Action of a Frobenius-like group

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Abstract

We call a finite group Frobenius-like if it has a nontrivial nilpotent normal subgroup \( F \) possessing a nontrivial complement \( H \) such that \([F, h] = F\) for all nonidentity elements \( h \in H \). We prove that any irreducible nontrivial \( FH \)-module for a Frobenius-like group \( FH \) of odd order over an algebraically closed field has an \( H \)-regular direct summand if either \( F \) is fixed point free on \( V \) or \( F \) acts nontrivially on \( V \) and the characteristic of the field is coprime to the order of \( F \). Some consequences of this result are also derived.

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Theorem A. Let \( G \triangleleft GA \) such that \( A \) normalizes a Sylow system of \( G \). Suppose that \( G' \neq G \) and \([G, a] = G\) for all nonidentity elements \( a \in A \). Let \( V \) be a nonzero vector space over an algebraically closed field \( k \) and let \( GA \) act on \( V \) as a group of linear transformations such that \( \text{char}(k) \) does not divide the order of \( A \). Then \( VA \) has a proper \( A \)-regular direct summand if one of the following holds:

(i) \( CV(G) = 0 \),

(ii) \([V, G] \neq 0 \) and \( \text{char}(k) \) does not divide the order of \( G \).

The proof of this theorem relies on the following result which can be regarded

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as a generalization of [[2], V.17.13], and is of independent interest too.

**Theorem B.** Let $H$ be a group in which each Sylow subgroup is cyclic. Assume that $H/F(H)$ is not a nontrivial 2-group. Let $P$ be an extraspecial group of order $p^{2m+1}$ for some prime $p$ not dividing $|H|$. Suppose that $H$ acts on $P$ in such a way that $H$ centralizes $Z(P)$, and $[P,h] = P$ for any nonidentity element $h \in H$. Let $k$ be an algebraically closed field of characteristic not dividing the order of $G = PH$ and let $V$ be a $kG$-module on which $Z(P)$ acts nontrivially and $P$ acts irreducibly. Let $\chi$ be the character of $G$ afforded by $V$. Then $|H|$ divides $p^m - \delta$ and $\chi_H = \frac{p^m - \delta}{|H|} \rho + \delta \mu$ where $\rho$ is the regular character of $H$, $\mu$ is a linear character of $H$ and $\delta \in \{-1,1\}$. In particular, $V_H$ contains the regular $kH$-module as a direct summand if $G$ is of odd order.

It should be noted that if $G$ is not of odd order, then the module $V_H$ need not contain the regular $kH$-module.

We want to draw the attention of the reader to Theorem 3.2 and Theorem 3.4 in the remarkable paper [8] of Turull which are very close to Theorem B and Theorem A respectively.

As applications of Theorem A and Theorem B we obtain the following:

**Corollary C.** Let $G$ be a finite solvable group acted on coprimely by a Frobenius-like group $FH$ of odd order so that $[G,F] \neq 1$. Then $C_G(H) \neq 1$.

**Corollary D.** Let $P$ be a $p$-group acted on coprimely by a Frobenius-like group $FH$ of odd order so that $[P,F] = P$. Then

(i) the nilpotency class of $P$ is at most $2 \log_p |C_P(H)|$,

(ii) $|P|$ is bounded in terms of $|F|$ and $|C_P(H)|$,

(iii) the rank of $P$ is bounded in terms of $|F|$ and the rank of $C_P(H)$.

In the present paper all groups are assumed to be finite. The notation and terminology are standard, and the rank of a finite group is the minimum number $r$ such that every subgroup is generated by $r$ elements.
1. Existence of regular modules

In this section we prove a technical result pertaining to the main result of this paper, which can be regarded as a generalization of [2, V.17.13]. We begin with a preliminary lemma.

