First name:________________

Last name: Sol'n

Student ID:________________

Signature:________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1. 30%

Consider the optimization problem

$$V_N(x) = \min_{u_0,\dots,u_{N-1}} x_N^T H x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k$$

$$\text{subject to} \begin{cases} x_0 = x \\ x_{k+1} = A x_k + B u_k \end{cases}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, the symmetric positive definite matrices H, Q, R are of appropriate dimensions, and $N \geq 0$ is an integer.

(15) (a) Write the recursive relation that allows one to compute Q_{N+1} from Q_N where $V_N(x) = x^T Q_N x$. What does Q_0 equal?

For the remainder of the question consider the scalar case with $A = B = Q = H = 1$ and $R = r$.

(10) (b) Compute the limit $\lim_{N\to\infty} V_N(x)$ in terms of r and x.

(5) (c) Take $r = 3/4$ and $N = \infty$. Compute the optimal control sequence $(u_0^*, u_1^*, \dots)$ in terms of x.

a) By principle of optimality:

$$V_{N+1}(x) = x^T Q x + \min_u \{ u^T R u + V_N(Ax+Bu) \}$$

$$= x^T Q x + \min_u \underbrace{\{ u^T R u + (Ax+Bu)^T Q_N (Ax+Bu) \}}_{\square} \quad (1)$$

$$0 = \nabla_u \square = 2Ru + 2B^T Q_N (Ax+Bu)$$

$$\Rightarrow u^* = -[R + B^T Q_N B]^{-1} B^T Q_N A x =: -K_N x \quad (2)$$

(1) & (2) imply

$$\boxed{\begin{aligned} Q_{N+1} &= Q + K_N^T R K_N + [A - BK_N]^T Q_N [A - BK_N] \\ \text{where } K_N &= [R + B^T Q_N B]^{-1} B^T Q_N A \end{aligned}} \quad (*)$$

$$\boxed{Q_0 = H}$$

b) ~~Let~~ let $p = Q_\infty$ (hence $V_\infty(x) = px^2$) Eq. (*) yields

$$K_\infty = \frac{p}{p+r} \quad \text{and}$$

$$p = 1 + \left(\frac{p}{p+r}\right)^2 r + p\left(1 - \frac{p}{p+r}\right)^2$$

$$= 1 + \frac{pr}{p+r} \Rightarrow p^2 + \cancel{pr} = p + r + \cancel{pr}$$

$$\Rightarrow \left(p - \frac{1}{2}\right)^2 = r + \frac{1}{4} \Rightarrow \boxed{p = \frac{1}{2} + \sqrt{r + \frac{1}{4}}} \quad (**)$$

c) Eq (**) yields $p = \frac{3}{2}$. Then

$$K_\infty = \frac{3/2}{3/2 + 3/4} = \frac{2}{3}. \text{ Therefore (2) yields}$$

$$x_{k+1} = x_k - \frac{2}{3} x_k = \frac{1}{3} x_k \Rightarrow x_k^* = \frac{1}{3^k} x$$

$$\Rightarrow u_k^* = -K_\infty x_k^* = \boxed{-\frac{2}{3^{k+1}} x}$$

Q2. 25%

Consider the first-order LTI system $\dot{x} = u$ and the associated optimization problem

$$J^*(t_0, x(t_0)) = \min_{u(\cdot)} \int_{t_0}^{\infty} \left(qx(t)^2 + ru(t)^2\right) dt$$

where q and r are positive constants.

(15) (a) Show that the candidate function $V(t, x) = cx^2$ (where c is a constant) satisfies the HJB equation for some c. What is this c?

(10) (b) Find the optimal trajectory $x^*(t)$ for $t_0 = 1$ and $x(1) = 7$.

a) $0 = \nabla_t V(t,x) + \min_{u} \left\{ qx^2 + ru^2 + \nabla_x V(t,x)\, u \right\}$

