First name:________________________

Last name:________________________

Student ID:________________________

Signature:________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 120 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1.

25%

Consider a unit mass with position y on a straight line, whose motion is subject to the force u through the Newton's law $\ddot{y} = u$. The initial position and velocity of the mass are $y(0) = 0.75$ and $\dot{y}(0) = -1$, respectively. Find the force $u^*(t)$ to be applied through the interval $t \in [0, 1]$ that brings the mass to rest ($\dot{y}(1) = 0$) at the origin ($y(1) = 0$) while minimizing the cost

$$J = \int_0^1 u^2 dt .$$

let $x_1 = y$ & $x_2 = \dot{y}$. Then

sys. $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$ $x_1(0) = 0.75$, $x_1(1) = 0$; $x_2(0) = -1$, $x_2(1) = 0$ $\quad H = u^2 + p_1 x_2 + p_2 u$

$0 = \nabla_u H = 2u + p_2 \Rightarrow u = -\frac{1}{2} p_2$

$\dot{p}_1 = -\nabla_{x_1} H = 0 \Rightarrow p_1(t) = c_1$ (constant)

$\dot{p}_2 = -\nabla_{x_2} H = -p_1 \Rightarrow p_2(t) = c_2 - c_1 t$

$\dot{x}_2 = u = -\frac{1}{2} p_2 = -\frac{1}{2}(c_2 - c_1 t)$

$\Rightarrow x_2(t) = x_2(0) - \frac{c_2}{2} t + \frac{c_1}{4} t^2 \quad (2)$

$\dot{x}_1 = x_2 \Rightarrow x_1(t) = x_1(0) + x_2(0) t - \frac{c_2}{4} t^2 + \frac{c_1}{12} t^3 \quad (1)$

Using (1), (2), and bound. cond. we have

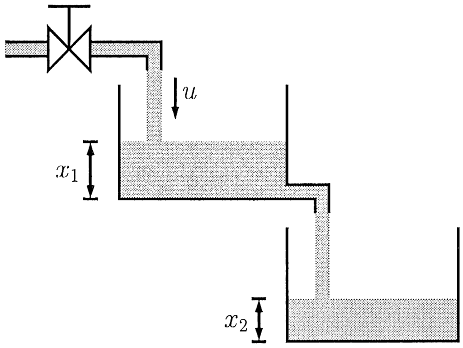
$0 = x_1(1) = 0.75 - 1 - \frac{c_2}{4} + \frac{c_1}{12} \quad (3)$

$0 = x_2(1) = -1 - \frac{c_2}{2} + \frac{c_1}{4} \quad (4)$

(3) & (4) $\Rightarrow c_1 = 6, \; c_2 = 1$

$\Rightarrow u^*(t) = -\frac{1}{2} p_2 = -\frac{1}{2}(1 - 6t) = \boxed{3t - \frac{1}{2}}$

Q2. 25%



Consider the cascaded water tanks shown in the figure, where x_1 and x_2 are the water levels in the tanks and $u \in [0,\ 1]$ is the inflow to the first tank. This system can be modelled as

$$\begin{aligned} \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= x_1\,. \end{aligned}$$

Let initially (at $t = 0$) both tanks be empty. Find the optimal control $u^*(t)$ that maximizes $x_2(1)$ while ensuring $x_1(1) = 0.5$.

$J = -x_2(1)$ (i.e. $h(x) = -x_2$ & $g(x,u) = 0$)

$H = p_1(-x_1+u) + p_2 x_1$

$u^* = \arg\min_{u\in[0,1]} H \Rightarrow u^* = \begin{cases} 0 & \text{for } p_1 > 0 \\ 1 & \text{for } p_1 < 0 \end{cases}$

$\dot{p}_1 = -\nabla_{x_1} H = p_1 - p_2$ (1)

$\dot{p}_2 = -\nabla_{x_2} H = 0$ (2)

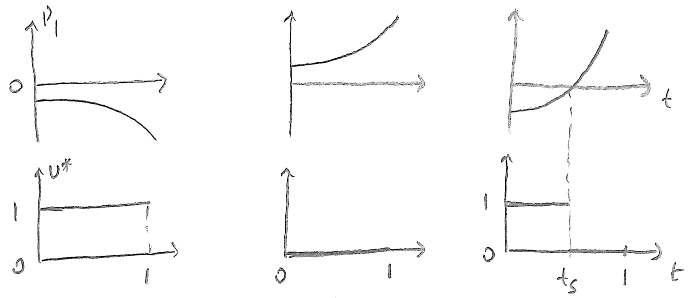
Bound. cond.

$\left[\frac{\partial h}{\partial x} - p^T\right] \begin{bmatrix} \cancel{\delta x_{1f}} \\ \delta x_{2f} \end{bmatrix} = 0 \Rightarrow \left.\underbrace{\frac{\partial h}{\partial x_2}}_{-1} - p_2\right|_{t=1} = 0$

$\Rightarrow p_2(1) = -1$ (3)

(1), (2), (3) imply $\dot{p}_1 = p_1 + 1 \Rightarrow p_1(t) = -1 + ce^t$

There are three possibilities (depending on c)



(I) (II) (III)

I & II does NOT satisfy the constraint $x_1(1) = \frac{1}{2}$.

Hence the optimal solution is as in case III.