**Lemma 1.1.** Let $FH$ be a group with $F \triangleleft FH$, $F' \neq F$ and $[F,h] = F$ for all nonidentity elements $h \in H$. Assume that all Sylow subgroups of $H$ are cyclic. Then

(i) the groups $H'$ and $H/H'$ are cyclic of coprime orders,
(ii) $H = H' \langle y \rangle$ with $H' \cap \langle y \rangle = 1$ for some $y \in H$ where $H_0$ denotes the Fitting subgroup of $H$, and $H_0 = H' \times C_{\langle y \rangle}(H')$ is cyclic,
(iii) $\pi(H_0) = \pi(H)$.

**Proof.** The group $FH/F'$ is Frobenius with Frobenius complement isomorphic to $H$. Then (i) follows by [[3], Theorem 5.16]. In particular, $H = H' \langle y \rangle$ for some $y \in H$ with $H' \cap \langle y \rangle = 1$. On the other hand the group $H$ has a unique subgroup of order $p$ for each prime $p$ dividing its order by the argument applied in the proof of Theorem 6.19 in [3] which relies on [[3], Theorem 6.9]. Hence $\pi(H_0) = \pi(H)$ as claimed in (iii). Let now $H_0$ denote the Fitting subgroup of $H$. Then $H_0 = H' \langle H_0 \cap \langle y \rangle \rangle$ and $[H_0 \cap \langle y \rangle, H'] = 1$, that is, $H_0 \cap \langle y \rangle \subseteq C_{\langle y \rangle}(H') \subseteq H_0$. This establishes the claim (ii). □

**Theorem B.** Let $H$ be a group in which each Sylow subgroup is cyclic. Assume that $H/F(H)$ is not a nontrivial 2-group. Let $P$ be an extraspecial group of order $p^{2m+1}$ for some prime $p$ not dividing $|H|$. Suppose that $H$ acts on $P$ in such a way that $H$ centralizes $Z(P)$, and $[P,h] = P$ for any nonidentity element $h \in H$. Let $k$ be an algebraically closed field of characteristic not dividing the order of $G = PH$ and let $V$ be a $kG$-module on which $Z(P)$ acts nontrivially and $P$ acts irreducibly. Let $\chi$ be the character of $G$ afforded by $V$. Then $|H|$ divides $p^m - \delta$ and $\chi_H = \frac{p^m - \delta}{|H|} \rho + \delta \mu$ where $\rho$ is the regular character of $H$, $\mu$ is a linear character of $H$ and $\delta \in \{-1,1\}$. In particular, $V_H$ contains the regular $kH$-module as a direct summand if $G$ is of odd order.
Proof. Since all Sylow subgroups of $H$ are cyclic and $G/Z(P)$ is a Frobenius group with a complement isomorphic to $H$, we see that $H$ has the properties described in Lemma 2.1. By [[2], V.17.13] we can assume that $H$ is not nilpotent and recall that $H/F(H)$ is not a 2-group by hypothesis.

Note that $\dim V = p^m$ as $\chi_\rho$ is a faithful irreducible character of $P$. Let $D$ be the representation of $G$ afforded by the module $V$ and let $M$ be the $k$-space of square matrices of size $p^m$ over $k$. We define a left $kH$-module structure on $M$ by letting 

$$h \cdot X := D(h)XD(h^{-1}), \text{ for any } X \in M \text{ and for any } h \in H.$$ 

It is known that $H$ acts on $\text{Hom}_k(V,V)$ via the multiplication $(h \cdot T)(v) = hT(h^{-1}v)$ for any $h \in H$, $T \in \text{Hom}_k(V,V)$, and $v \in V$. Then clearly $M$ is isomorphic to the $k[H]$-module $\text{Hom}_k(V,V)$. Furthermore $\text{Hom}_k(V,V)$ and $V^* \otimes V$ are isomorphic as $k[H]$-modules. So by letting $\text{Irr}(H) = \{\psi_1, \psi_2, \ldots, \psi_s\}$ and $\chi_H = \sum_{i=1}^s n_i \psi_i$ with nonnegative integers $n_i, i = 1, \ldots, s$, we have $\Psi = \sum_{k,l=1}^s n_k n_l \psi_k \psi_l$ where $\Psi$ is the character of $H$ afforded by $M$.

