First name:____________________________

Last name:____________________________

Student ID:____________________________

Signature:____________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 120 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

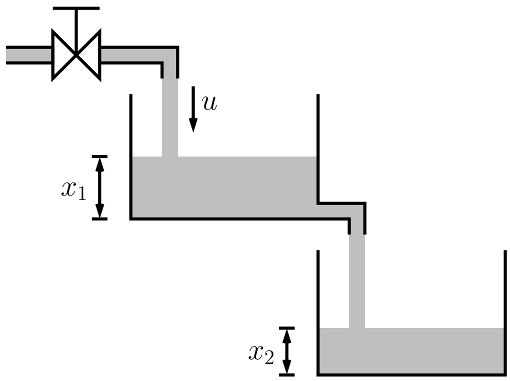
Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1. 25%

Consider a unit mass with position y on a straight line, whose motion is subject to the force u through the Newton's law $\ddot{y} = u$. The initial position and velocity of the mass are $y(0) = 0.75$ and $\dot{y}(0) = -1$, respectively. Find the force $u^*(t)$ to be applied through the interval $t \in [0,\, 1]$ that brings the mass to rest ($\dot{y}(1) = 0$) at the origin ($y(1) = 0$) while minimizing the cost

$$J = \int_0^1 u^2 dt\,.$$

Q2. 25%



Consider the cascaded water tanks shown in the figure, where x_1 and x_2 are the water levels in the tanks and $u \in [0, 1]$ is the inflow to the first tank. This system can be modelled as

$$\begin{aligned}\dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= x_1\,.\end{aligned}$$

Let initially (at $t = 0$) both tanks be empty. Find the optimal control $u^*(t)$ that maximizes $x_2(1)$ while ensuring $x_1(1) = 0.5$.

Q3. 25%

Consider the second-order (normalized) antenna position system with friction

$$\ddot{y} + \dot{y} = u$$

where y is the (angular) position of the antenna and u is the torque applied to the system. Cost to be minimized is

$$J = \int_0^\infty (y^2 + u^2) dt \, .$$

(a) Find $J_{\min}$ in terms of $y(0)$ and $\dot{y}(0)$.

(b) Let the optimal control law be $u_{\min} = -k_1 y - k_2 \dot{y}$. Find k_1 and k_2.

Q4. 25%

For the system

$$x^+ = f(x,\, u)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, let $h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\kappa_{\mathrm{f}} : \mathbb{R}^n \to \mathbb{R}^m$ satisfy for all x

$$h(f(x,\, \kappa_{\mathrm{f}}(x))) - h(x) \leq -\|x\|^2\,.$$

Consider the following optimization problem

$$\mathrm{Prob}(x,\, N):\ V_N(x) = \min_{(v_0,\,\dots\,,\,v_{N-1})} h(\xi_N) + \sum_{k=0}^{N-1} \|\xi_k\|^2 \quad \text{subject to} \quad \begin{cases} \xi_0 = x\,, \\ \xi_{k+1} = f(\xi_k,\, v_k)\,. \end{cases}$$

Let the feedback law $\kappa_N : \mathbb{R}^n \to \mathbb{R}^m$ satisfy $\kappa_N(x) = v_0^*$, where $(v_0^*,\, \dots,\, v_{N-1}^*)$ is a minimizing control sequence for $\mathrm{Prob}(x,\, N)$.

(a) Show that

$$V_N(f(x,\, \kappa_N(x))) \leq V_{N-1}(f(x,\, \kappa_N(x)))\,. \tag{1}$$

(b) Show that (1) implies

$$V_N(f(x,\, \kappa_N(x))) - V_N(x) \leq -\|x\|^2\,.$$