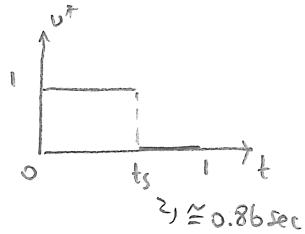
$t_s = ?$

$x_1(t) = 1 - e^{-t}$ for $t \in [0, t_s]$

$\Rightarrow x_1(t) = [1 - e^{-t_s}]\, e^{-(t-t_s)}$ for $t \geq t_s$

$x_1(1) = \frac{1}{2} = [1 - e^{-t_s}]\, e^{-(1-t_s)} \Rightarrow \boxed{t_s = \ln\left(1 + \frac{e}{2}\right)}$

Hence,



Q3. 25%

Consider the second-order (normalized) antenna position system with friction

$$\ddot{y} + \dot{y} = u$$

where y is the (angular) position of the antenna and u is the torque applied to the system. Cost to be minimized is

$$J = \int_0^\infty (y^2 + u^2)dt\,.$$

(a) Find $J_{\min}$ in terms of $y(0)$ and $\dot{y}(0)$.

(b) Let the optimal control law be $u_{\min} = -k_1 y - k_2 \dot{y}$. Find k_1 and k_2.

Let $x_1 = y$ & $x_2 = \dot{y}$. Then

Sys. $\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$

$J = \int_0^\infty (x^T Q x + u^T R u)dt$ with $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $R = 1$.

Let $P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

ARE: $\underbrace{A^T P + PA}_{\begin{bmatrix} 0 & a-b \\ a-b & 2(b-c) \end{bmatrix}} + Q - \underbrace{PBR^{-1}B^T P}_{\begin{bmatrix} b^2 & bc \\ bc & c^2 \end{bmatrix}} = 0$

$\Rightarrow \begin{bmatrix} 1-b^2 & a-b-bc \\ a-b-bc & 2(b-c)-c^2 \end{bmatrix} = 0$

$\Rightarrow a = \sqrt{3}, \; b = 1, \; c = \sqrt{3} - 1$

$\Rightarrow P = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3}-1 \end{bmatrix}$

a) $J_{min} = x(0)^T P x(0)$

$= \boxed{\sqrt{3}\, y(0)^2 + 2y(0)\dot{y}(0) + (\sqrt{3}-1)\dot{y}(0)^2}$

b) $[k_1 \; k_2] = R^{-1} B^T P$

$= \boxed{[1 \quad \sqrt{3}-1]}$

Q4. 25%

For the system

$$x^+ = f(x, u)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, let $h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\kappa_f : \mathbb{R}^n \to \mathbb{R}^m$ satisfy for all x

$$h(f(x, \kappa_f(x))) - h(x) \leq -\|x\|^2 .$$

Consider the following optimization problem

$$\text{Prob}(x, N) : \; V_N(x) = \min_{(v_0, \cdots, v_{N-1})} h(\xi_N) + \sum_{k=0}^{N-1} \|\xi_k\|^2 \quad \text{subject to} \quad \begin{cases} \xi_0 = x , \\ \xi_{k+1} = f(\xi_k, v_k) . \end{cases}$$

Let the feedback law $\kappa_N : \mathbb{R}^n \to \mathbb{R}^m$ satisfy $\kappa_N(x) = v_0^*$, where $(v_0^*, \ldots, v_{N-1}^*)$ is a minimizing control sequence for $\text{Prob}(x, N)$.

(a) Show that

$$V_N(f(x, \kappa_N(x))) \leq V_{N-1}(f(x, \kappa_N(x))) . \tag{1}$$

(b) Show that (1) implies

$$V_N(f(x, \kappa_N(x))) - V_N(x) \leq -\|x\|^2 .$$

Given x, let $(\xi_0^*, \cdots, \xi_N^*)$ & $(v_0^*, \cdots, v_{N-1}^*)$ be corresponding optimal sequences. We can write by principle of optimality:

$$V_N(x) = \|\xi_0^*\|^2 + V_{N-1}(\xi_1^*) \qquad \xi_0^* = x, \; v_0^* = \kappa_N(x)$$

$$= \|x\|^2 + V_{N-1}(f(x, \kappa_N(x))) \qquad (2)$$

In particular, we have

$$V_{N-1}(f(x, \kappa_N(x))) = \sum_{k=1}^{N-1} \|\xi_k^*\|^2 + h(\xi_N^*) \qquad (3)$$

a) Construct the feasible sequences

$$(\hat{\xi}_0, \cdots, \hat{\xi}_{N-1}, \hat{\xi}_N) = (\xi_1^*, \cdots, \xi_N^*, f(\xi_N^*, \kappa_f(\xi_N^*)))$$

$$(\hat{v}_0, \cdots, \hat{v}_{N-1}) = (v_1^*, \cdots, v_{N-1}^*, \kappa_f(\xi_N^*))$$

Now, let's use these to upperbound $V_N(f(x, \kappa_N(x)))$

$$V_N(f(x, \kappa_N(x))) \leq \sum_{k=0}^{N-1} \|\hat{\xi}_k\|^2 + h(\hat{\xi}_N)$$

$$= \sum_{k=1}^{N-1} \|\xi_k^*\|^2 + \underbrace{\|\xi_N^*\|^2 + h(f(\xi_N^*, \kappa_f(\xi_N^*)))}_{\leq h(\xi_N^*)}$$

Hence,

$$V_N(f(x, \kappa_N(x))) \leq \sum_{k=1}^{N-1} \|\xi_k^*\|^2 + h(\xi_N^*) \qquad (3)$$

$$= V_{N-1}(f(x, \kappa_N(x))) \qquad \blacksquare$$

b) Combining (1) & (2) gives us the desired inequality. $\blacksquare$