Choose a transversal $T$ for $Z(P)$ in $P$. Then the set $\{D(x)|x \in T\}$ forms a basis for $M$ by a result of Burnside [[2], V.5.14] and the fact that $D(zx) = \lambda(z)D(x)$ for any $x \in T$ and $z \in Z(P)$. Notice that $P/Z(P)$ is the union of one $H$-orbit of length 1 and $d = \frac{p^{2m} - 1}{|H|}$ orbits of length $|H|$. Thus we have $M = \langle I \rangle \oplus M_1 \oplus \cdots \oplus M_d$ with $M_i \cong k[H]$ as $H$-module for any $i = 1, 2, \ldots, d$. So we get

$$\Psi = 1_H + \sum_{i=1}^s \frac{p^{2m} - 1}{|H|} \psi_i(1) \psi_i = \sum_{k,l=1}^s n_k n_l \psi_k \psi_l.$$ 

Thus the multiplicity of the principal character $1_H$ in $\Psi$ is

$$[1_H, \Psi]_H = 1 + \frac{p^{2m} - 1}{|H|} = \sum_{k=1}^s n_k^2$$

and the multiplicity of any nonprincipal $\alpha \in \text{Irr}(H)$ in $\Psi$ is

$$[\alpha, \Psi]_H = \frac{p^{2m} - 1}{|H|} \alpha(1) = \sum_{k,l=1}^s n_k n_l (\psi_1, \psi_k \alpha).$$

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In particular for any nonprincipal linear character $\gamma$ of $H$ we have

$$\frac{p^{2m-1}}{|H|} = \sum_{\alpha \in \text{Irr}(H)} n_{\alpha} n_{\alpha \gamma}.$$ 

This gives

$$1 = \sum_{\alpha \in \text{Irr}(H)} n_{\alpha}^2 - \sum_{\alpha \in \text{Irr}(H)} n_{\alpha} n_{\alpha \gamma},$$

and hence

$$2 = \sum_{\alpha \in \text{Irr}(H)} (n_{\alpha} - n_{\alpha \gamma})^2$$

for any nonprincipal linear character $\gamma$ of $H$.

The group $\hat{H}/H'$ of characters of the abelian group $H/H'$ is isomorphic to $H/H'$. In particular it is cyclic. Let $\vartheta$ be a generator of $\hat{H}/H'$. It acts on $\text{Irr}(H)$ by multiplication. Let $\Phi_i, i = 1, \ldots, b$ be the orbits of $\vartheta$ on $\text{Irr}(H)$ and let $m_i = |\Phi_i|$. Then we have

$$2 = \sum_{i=1}^{b} \sum_{\alpha \in \Phi_i} (n_{\alpha} - n_{\alpha \vartheta})^2.$$ 

So there are exactly two elements $\beta$ and $\gamma$ in $\text{Irr}(H)$ such that $|n_{\beta} - n_{\beta \vartheta}| = 1 = |n_{\gamma} - n_{\gamma \vartheta}|$, and we have $n_{\alpha} = n_{\alpha \vartheta}$ for any $\alpha \in \text{Irr}(H) - \{\beta, \gamma\}$. If $\beta \in \Phi_i$ and $\gamma \notin \Phi_i$, then $n_{\beta} \neq n_{\beta \vartheta} = n_{\beta \vartheta^2} = \cdots = n_{\beta \vartheta^{m_i-1}} = n_{\beta}$, which is not possible. So if necessary by reindexing the orbits, we can assume that $\beta$ and $\gamma$ are both elements of $\Phi_b$, and $n_{\alpha} = n_{\alpha \vartheta}$ for any $i = 1, 2, \ldots, b-1$ and any $\alpha \in \Phi_i$.

