First name:____________________

Last name: KEY

Student ID:____________________

Signature:____________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 120 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1.

25%

For the scalar system $\dot{x} = \alpha x - u$ starting from the initial condition $x(0) = 1$ the below cost is to be minimized

$$J = \int_0^T e^{-t}\sqrt{u(t)}\, dt$$

while ensuring the terminal constraint $x(T) = 0$, where $T > 0$ is fixed. Find (in terms of T) the optimal control $u^*(t)$

(a) when $\alpha = 1$,

(b) when $\alpha = 2$.

$$H = e^{-t}\sqrt{u} + p(\alpha x - u)$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\alpha p \Rightarrow p(t) = c_1 e^{-\alpha t}$$

$$0 = \frac{\partial H}{\partial u} = \frac{e^{-t}}{2\sqrt{u}} - p \Rightarrow \sqrt{u} = \frac{e^{-t}}{2p} = \frac{e^{-t}}{2c_1 e^{-\alpha t}}$$

$$\Rightarrow \sqrt{u} = \frac{e^{(\alpha-1)t}}{2c_1} \Rightarrow u(t) = c_2 e^{2(\alpha-1)t}$$

a) $\alpha = 1 \Rightarrow u(t) \equiv \text{constant}$

$$\Rightarrow \dot{x} = x - c_2 \Rightarrow x(t) = c_3 e^{t} + c_4$$

$$\left.\begin{array}{l} x(0) = 1 \Rightarrow c_3 + c_4 = 1 \\ x(T) = 0 \Rightarrow c_3 e^{T} + c_4 = 0 \end{array}\right\} c_3 = \frac{1}{1-e^{T}},\ c_4 = \frac{e^{T}}{e^{T}-1}$$

$$u = x - \dot{x} = c_4 \Rightarrow \boxed{u(t) = \frac{e^{T}}{e^{T}-1}}$$

b) $\alpha = 2 \Rightarrow u(t) = c_2 e^{2t}$

$$\Rightarrow \dot{x} = 2x - c_2 e^{2t} \Rightarrow x(t) = (c_5 + c_6 t)e^{2t}$$

$$x(0) = 1 \Rightarrow c_5 = 1$$

$$x(T) = 0 \Rightarrow (1 + c_6 T)e^{2T} = 0 \Rightarrow c_6 = -\frac{1}{T}$$

$$u = 2x - \dot{x} = 2(c_5 + c_6 t)e^{2t} - \{2(c_5 + c_6 t)e^{2t} + c_6 e^{2t}\}$$

$$= -c_6 e^{2t}$$

$$\Rightarrow \boxed{u(t) = \frac{e^{2t}}{T}}$$

Q2. 25%

For the below second order system

$$\dot{x}_1 = u_1$$
$$\dot{x}_2 = u_2$$

the following cost is to be minimized.

$$J = \int_0^{\frac{\pi}{4}} (x_1^2 + u_1 u_2 + u_2^2)\, dt \quad \text{subject to} \quad \begin{cases} x_1(0) = 1, & x_1(\frac{\pi}{4}) = 2, \\ x_2(0) = \frac{3}{2}, & x_2(\frac{\pi}{4}) \text{ free.} \end{cases}$$

Find the optimal trajectories $x_1^*(t)$ and $x_2^*(t)$.

$H = x_1^2 + u_1 u_2 + u_2^2 + P_1 u_1 + P_2 u_2$

$\dot{P}_1 = -\frac{\partial H}{\partial x_1} = -2x_1 \quad (1)$

$\dot{P}_2 = -\frac{\partial H}{\partial x_2} = 0 \Rightarrow P_2(t) = \text{constant} \quad (2)$

Bound. cond.

$\underbrace{\frac{\partial h}{\partial x_2}}_{0} - P_2 \Big|_{t=\frac{\pi}{4}} = 0 \Rightarrow P_2(\frac{\pi}{4}) = 0 \quad \Big\} \text{ by } (2)$

$\Rightarrow P_2(t) = 0 \quad (3)$

$0 = \frac{\partial H}{\partial u_1} = u_2 + P_1 \quad (4)$

$0 = \frac{\partial H}{\partial u_2} = u_1 + 2u_2 + \underbrace{P_2}_{0} \overset{(3)}{\Rightarrow} u_1 + 2u_2 = 0 \quad (5)$

$\dot{x}_1 = u_1 \overset{(5)}{=} -2u_2 \overset{(4)}{=} 2P_1$

$\Rightarrow \ddot{x}_1 = 2\dot{P}_1 \overset{(1)}{=} -4x_1 \Rightarrow x_1(t) = c_1 \cos 2t + c_2 \sin 2t$

$x_1(0) = 1 \Rightarrow c_1 = 1$

$x_1(\frac{\pi}{4}) = 2 \Rightarrow c_2 = 2$

$\boxed{x_1^*(t) = \cos 2t + 2\sin 2t}$

$\dot{x}_2 = u_2 \overset{(5)}{=} -\frac{1}{2} u_1 = -\frac{1}{2}\dot{x}_1$

$\Rightarrow x_2(t) = -\frac{1}{2} x_1(t) + c_3$

$= -\frac{1}{2}\cos 2t - \sin 2t + c_3$

$x_2(0) = \frac{3}{2} \Rightarrow c_3 = 2$

$\Rightarrow \boxed{x_2^*(t) = 2 - \frac{1}{2}\cos 2t - \sin 2t}$

Q3. 25%

Consider the Duffing map

$$\left.\begin{aligned} x_1^+ &= x_2 \\ x_2^+ &= -x_1 + 3x_2 - x_2^3 \end{aligned}\right\} =: f(x)$$
$$y = x_2. \quad =: h(x)$$

(a) Find the equilibrium point(s) of this system.

