dt$

$\frac{\partial g}{\partial x_1} = 2x_1 + x_2$, $\frac{\partial g}{\partial \dot{x}_1} = 2\dot{x}_2$, $\frac{d}{dt}\frac{\partial g}{\partial \dot{x}_1} = 2\ddot{x}_2$

$\frac{\partial g}{\partial x_2} = 2x_2 + x_1$, $\frac{\partial g}{\partial \dot{x}_2} = 2\dot{x}_1$, $\frac{d}{dt}\frac{\partial g}{\partial \dot{x}_2} = 2\ddot{x}_1$

$$\Rightarrow \boxed{\delta J = \int_{t_0}^{t_f} \{ [2x_1 + x_2 - 2\ddot{x}_2]\delta x_1 + [2x_2 + x_1 - 2\ddot{x}_1]\delta x_2 \} dt}$$

b) $\frac{\partial g}{\partial x} = 2x + 2\dot{x}$

$\frac{\partial g}{\partial \dot{x}} = 2x - 2\dot{x} \Rightarrow \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} = 2\dot{x} - 2\ddot{x}$

Euler-Lgr. Eqn. $\Rightarrow \cancel{2\dot{x}} - 2\ddot{x} = 2x + \cancel{2\dot{x}}$

$\Rightarrow \ddot{x} + x = 0$

$\Rightarrow x(t) = c_1 \cos t + c_2 \sin t$

$x(0) = 1 \Rightarrow c_1 = 1$

$x(7) = 0 \Rightarrow \cos 7 + c_2 \sin 7 = 0 \Rightarrow c_2 = -\frac{\cos 7}{\sin 7}$

$\Rightarrow x^*(t) = \cos t - \frac{\cos 7}{\sin 7}\sin t = \boxed{-\frac{1}{\sin 7}\sin(t-7)}$

c) $\frac{\partial g}{\partial x} = \dot{x} + 1$

$\frac{d}{dt}\left\{\frac{\partial g}{\partial \dot{x}}\right\} = \frac{d}{dt}\{\dot{x} + x + 1\} = \ddot{x} + \dot{x}$

Eul.-Lgr. Eqn. $\Rightarrow \ddot{x} + \dot{x} = \dot{x} + 1 \Rightarrow \ddot{x} = 1$

$\Rightarrow x(t) = \frac{1}{2}t^2 + c_1 t + c_0$

$x(0) = \frac{1}{2} \Rightarrow c_0 = \frac{1}{2}$

$x(1)$ free $\Rightarrow \frac{\partial}{\partial \dot{x}} g \Big|_{t=1} = 0 \Rightarrow \dot{x}(1) + x(1) + 1 = 0$

$\Rightarrow \{1 + c_1\} + \{\frac{1}{2} + c_1 + \frac{1}{2}\} + 1 = 0 \Rightarrow c_1 = -\frac{3}{2}$

$\Rightarrow \boxed{x^*(t) = \frac{1}{2}t^2 - \frac{3}{2}t + \frac{1}{2}}$

Q2. 25%

For the scalar system $\dot{x} = u$ starting from the initial condition $x(0) = 1$ the following cost

$$J(u) = x(1)^2 + \int_0^1 (x^2 + u^2)\, dt$$

is to be minimized. Find the optimal trajectory $x^*(t)$.

$H = x^2 + u^2 + pu$

$\dot{p} = -\frac{\partial H}{\partial x} = -2x$

$0 = \frac{\partial H}{\partial u} = 2u + p \Rightarrow u = -\frac{1}{2}p$

$\dot{x} = u = -\frac{1}{2}p \Rightarrow \ddot{x} = -\frac{1}{2}\dot{p} = -\frac{1}{2}\{-2x\}$

$\Rightarrow \ddot{x} - x = 0 \Rightarrow x(t) = c_1 e^{t} + c_2 e^{-t}$

$x(0) = 1 \Rightarrow c_1 + c_2 = 1 \quad (1)$

bound. cond. $\left.\frac{\partial h}{\partial x} - p\right|_{t=1} = 0 \Rightarrow 2x(1) - p(1) = 0$