Suppose that $\gamma = \beta \vartheta^u$ for some $u \in \{1, 2, \ldots, m_b - 1\}$. We have

$$n_{\beta} \neq n_{\beta \vartheta} = \cdots = n_{\beta \vartheta^u} \neq n_{\beta \vartheta^{u+1}} = \cdots = n_{\beta \vartheta^{m_b-1}} = n_{\beta}.$$ 

Since each $\Phi_i$ is either a $\vartheta^2$-orbit or the union of two $\vartheta^2$-orbits of the same size we get

$$2 = \sum_{i=1}^{b} \sum_{\alpha \in \Phi_i} (n_{\alpha} - n_{\alpha \vartheta^2})^2 = \sum_{\Phi_b} (n_{\alpha} - n_{\alpha \vartheta^2})^2.$$ 

So the differences $n_{\beta} - n_{\beta \vartheta^2}, n_{\beta \vartheta^u} - n_{\beta \vartheta}, n_{\gamma} - n_{\gamma \vartheta^2}$ are all nonzero if $u \in \{2, \ldots, m_b - 2\}$, which is a contradiction. If necessary by replacing $\vartheta$ by $\vartheta^{-1}$ we can assume that $n_{\beta} \neq n_{\beta \vartheta} \neq n_{\beta \vartheta^2} = \cdots = n_{\beta \vartheta^{m_b-1}} = n_{\beta}$. We let $n_{\beta \vartheta} = n_{\beta} + \delta$, with some $\delta \in \{-1, 1\}$. Choose an element $\alpha_i$ from $\Phi_i, i = 1, 2, \ldots, b-1$, and let $\alpha_b = \beta$. Then

$$\chi_i = \sum_{i=1}^{b} n_{\alpha_i} (\alpha_i + \alpha_i \vartheta + \cdots + \alpha_i \vartheta^{m_i-1}) + \delta \mu,$$

where $\mu = \alpha_b \vartheta$. 

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By [2], V.17.13 we have
\[\chi_{\nu} = \frac{p^m - \delta}{|H'|} \rho' + \delta' \mu' = \left(\sum_{i=1}^{b} n_i \alpha_i \alpha_i\right)_{\nu} + \delta \alpha_{\nu},\]
for some \(\delta' \in \{-1, 1\}\) and \(\mu' \in \text{Irr}(H')\) where \(\rho'\) is the regular character of \(H'\).

It follows by ([4], Exercise 6.2) that if \(i \neq j\) then the sets of irreducible constituents of the restrictions of \(\alpha_i\) and \(\alpha_j\) are disjoint. By Clifford’s theorem we have
\[\alpha_i H' = e_i \sum_{j=1}^t \lambda_{i,j}\] where \(I_H(\lambda_{i,1}) = T_i, t_i = [H : T_i], H = \bigcup_{j=1}^t T_i x_{i,j},\) and \(\lambda_{i,j} = \lambda_{i,j}^{\pm}, j = 1, 2, \ldots, t_i; i = 1, 2, \ldots, b.\) Now \(\{\lambda_{i,j}|j = 1, 2, \ldots, t_i; i = 1, 2, \ldots, b\} = \text{Irr}(H').\)

It is known that there is a unique \(\xi_i \in \text{Irr}(T_i)\) such that \(\xi_i H = \alpha_i\) and \(\xi_{i\nu} = e_i \lambda_{i,1}.\) On the other hand as \(T_i/H'\) is cyclic, \(\lambda_{i,1}\) has an extension, say \(\varphi_i\) to \(T_i.\) But then \(\varphi H\) must belong to the \(\vartheta\)-orbit containing \(\alpha_i\) which implies \(\alpha_{i\nu} = (\varphi H)_{\nu}.\) Therefore we have
\[e_i = [\alpha_{i\nu}, \lambda_{i,1}] = [(\varphi H)_{\nu}, \lambda_{i,1}] = [\varphi_{\nu}, \lambda_{i,1}] = 1\] for any \(i = 1, 2, \ldots, b.\)