(b) For the initial condition $x(0) = [1 \quad 0]^T$ find $x(3)$.

(c) For this system find the minimum-time observer dynamics

$$z^+ = ?, \quad \hat{x} = ?$$

by applying Glad's algorithm.

(d) For the initial conditions $x(0) = [1 \quad 0]^T$ and $z(0) = [7 \quad -3]^T$ find $\hat{x}(3)$.

a) $f(x_e) = x_e$

$\Rightarrow \left.\begin{aligned} x_2 &= x_1 \\ -x_1 + 3x_2 - x_2^3 &= x_2 \end{aligned}\right\} -x_1 + 3x_1 - x_1^3 = x_1$

$\Rightarrow x_1^3 - x_1 = 0 \Rightarrow x_1(x_1 - 1)(x_1 + 1) = 0$

$\Rightarrow x_1 \in \{-1, 0, 1\}$

$\Rightarrow x_e \in \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

b) $x(1) = \begin{bmatrix} x_2 \\ -x_1 + 3x_2 - x_2^3 \end{bmatrix}_{x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$x(2) = \begin{bmatrix} x_2 \\ -x_1 + 3x_2 - x_2^3 \end{bmatrix}_{x = \begin{bmatrix} 0 \\ -1 \end{bmatrix}} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

$x(3) = \begin{bmatrix} x_2 \\ -x_1 + 3x_2 - x_2^3 \end{bmatrix}_{x = \begin{bmatrix} -1 \\ -2 \end{bmatrix}} = \begin{bmatrix} -2 \\ 1 - 6 + 8 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$\Rightarrow \boxed{x(3) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}}$

c) $h\eta = hz \Rightarrow \eta_2 = z_2$

$hf\eta = y \Rightarrow -\eta_1 + 3\eta_2 - \eta_2^3 = y$

$\Rightarrow \eta_1 = -y + 3z_2 - z_2^3$

$z^+ = f\eta \Rightarrow \left\{ \begin{aligned} z_1^+ &= \eta_2 = z_2 \\ z_2^+ &= -\eta_1 + 3\eta_2 - \eta_2^3 \\ &= -\{-y + 3z_2 - z_2^3\} + 3z_2 - z_2^3 \\ &= y \end{aligned} \right.$

$\Rightarrow \boxed{\begin{aligned} z_1^+ &= z_2 \\ z_2^+ &= y \end{aligned}}$

$\hat{x} = f(z) \Rightarrow \boxed{\begin{aligned} \hat{x}_1 &= z_2 \\ \hat{x}_2 &= -z_1 + 3z_2 - z_2^3 \end{aligned}}$

d) Glad's observer is dead beat. Hence for $k \geq 2$ $\hat{x}(k) = x(k)$. Therefore

$\boxed{\hat{x}(3) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}}$

Q4. 25%

Consider the single-input single-output LTI system

$$\dot{x} = Ax + Bu$$
$$y = B^T x$$

where the pair (A, B) is controllable and the matrix A is skew-symmetric, i.e., $A + A^T = 0$.

Claim. All the eigenvalues of A lie on the imaginary axis.
Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A with the eigenvector v. We can write

$$0 = v^*(A + A^T)v = v^*Av + v^*A^Tv = v^*(Av) + (Av)^*v$$
$$= \dots \quad v^*(\lambda v) + (\lambda v)^*v = \lambda \|v\|^2 + \lambda^* \|v\|^2 = 2\mathrm{Re}\{\lambda\} \cdot \underbrace{\|v\|^2}_{>0}$$

(a) Complete the proof. $\Rightarrow \mathrm{Re}\{\lambda\} = 0$ ▨

(b) Show that this system is observable.

Consider the following cost to be minimized for the system given above.

$$J = \int_0^\infty (y^2 + u^2)\, dt\,.$$

(c) Show that $J_{\min} = \|x(0)\|^2$.

(d) Show that the optimal control is in output feedback form, i.e., $u_{\min} = -\alpha y$ for some $\alpha \in \mathbb{R}$.

b) Suppose not. Then by eigenvector test. There exist an eigenvector v of A that belongs to null (B^T). Note that $Av = \lambda v$ implies

$-Av = -\lambda v \Rightarrow A^Tv = -\lambda v \Rightarrow v$ is an eigenvector of A^T. Hence we have

$A^Tv = \mu v \quad (\mu = -\lambda)$

& $B^Tv = 0$

But this contradicts that (A, B) is controllable. ▨

c) $J_{\min} = x(0)^T P x(0) = \|x(0)\|^2 = x(0)^T I x(0)$

Hence $P = I$. If we can show that $P = I$ satisfies ARE then we're done.

ARE: $A^TP + PA + Q - PBR^{-1}B^TP \overset{?}{=} 0$ | $P = I$, $Q = BB^T$, $R = I$

$\Rightarrow \underbrace{A^T + A}_{0} + BB^T - BB^T = 0$

▨

d) $u_{\min} = -R^{-1}B^TPx \quad (R = I,\ P = I)$

$= -B^Tx$

$= -y$

$\Rightarrow u_{\min} = -y \quad$ (i.e. $\alpha = 1$) ▨