Hence for $V(t,x) = cx^2$ we have

$$0 = \min_{u} \left\{ qx^2 + ru^2 + 2cxu \right\}$$

let $c = \sqrt{qr}$

$$= \min_{u} \left\{ \left(\sqrt{q}\, x + \sqrt{r}\, u\right)^2 \right\}$$

$$= 0$$

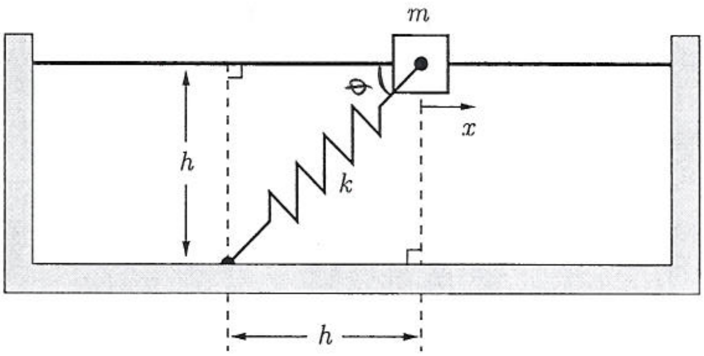
Hence $\boxed{c = \sqrt{qr}}$

b) $\arg\min_{u} \left\{ \left(\sqrt{q}\, x + \sqrt{r}\, u\right)^2 \right\} = -\sqrt{q/r}\, x$

$\Rightarrow \dot{x}^* = -\sqrt{q/r}\, x^* \Rightarrow x^*(t) = x(t_0)\, e^{-\sqrt{q/r}\,(t - t_0)}$

$$\Rightarrow \boxed{x^*(t) = 7\, e^{-\sqrt{q/r}\,(t-1)}}$$

Q3. 20%



Consider the frictionless mass-spring system shown at rest in the figure. The mass m, connected to the spring with spring constant k, is allowed to slide (horizontally) through the bar, where the displacement from the equilibrium is denoted by x. Obtain the dynamics of this system using the Euler-Lagrange equation.

$$KE = \frac{1}{2} m \dot{x}^2$$

$$PE = \frac{1}{2} k (\Delta l)^2 \quad \text{where} \quad \Delta l = \sqrt{(h+x)^2 + h^2} - h\sqrt{2}$$

Euler Lagrange eq:

$$\underbrace{\frac{d}{dt}\left\{\underbrace{\frac{\partial}{\partial \dot{x}}\left[KE - PE\right]}_{m\dot{x}}\right\}}_{m\ddot{x}} = \underbrace{\frac{\partial}{\partial x}\left[KE - PE\right]}_{-\frac{\partial}{\partial x} PE}$$

$$\frac{\partial}{\partial x} PE = k\,\Delta l \cdot \underbrace{\frac{\partial}{\partial x}\Delta l}_{\frac{h+x}{\sqrt{(h+x)^2 + h^2}}}$$

Hence,

$$\boxed{m\ddot{x} = -\underbrace{k \underbrace{\left[\sqrt{(h+x)^2 + h^2} - h\sqrt{2}\right]}_{\Delta l} \cdot \underbrace{\frac{h+x}{\sqrt{(h+x)^2 + h^2}}}_{\cos\theta}}}$$

Q4. 25%

The system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2) + u$$

under the constraint $|u| \leq 1$, is to be controlled to minimize the performance measure

$$J = \int_0^1 [2x_1(t)^2 + x_2(t)^2 + u(t)^2]dt\,.$$

The initial and final state values are specified.

(10) **(a)** Determine the costate equation for the system.

(15) **(b)** Determine the optimal control $u^*(t)$ (in terms of the optimal state and/or optimal costate solutions).

a) $H = 2x_1^2 + x_2^2 + u^2 + p_1 x_2 + p_2(-x_1 + x_2(1-x_1^2) + u)$

$$\frac{\partial H}{\partial x_1} = 4x_1 - p_2 - 2x_1 x_2 p_2$$

$$\frac{\partial H}{\partial x_2} = 2x_2 + p_1 + (1-x_1^2)p_2$$

Hence,

$$\boxed{\begin{aligned} \dot{p}_1 &= -4x_1 + p_2 + 2x_1 x_2 p_2 \\ \dot{p}_2 &= -2x_2 - p_1 - (1-x_1^2)p_2 \end{aligned}}$$

b) $u^*(t) = \arg\min_{|u| \leq 1} H(x^*(t), u, p^*(t))$

$$= \arg\min_{|u|\leq 1} u^2 + p_2^*(t)u$$

$$= \begin{cases} -\frac{1}{2}p_2^*(t) & \text{if } |p_2^*(t)| \leq 2 \\ -1 & \text{if } p_2^*(t) > 2 \\ +1 & \text{if } p_2^*(t) < -2 \end{cases}$$

Hence,

$$\boxed{u^*(t) = -\text{sat}\left(\frac{1}{2}p_2^*(t)\right)}$$ where