Let now \(e = \frac{p^m - \delta}{|H'|}\) and \(\mu' = \lambda_{i_0 j_0}.\) Then for any \(v \in \hat{H'}\) we have
\[\left[\chi_{\nu}, v\right]_{\nu} = \begin{cases} e & \text{if } v \neq \mu' \\ e + \delta' & \text{if } v = \mu'. \end{cases}\]

Set \(H_0 = F(H).\) Applying [2], V.17.13 to the action of \(PH_0\) on \(V\) we see in particular that \(|H_0|\) divides \(p^m - \delta^*\) for some \(\delta^* \in \{-1, 1\}.\) Then \(|H'|\) divides \(\delta - \delta^* = (p^m - \delta^*) - (p^m - \delta)\) and so we have either \(\delta = \delta^*\) or \(|H'| = 2.\) If the latter holds then \(H' \leq Z(H)\) and hence \(H\) is abelian, which is not the case. Thus \(|H_0/H'|\) divides \(e.\) In particular \(e > 1\) and so \(e + \delta' > 0\) which shows that \([\chi_{\nu}, \alpha_{i_0}]_{\nu} \neq 0.\)

If \(t_{i_0} \neq 1,\) then there exists \(j_1 \neq j_0\) such that
\[e = [\chi_{\nu}, \lambda_{i_0, j_1}]_{\nu} = [\chi_{\nu}, \lambda_{i_0, j_0}]_{\nu} = e + \delta'\]
which is not possible. Then \(t_{i_0} = 1\) and hence \(\mu'\) is \(H\)-invariant. This yields that \(\alpha_{i_0 H'} = \mu' = \lambda_{i_0, 1}.\) In particular \(\alpha_{i_0}\) is a linear character of \(H\) and so \(m_{i_0} = |H/H'|.\)
Furthermore we have
\[ e + \delta' = \begin{cases} 
  n_{\alpha_{i_0}} m_{i_0} & \text{if } i_0 < b \\
  n_{\alpha_i} m_b + \delta & \text{if } i_0 = b
\end{cases}. \]

Now \(|H_0/H'|\) divides the greatest common divisor of \(e\) and \(m_{i_0}\) which forces that \(i_0 = b\) as \(H_0/H'\) is nontrivial. Furthermore if \(\delta \neq \delta'\) we have \(|H_0/H'| = 2\), which implies by Lemma 2.1 that \(H/H'\) is a 2-group. This contradiction shows that \(\delta = \delta'\) and hence \(n_{\alpha_i} m_b = e\) by the above formula. In particular \(\frac{p^{m-\delta}}{|H|} = n_{\alpha_b}\) is an integer. On the other hand we also have \(e = [\chi_{\mu}, \lambda_{i,1}]_{\mu'} = n_{\alpha_i} m_i\) if \(i < b\).

Set next \(r_i = |T_i/H'|\). As \(\hat{T}_i/\hat{H}' = \langle \vartheta_{T_i} \rangle\) we obtain \(T_i \leq \text{Ker} \vartheta^r\). As \(\alpha_i = \xi_i^H\) for some \(\xi_i\) of \(T_i\) and \(T_i\) is normal in \(H\), we observe that \(\alpha_i(x) = 0\) for any \(x \notin T_i\). Combining these two observations we get \(\vartheta^r : \alpha_i = \alpha_i\). Thus \(m_i\) divides \(r_i\) and hence \(|H/H'| = r_i t_i = m_i c_i t_i\) for some positive integer \(c_i\). It follows now that \(n_{\alpha_i} m_i c_i t_i = e c_i \alpha_i(1)\) and hence \(n_{\alpha_i} = \frac{p^{m-\delta}}{|H|} c_i \alpha_i(1) \geq \frac{p^{m-\delta}}{|H|} \alpha_i(1)\). Thus \(\frac{p^{m-\delta}}{|H|} \rho + \delta \mu\) occurs in \(\chi_{\mu}\). As the degrees of these characters are the same we see that they are equal. This completes the proof of the theorem. \(\square\)

The next example shows that the hypothesis about the structure of \(H\) cannot be avoided.

