ON FINITE GROUPS ADMITTING
A SPECIAL NONCOPRIME ACTION

GÜLİN ERCAN

(Communicated by Jonathan I. Hall)

ABSTRACT. An important result of Turull (1984) is the following:

Let $G+A$ be a finite solvable group, $G < GA$ and $\left( |G|, |A| \right) = 1$. Then
\[ f(G) \leq f(C_{G}(A)) + 2(\ell), \]
where $f$ denotes the Fitting height and $\ell$ denotes the composition length.

The purpose of this work is to give a treatment of the minimal configuration
in this framework with additional conditions, yet without the coprimeness
condition.

Here we will prove (see Theorem 2) the following:

Let $G$ be a finite solvable group and let $\alpha$ be an automorphism of $G$ of order $p$
for some prime $p$. Assume that the orders of elements of $H = G\langle \alpha \rangle$ lying outside
of $G$ are not divisible by $p^2$. If $C_{S}(x)$ is nilpotent for any $x \in H - G$ of order $p$ and
for any $x$-invariant section $S$ of $G$, then $f(G) \leq 3$. Furthermore, if the nilpotency
condition is replaced by abelianess, then $f(G) \leq 2$.

An immediate consequence of this theorem is a particular case of Turull’s result
(see also [1] and [4]):

Let $G$ be a finite solvable group and let $\alpha$ be an automorphism of $G$ of order $p$
for some prime $p$ where $\left( |G|, |\alpha| \right) = 1$. If $C_{G}(\alpha)$ is nilpotent, then $f(G) \leq 3$.
Furthermore if $C_{G}(\alpha)$ is abelian, then $f(G) \leq 2$.

Although our main purpose is the proof of Theorem 2, and Theorem 1 below
makes its appearence as an auxiliary, it should be pointed out that Theorem 1
is of independent interest, too. Theorem 1 is, in its turn, a generalization of the
following Lemma.

Lemma ([3, Lemma 1]). Let $G = ST$ be a group where $S \triangleleft G$, $S$ is a $p$-group
and $T$ is a $t$-group for distinct primes $p$ and $t$, and let $\alpha$ be an automorphism of $G$
of order $p^n$ which leaves $T$ invariant. Assume that $C_{T/T_0}(z) = 1$, where $T_0 = C_{T}(S)$
and $z = \alpha^{p^n-1}$. Let $V$ be a $kG\langle \alpha \rangle$-module on which $S$ acts faithfully and $k$ is a field
of characteristic different from $p$. If $[C_{V}(z), C_{S}(z)] = 1$, then $[S, T] = 1$.

Theorem 1. Let $\langle \alpha \rangle$ be a cyclic group of order $p^n$ for some prime $p$, and let $G$
be a group acted on by $\langle \alpha \rangle$. Suppose that $S \triangleleft G\langle \alpha \rangle$ is an $s$-group and $T$ is an
$\langle \alpha \rangle$-invariant $t$-subgroup of $G$ for distinct primes $s$ and $t$, such that $[S, T] \neq 1$. Let
$V$ be a $kG\langle \alpha \rangle$-module on which $S$ acts faithfully, where $k$ is a field of characteristic

Received by the editors May 17, 2004.

2000 Mathematics Subject Classification. Primary 20D10, 20F28.

Key words and phrases. Noncoprime action, fitting height.
not dividing $s$. Let $z = \alpha^p n - 1$. Then either $[C_V(z), C_S(z)] \neq 1$

or $[C_V(z), C_T(z)] \neq 1$

or $[C_S(x), C_{T/T_0}(x)] \neq 1$ for some $\mathfrak{T} \in (T/T_0)(\alpha) - (T/T_0)$ of order $p$,

where $T_0 = C_T(S)$.

Proof. Set $H = G(\alpha)$ and use induction on $|H| + \dim_k V$. We may assume that $n = 1$ and $G = \text{ST}$.

(1) $\Phi(T/T_0) = 1$ and $\langle \alpha \rangle$ acts irreducibly on $T/T_0$.

This is an immediate consequence of induction argument applied to $ST_1(\alpha)$ on $V$ for a minimal $\langle \alpha \rangle$-invariant subgroup $T_1/T_0$ of $T/T_0$.

(2) \(t \neq p\).

Assume the contrary. Then $T/T_0$ is centralized by any $1 \neq \mathfrak{t} \in (T/T_0)(\alpha) - (T/T_0)$. Let $U$ be an irreducible $T(\alpha)$-submodule of $S/\Phi(S)$ on which $T$ acts nontrivially, and let $\mathfrak{t} \in T/T_0$ such that $[U, \mathfrak{t}] \neq 1$. Then $C_U(\mathfrak{t}) = 1$. This yields a contradiction as $U = \langle C_U(\alpha) | 1 \neq a \in \langle \mathfrak{t}, \alpha \rangle \rangle$ by (6 5.3.16) and $[C_U(\mathfrak{t}), T/T_0] = 1$ for any $1 \neq \mathfrak{t} \in \langle \mathfrak{t}, \alpha \rangle - \langle \mathfrak{t} \rangle$.