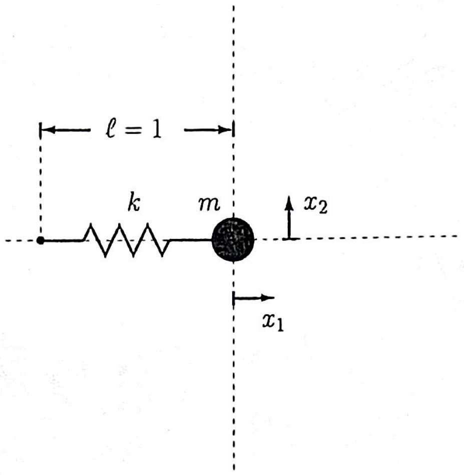
$\Rightarrow \dot{x}(1) = -\frac{1}{2}p(1) = -\frac{1}{2}\{2x(1)\} = -x(1)$

$\Rightarrow c_1 e - c_2 e^{-1} = -(c_1 e + c_2 e^{-1}) \Rightarrow c_1 = 0$

$(1) \Rightarrow c_2 = 1$

$\Rightarrow \boxed{x^*(t) = e^{-t}}$

Q3. 20%



Consider the frictionless planar mass-spring system shown at rest in the figure. (Note that the equilibrium shown in the figure is not the only equilibrium.) The body with mass m is connected to the spring with spring constant k. The other end of the spring is fixed to the frame of reference as shown. This body is allowed to move on the plane, where the displacement from the shown equilibrium to east is denoted by x_1 and to north by x_2. Obtain the dynamics of this system using the Euler-Lagrange equation.

$$KE = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2)$$

$$PE = \frac{1}{2} k \left[\sqrt{(x_1+1)^2 + x_2^2} - 1 \right]^2$$

$$\mathcal{L} = KE - PE$$

E.-L. eqn $\Rightarrow$ $\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$

$$\frac{\partial \mathcal{L}}{\partial x_1} = -k \left[\sqrt{(x_1+1)^2 + x_2^2} - 1 \right] \cdot \frac{x_1+1}{\sqrt{(x_1+1)^2 + x_2^2}}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = -k \left[\sqrt{(x_1+1)^2 + x_2^2} - 1 \right] \cdot \frac{x_2}{\sqrt{(x_1+1)^2 + x_2^2}}$$

$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m\ddot{x}_1$ & $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m\ddot{x}_2$

Hence,

$$\boxed{\begin{aligned} m\ddot{x}_1 &= \frac{\partial \mathcal{L}}{\partial x_1} \\ m\ddot{x}_2 &= \frac{\partial \mathcal{L}}{\partial x_2} \end{aligned}}$$

Q4. 25%

Consider the scalar system $\dot{x} = ax + u$ and the associated optimal cost

$$J^*(t_0, x(t_0)) = \min_{u(\cdot)} \int_{t_0}^{\infty} (x(t)^2 + u(t)^2)\, dt\,.$$

It is known that $J^*(t, x) = px^2$ for some constant p.

(a) Find p (in terms of a) using HJB equation.

(b) Show that optimal trajectories satisfy $|x^*(t)| \le |x(0)|e^{-t}$ for all a.

a) $0 = \frac{\partial J}{\partial t} + \min_u \left\{ x^2 + u^2 + \frac{\partial J}{\partial x}(ax+u) \right\}$

$= \min_u \left\{ x^2 + u^2 + 2px(ax+u) \right\}$

$\Rightarrow 2u + 2px = 0 \Rightarrow u = -px$

$\Rightarrow x^2 + u^2 + 2px(ax+u) \Big|_{u=-px} = 0$

$\Rightarrow 0 = x^2 + p^2x^2 + 2px(a-p)x$

$= [\underbrace{-p^2 + 2ap + 1}_{=0}]x^2$

$\boxed{p = a + \sqrt{1+a^2}}$ (p cannot be negative!)

b) $\dot{x} = ax + u$

$= ax - px = ax - (a + \sqrt{1+a^2})x$

$= -\sqrt{1+a^2}\, x$

$\rightarrow x^*(t) = x(0)\, e^{-\sqrt{1+a^2}\cdot t}$

$\Rightarrow |x^*(t)| = |x(0)| \cdot e^{-\sqrt{1+a^2}\, t}$ $\quad \sqrt{1+a^2} \ge 1 \quad \forall a$

$\le |x(0)| \cdot e^{-t}$

$\blacksquare$