**Example.** Let \(V\) be the \(GF(3^4)\)-space \(GF(3^4)\). We define the map \((\cdot | \cdot) : V \times V \to GF(3)\) by 
\[(\cdot | \cdot)(x, y) = \text{Tr}(d \cdot (xy^9 - x^9 y))\] for \(x, y \in V\), where \(d\) is an element of order 16 in \(GF(3^4)^*\). One can check that \((\cdot | \cdot)\) is a nonsingular symplectic form on \(V\).

Let \(b \in GF(3^4)^*\) be an element of order 5 and \(c \in GF(3^4)^*\) be an element of order 4. We define \(\tau_b : V \to V\) by \(\tau_b(x) = b \cdot x\) and \(\sigma : V \to V\) by \(\sigma(x) = c \cdot x^9\).

Then \(H = \langle \tau_b, \sigma \rangle\) is a subgroup of \(GL(4, 3)\) preserving the symplectic form, with \(|H| = 20\), \(H' = \langle \tau_b \rangle\) of order 5, and \(F(H) = H' \times \langle \sigma^2 \rangle\) of order 10. Furthermore \(h(v) = v\) for some \(0 \neq v \in V\) and \(h \in H\) implies that \(h = 1\). So if \(P\) is the extraspecial group of order \(3^3\) and exponent 3, then it admits \(H\) as a subgroup of automorphisms of \(P\), centralizing \(Z(P)\) and satisfying \([P, h] = P\) for any nonidentity element \(h \in H\). Let \(\chi\) be any irreducible character of the group \(PH\) which does not contain \(Z(P)\) in its kernel. Clearly, we have \(\chi_{\mu} \neq \frac{p^{m-\delta}}{|H|} \rho + \delta \mu\) for
the regular $H$-character $\rho$ and any $\delta \in \{-1,1\}$ and $\mu \in \text{Irr}(H)$, because $\frac{2^2 - \delta}{|H|}$ is not an integer.

2. Action of a Frobenius-like group

We define a slight generalization of Frobenius groups which we call Frobenius-like groups and prove the main result of this paper.

**Definition 2.1.** Let $F$ and $H$ be nontrivial finite groups such that $H$ acts on $F$ via automorphisms. Assume that $F$ is nilpotent and $[F, h] = F$ for all nonidentity elements $h \in H$. We call the semidirect product $FH$ a "Frobenius-like group" with kernel $F$ and complement $H$.

**Lemma 2.2.** Let $FH$ be a group with $F \trianglelefteq FH$, and $[F, h] = F$ for all nonidentity elements $h \in H$. Let $FH$ act on the set $X$. If $F$ acts nontrivially on $X$ then $H$ acts faithfully on $X$.

**Proof.** Let $K$ denote the kernel of $FH$ on $X$. If $K \cap H \neq 1$ then we have $F = [F, K \cap H] \leq K$. This contradiction proves the claim. $\Box$

**Theorem A.** Let $V$ be a nonzero vector space over an algebraically closed field $k$ and let $FH$ be a Frobenius-like group of odd order acting on $V$ as a group of linear transformations such that $\text{char}(k)$ does not divide the order of $H$. Then $V_H$ has a proper $H$-regular direct summand if one of the following holds:

(i) $C_V(F) = 0$,

(ii) $[V, F] \neq 0$ and $\text{char}(k)$ does not divide the order of $F$.

**Proof.** Assume that the theorem is false and choose a counter-example with minimum $\text{dim}V + |FH|$. We shall proceed in several steps.

(1) We may assume that $\text{char}(k)$ does not divide the order of $F$ and $F$ is a $q$-group for some prime $q$ with $C_V(F) = 0$.