(3) $S/\Phi(S)$ is an irreducible $T(\alpha)$-module with $[S, T] = S$, $[\Phi(S), T] = 1$ and $S$ is special.

Let $S_1$ be a normal subgroup of $H$ properly contained in $S$ on which $T$ acts nontrivially. Put $T_1 = C_T(S_1)$. By induction, there exists $\mathfrak{T} \in (T/T_1)(\alpha) - (T/T_1)$ such that $[C_{S_1}(x), C_{T/T_1}(x)] \neq 1$. As $t \neq p$, this yields that $[C_{S_1}(x), C_{T/T_0}(x)] \neq 1$ which is not the case. Thus $T(\alpha)$ acts irreducibly on $S/\Phi(S)$, $[S, T] = S$, $[\Phi(S), T] = 1$ and $S$ is special.

(4) $[T, \alpha] = 1$.

Assume the contrary. Then $C_{T/T_0}(\alpha) = 1 = 1$ and so $C_{S/\Phi(S)}(\alpha) \neq 1$. Now $s \neq p$, because otherwise $[C_V(\alpha), C_S(\alpha)] \neq 1$ by the Lemma.

Let $M$ be an irreducible $ST(\alpha)$-submodule of $V$ on which $S$ acts nontrivially. Then $[M, S] = M$ and so $[M, T] \neq 1$. Set $\mathfrak{S} = S/C_S(M)$. By Clifford’s theorem applied to $ST$ on $M$, we have that $M = W_1 \oplus \cdots \oplus W_r$, where the $W_i$'s are homogeneous $\mathfrak{ST}$-modules. Here $N(\alpha)(W_i) = \langle W_i \rangle$ for each $i = 1, \ldots, r$ and so either $r = 1$ or $r = p$. If the latter holds, then $[W_i, C_\mathfrak{S}(\alpha)] = 1$ for each $i$, because $[C_S(\alpha), C_S(\alpha)] = 1 = s \neq p$. It follows that $C_S(\alpha) \leq C_S(M)$ and so $C_S(\alpha) \leq \Phi(S)$ which is not the case. Thus $M$ is a homogeneous $\mathfrak{ST}$-module, i.e. $M = M_1 \oplus \cdots \oplus M_t$ with $M_i \cong M_t$ irreducible $\mathfrak{ST}$-modules.

If $\mathfrak{S}$ is nonabelian, then $[\Phi(\mathfrak{S}), \alpha] = 1$ and so $C_M(\alpha) = C_M(\Phi(S)) = 1$. This shows that $\mathfrak{S}$ acts irreducibly $\mathfrak{S}$, $[\mathfrak{S}, \alpha] \neq 1$, because otherwise $[\mathfrak{S}, T] = 1$ by the three subgroup lemma. By (5) applied to the action of both $\mathfrak{S}, \alpha, [\alpha]$ and $T(\alpha)$ on $M$, we conclude that $s = 2 = t$, which is impossible.

Thus $\mathfrak{S}$ is abelian. The number of homogeneous components of $M_1|\mathfrak{S}$ is a power of $t$ and so the number of homogeneous components of $M|\mathfrak{S}$ is also a power of $t$. Since $t \neq p$, $\alpha$ fixes a homogeneous component $W$ of $M|\mathfrak{S}$. If $U$ is a homogeneous component of $M|\mathfrak{S}$ which is $\alpha$-invariant and different from $W$, then $W = U^a$ for some $a \in T(\alpha)$. Now $[a, \alpha] \in N_T(W)$ and so $1 \neq C_T/N_T(W)(\alpha) \cong C_T(T/T_0)(N_T(W)/T_0)(\alpha)$, i.e. $C_{T/T_0}(\alpha) \neq 1$, which is not the case. Thus $\alpha$ fixes exactly one homogeneous component $W$ of $M|\mathfrak{S}$. Observe that either $N_T(W) = T$ or $N_T(W) \leq T_0$. If the first holds, then $[\mathfrak{S}, T, W] = [\mathfrak{S}, W] = 1$ implying that $C_M(\mathfrak{S}) = 1$, which is not the case. Hence $N_T(W) \leq T_0$. Also note that $[\mathfrak{S}, \alpha] \neq 1$, because otherwise $[\mathfrak{S}, T] = 1$ by the three subgroup lemma. Now $[[\mathfrak{S}, \alpha], W] = 1$, and so there exists
a homogeneous component $U$ of $M_S$ such that $U \neq U^\alpha$. Here note that

$$C_{\overline{S}}(\alpha) \leq \text{Ker}(\overline{S} \text{ on } U),$$

as $[C_U(\alpha), C_S(\alpha)] = 1$, and so

$$C_{\overline{S}}(\alpha) \cap \text{Ker}(\overline{S} \text{ on } W) \leq \text{Ker}(\overline{S} \text{ on } M) = 1.$$

