First name:__________________________

Last name:__________________________

Student ID:__________________________

Signature:__________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 120 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1. 25%

For the scalar system $\dot{x} = \alpha x - u$ starting from the initial condition $x(0) = 1$ the below cost is to be minimized

$$J = \int_0^T e^{-t} \sqrt{u(t)} \, dt$$

while ensuring the terminal constraint $x(T) = 0$, where $T > 0$ is fixed. Find (in terms of T) the optimal control $u^*(t)$

(a) when $\alpha = 1$,

(b) when $\alpha = 2$.

Q2. 25%

For the below second order system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \end{aligned}$$

the following cost is to be minimized.

$$J = \int_0^{\frac{\pi}{4}} \left(x_1^2 + u_1 u_2 + u_2^2\right) dt \qquad \text{subject to} \qquad \begin{cases} x_1(0) = 1, & x_1(\frac{\pi}{4}) = 2, \\ x_2(0) = \frac{3}{2}, & x_2(\frac{\pi}{4}) \text{ free.} \end{cases}$$

Find the optimal trajectories $x_1^*(t)$ and $x_2^*(t)$.

Q3. 25%

Consider the Duffing map

$$\begin{aligned} x_1^+ &= x_2 \\ x_2^+ &= -x_1 + 3x_2 - x_2^3 \\ y &= x_2\,. \end{aligned}$$

(a) Find the equilibrium point(s) of this system.

(b) For the initial condition $x(0) = [1 \;\; 0]^T$ find $x(3)$.

(c) For this system find the minimum-time observer dynamics

$$z^+ = ?\,, \quad \hat{x} = ?$$

by applying Glad's algorithm.

(d) For the initial conditions $x(0) = [1 \;\; 0]^T$ and $z(0) = [7 \;\; -3]^T$ find $\hat{x}(3)$.

Q4. 25%

Consider the single-input single-output LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= B^T x \end{aligned}$$

where the pair (A, B) is controllable and the matrix A is skew-symmetric, i.e., $A + A^T = 0$.

Claim. All the eigenvalues of A lie on the imaginary axis.
Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A with the eigenvector v. We can write

$$\begin{aligned} 0 &= v^*(A + A^T)v \\ &= \ldots \end{aligned}$$

(a) Complete the proof.

(b) Show that this system is observable.

Consider the following cost to be minimized for the system given above.

$$J = \int_0^\infty \left(y^2 + u^2\right) dt\,.$$

(c) Show that $J_{\min} = \|x(0)\|^2$.

(d) Show that the optimal control is in output feedback form, i.e., $u_{\min} = -\alpha y$ for some $\alpha \in \mathbb{R}$.