Set $\text{char}(k) = p$. As $F$ is nilpotent, we have $F = F_p \times F_p'$. If (i) holds then $C_V(F_p') = 0$. Notice also that $[F_p', h] = F_p'$ for every nonidentity element.
So by an induction argument applied to the action of $F_{p'} H$ on $V$ we see that $F$ is a $p'$-group.

Let now $q$ be a prime dividing the order of $F$ such that $[V, F_q] \neq 0$. As the action of $F_q H$ on $[V, F_q]$ satisfies the hypothesis of the theorem it follows by induction that $V = [V, F_q]$ and $F = F_q$.

(2) The group $F H$ acts irreducibly and faithfully on $V$.

Let $U$ be an irreducible $F H$ submodule of $V$. Note that $C_{U}(F) \subseteq C_{V}(F) = 0$. It follows now by induction that $U$ has a proper $H$-regular direct summand and hence so does $V$. Therefore $V$ is an irreducible $F H$-module as claimed.

Notice next that $C_{F H}(V) = C_{F}(V)C_{H}(V)$. As a consequence of Lemma 2.2 we have $C_{H}(V) = 1$. Now an induction argument applied to the action of $(F/C_{F}(V))H$ on $V$ yields that $C_{F}(V) = 1$ which completes the proof of the claim.

(3) $V_{F}$ is homogeneous and hence $F$ is nonabelian.

By Clifford’s theorem the module $V$ is a direct sum of homogeneous $F$-modules permuted transitively by $H$. We pick now an $F$-homogeneous component $W$ of $V$ and set $H_1 = Stab_{H}(W)$. If $H_1 = 1$, then $V$ is free as a $kH$-module obviously. Thus we may assume that $H_1 \neq 1$. Applying induction to the action of $F H_1$ on $W$ we conclude that $W$ has a proper $H_1$-regular direct summand and hence $V$ has a proper $H$-regular direct summand, as desired. This forces now that $H_1 = H$, that is, $V_{F}$ is homogeneous.

Assume next that $F$ is abelian. Then $F$ acts by scalars on $V$ and so we have $F = [F, h] = 1$ for every $h \in H$ which is not the case. Therefore $F$ is a nonabelian group as claimed.

(4) $V_{F}$ is irreducible.

By (3), $V \cong X \oplus \cdots \oplus X$ for some irreducible $kF$-module $X$. Note also that for every $h \in H$ $V^h = V$ and hence $X^h$ and $X$ are isomorphic as $kF$-modules. As $H$ acts coprimely on $F$, Corollary 8.16 in [4] yields that the module $X$ can be extended to an $F H$-module $Y$ in a unique way subject to the condition that if $x \in H$ then $det_{Y}(x) = 1$. Then by Corollary 6.17 in [4] there is a $k(FH/F)$-module $U$ where $V \cong Y \otimes U$. It should be noted that $H$ acts faithfully on
So for all \( i \), \( \dim_k V = dim_k Y \). Thus \( \dim_k U = 1 \) and hence \( X = V_F \) establishing the claim.

(5) **\( V_M \) is homogeneous for every maximal \( FH \)-invariant subgroup \( M \) of \( F \).**

Pick a maximal \( FH \)-invariant subgroup \( M \) of \( F \). Then \( F/M \) is an elementary abelian group on which \( H \) acts irreducibly. Furthermore, the group \( (F/M)H \) is Frobenius as \( C_{F/M}(h) = 1 \) for every nonidentity element \( h \in H \). By (4), \( V_F \) is irreducible. By Clifford’s theorem there is a collection \( \{U_1, \ldots, U_s\} \) of homogeneous \( M \)-modules permuted transitively by \( F \) such that \( V_M = \bigoplus_{i=1}^s U_i \). On the other hand \( M \triangleleft FH \) and the components \( U_1, \ldots, U_s \) are permuted transitively also by \( FH \). Then by setting \( F_0 = Stab_F(U_1) \), we see that \( s = |F : F_0| = |FH : Stab_{FH}(U_1)| \). It follows now that \( s = \frac{|F| |H|}{|Stab_{FH}(U_1)|} = \frac{|F|}{|F_0|} \) whence