Then $C_{\overline{S}}(\alpha) \cap \langle C_{\overline{S}}(\alpha) \rangle T \neq \overline{S}$, where $T = T/C_T(M)$ and so $C_{\overline{S}}(\alpha) \cap \langle C_{\overline{S}}(\alpha) \rangle T \neq \overline{S}$, since $\overline{S}$ is an irreducible $T(\alpha)$-module. It follows that $|\overline{S}| = |C_{\overline{S}}(\alpha)|$. On the other hand $|\overline{S}, [T/T_0, \alpha]| = \overline{S}$ and so $|\overline{S}| = |C_{\overline{S}}(\alpha)|$ by Lemma 4.5 in [7]. As $t \neq p$, we get a contradiction. Therefore $[T/T_0, \alpha] = 1$, i.e. $C_T(\alpha)T_0 = T$. By induction we see that $C_T(\alpha) = T$.

(5) $[S, \alpha] = S$ and so $s \neq p$.

$[S, \alpha]$ is either trivial or the whole of $S$. If it is trivial, then $[S, T] = 1$ as $[C_{\overline{S}}(\alpha), C_T(\alpha)] = 1$, a contradiction.

(6) $S$ is abelian.

Assume the contrary. Then $1 \neq \Phi(S) = Z(S)$. Let $M$ be an irreducible $ST(\alpha)$-submodule of $V$ on which $\Phi(S)$ acts nontrivially. Set $\overline{S} = S/C_S(M)$. We consider $M|ST = W_1 + \cdots + W_r$, where $W_i$'s are homogeneous $ST$-components of $M$. If $r = p$, then $[W_i, T] = 1$ for each $i$, as $[C_M(\alpha), T] = 1$, and so $[M, T] = 1$, which is not the case. Then $r = 1$. It follows that $\Phi(S)$ is not cyclic, then there exists $1 \neq \alpha \in \Phi(S)$ such that $C_M(\alpha) \neq 1$ by (5.3.16), implying that $C_{\overline{S}}(\alpha) 

Hence $\Phi(S)$ is cyclic and so $\overline{S}$ is extraspecial, where $|\overline{S}| = 2^{2n+1}$ and $p = 2n + 1$ for some $n \geq 1$, by [6].

By [6] 5.5.2, the number of distinct cyclic subgroups of order 4 in $\overline{S}$ is

$$\frac{1}{2}(2^{2n} + (-2)^n).$$

Since each cyclic group of order 4 contains two elements of order 4, and distinct cyclic subgroups of order 4 have no element of order 4 in common, there are $2^{2n} + (-2)^n = 2^{2n} + 1$ elements of order 4 in $\overline{S}$. As $T(\alpha)$ acts irreducibly on $\overline{S}/\Phi(S)$ and $[\overline{S}, T] = \overline{S}$, we have $C_T(\alpha) \leq \Phi(S)$. It follows that $C_T(\alpha) = \Phi(S)$, since $[\Phi(S), T] = 1$. Now $\Phi(S)$ contains no element of order 4, since it is cyclic of order 2. Thus $T(\alpha)$ permutes the elements of $\overline{S}$ of order 4, without fixing any, in orbit of length $|T(\alpha)/\Phi(T)| = tp$. Therefore $tp$ divides $2^{2n} (2n + 1)$. But as $t \neq s = 2$ and $p = 2n + 1$, $tp$ divides $2n + 1 = p$ which yields that $t = 1$, a contradiction.

(7) Finally, let $M$ be an irreducible $ST(\alpha)$-submodule of $V$ on which $S$ acts nontrivially. Set $\overline{S} = S/C_S(M)$. Let $\Omega = \{W_1, \cdots, W_r\}$ be the set of all homogeneous $\overline{S}$-components of $M$. Since $[S, \alpha] = S$, no $W_i$ is $\alpha$-invariant. Because otherwise as $[S, W_i] = 1$ for each $i$, we have $C_M(S) \neq 1$, a contradiction.

Let $\mathcal{O} = \{W, W^\alpha, \cdots, W^{p-1}\}$ be an $\alpha$-orbit. Set $T = T/C_T(M)$ and $X = \bigoplus_{i=0}^{p-1} W^\alpha$. As $[C_X(\alpha), T(\alpha)] = 1$, we have $[W, N_T(\alpha)(W)] = 1$. Let $t \in T$. If $Y = X^t$, then $C_Y(\alpha) = C_X(\alpha)^t = C_X(\alpha)$ and so $X \cap Y \neq 0$, i.e. $X = Y$. Hence $T$ acts on $\mathcal{O}$ and $\mathcal{O} = \Omega$. This gives that $p = |\Omega| = |T(\alpha): N_T(\alpha)(W)|$. Then $N_T(\alpha)(W) = T$ because $T$ is the unique subgroup of $T(\alpha)$ of index $p$. This yields that $[W, T] = 1$ and so $[M, T] = 1$, a contradiction which completes the proof of Theorem 1.
As a consequence of Theorem 1, we have

**Theorem 2.** Let $G$ be a finite solvable group and let $\alpha$ be an automorphism of $G$ of order $p$ for some prime $p$. Assume that the orders of elements of $H = G(\alpha)$ lying outside $G$ are not divisible by $p^2$. If $C_2(x)$ is nilpotent for any $x \in H - G$ of order $p$ and for any $x$-invariant section $S$ of $G$, then $f(G)$ is at most 3. Furthermore, if the nilpotency condition is replaced by abelianness, then $f(G) \leq 2$.

**Proof.** Let $H = G(\alpha)$ be a minimal counterexample to the theorem. We may assume that $f(G) = 4$. Then by Lemma 1 in [2] there exist subgroups $C_i$ of $G$ and subgroups $D_i \triangleleft C_i$ for $i = 1, 2, 3, 4$ and an element $x \in H - G$ of order $p$ such that the following are satisfied:

(i) $C_i$ is a $p_i$-subgroup for some prime $p_i$, i.e. $\pi(C_i) = \{p_i\}$ for any $i$ and $p_i \neq p_{i+1}$ for $i = 1, 2, 3$.

(ii) $C_i$ and $D_i$ are $(\prod_{j>i} C_j)$-invariant for any $i$.

(iii) $C_i = C_i/D_i$ is a special group on the Frattini factor group of which $(\prod_{j>i} C_j)(\alpha)$ acts irreducibly and $C_{i+1}$ acts trivially on $\Phi(C_i)$ for any $i$.

(iv) $[C_i, C_{i+1}] = C_i$ for $i = 1, 2, 3$.

(v) $C_{i+1}(C_i/\Phi(C_i)) = C_{i+1}(C_i)$ is contained in $\Phi(C_{i+1} mod D_{i+1})$ for $i = 1, 2, 3$.

(vi) $[C_j, C_i]$ is not contained in $\Phi(C_j)$ for any $i = 2, 3, 4$ and any $1 \leq j < i$.

Put $K = C_1C_2C_3C_4$. Now $K(x)$ satisfies the hypothesis of the theorem.

Applying Theorem 1 to the action of $C_3C_4(x)$ on the Frattini factor group $\hat{C}_2$ of $C_2$ we see that $[C_{C_2}(x), C_{C_2}(x)] \neq 1$ with the requirement $\pi(C_2) = \pi(C_1)$. Also applying Theorem 1 to the action of $C_2C_3(x)$ on $C_1$ we see that $[C_{C_1}(x), C_{C_3}(x)] \neq 1$ with the requirement $\pi(C_1) = \pi(C_3)$. Now $D_4 = C_{C_1}(\hat{C}_2)$ and so $C_{C_1}(x) \not\subset D_4$, i.e. $[C_{C_4}, x] = 1$ this forces that $C_{C_4}(x) \leq \Phi(C_3)$, because otherwise $[C_3C_4, x] = 1$, which is not the case. Then $C_{C_4}(x) \leq Z(C_3C_4(x))$ and so $C_{C_2}(C_{C_4}(x))$ is either trivial or $\hat{C}_2$. If it is trivial, then $C_{\hat{C}_2}(x) = 1$, which is not the case. Hence $C_{\hat{C}_2}(x) = 1$, i.e. $C_{C_1}(x) \leq D_3 = C_{C_1}(C_1)$ as $\pi(C_1) = \pi(C_3)$, a contradiction. This completes the proof of the first claim.

The last claim can be easily shown by an application of Theorem 1 to $C_1C_2C_3C_4(x)$, where $C_i$ are subgroups of $H$ and $D_i \triangleleft C_i$, $i = 1, 2, 3$, satisfying (i)–(vi). \hfill \Box

**Corollary.** Let $G$ be a finite solvable group and let $\alpha$ be an automorphism of $G$ of order $p$ for some prime $p$ where $|\langle G, \alpha \rangle| = 1$. If $C_G(\alpha)$ is nilpotent, then $f(G) \leq 3$. Furthermore if $C_G(\alpha)$ is abelian, then $f(G) \leq 2$.

**References**

3. Ercan, G, Güloğlu, I.: On finite groups admitting a fixed point free automorphism of order $pq$. J. Group Theory 7 (2004), no. 4, 437-446. MR2080444

Department of Mathematics, Middle East Technical University, 06531, Ankara, Turkey