\[ |Stab_{FH}(U_1)| = |F_0| |H| \]. As \( |F_0|, |Stab_{FH}(U_1)/F_0| \) = 1, a complement \( H_0 \) of \( F_0 \) in \( Stab_{FH}(U_1) \) exists. Therefore without loss of generality we may assume that \( H \leq Stab_{FH}(U_1) \), that is, \( Stab_{FH}(U_1) = F_0 H \).

On the other hand, \( F_0/M \) is either trivial or equal to \( F/M \) due to the irreducible action of \( H \) on \( F/M \). If trivial, then \( F/M \) acts regularly on \( \{U_1, \ldots, U_s\} \). So for all \( i = 1, \ldots, s \), there is a unique \( \bar{x}_i \in F/M \) such that \( U_i = U_1^{\bar{x}_i} \). Then we have \( U_1^{\bar{x}_i h} = U_1^{\bar{x}_i h} \) for all \( h \in H \). This means that \( H \) acts regularly on \( \{U_2, \ldots, U_s\} \) and hence for any \( 0 \neq w \in U_2 \), the set \( \{w^h | h \in H\} \) forms a basis for a free \( kH \)-module. Thus we may assume that \( F_0 = F \). In particular \( U_1 \) is \( FH \)-invariant and hence \( V_M \) is homogeneous as claimed.

(6) **\( V_S \) is homogeneous for every \( FH \)-invariant subgroup \( S \) of \( F \).**

Let \( S \) be an \( FH \)-invariant subgroup of \( F \). Now by (4) \( V_F \) is irreducible. Then, by Clifford’s theorem we have \( V_S = \bigoplus_{i=1}^s U_i \) for a collection of \( S \)-homogeneous modules \( \{U_1, \ldots, U_s\} \) permuted transitively by \( F \). On the other hand \( S \triangleleft FH \) and the components \( U_1, \ldots, U_s \) are permuted transitively also by \( FH \). We set now \( F_0 = Stab_F(U_1) \). Then \( s = |F : F_0| = |FH : Stab_{FH}(U_1)| \) and hence we may assume by a similar argument as in the proof of claim (5) that \( H \) stabilizes \( U_1 \).
If \( s \neq 1 \), \( F_0 \) is contained in a maximal subgroup, say \( K \), of \( F \). However, every maximal subgroup and hence \( K \) is normal in \( F \) as \( F \) is nilpotent. In fact \( F_0 \) is \( H \)-invariant and hence \( F_0 \leq \cap_{h \in H} K^h \triangleleft FH \). Now \( \cap_{h \in H} K^h \) is contained in a maximal \( FH \)-invariant subgroup, say \( M \), of \( F \). It follows then by (5) that \( V_M \) is homogeneous, that is, \( V \cong X \oplus \cdots \oplus X \) for some irreducible \( M \)-module \( X \).

We consider the decomposition of \( X \) into its \( S \)-homogeneous components; more precisely we have \( X_S = Y_1 \oplus \cdots \oplus Y_r \) for \( S \)-homogeneous modules \( Y_1, \ldots, Y_r \) by Clifford’s theorem. Clearly \( r = s \) and \( Y_i = X \cap Y \) for each \( i = 1, \ldots, s \). Since \( M \) acts transitively on the set \( \{Y_1, \ldots, Y_s\} \), its action on the set \( \{U_1, \ldots, U_s\} \) has to be transitive also. So \( |M : Stab_M(U_1)| = s = |F : F_0| \). As \( F_0 \leq M \) we have the equality \( F = M \), which is a contradiction. Therefore \( s = 1 \), that is, \( V_S \) is homogeneous as claimed.

(7) \( F \) is extraspecial such that \( Z(F) \leq Z(FH) \), and the theorem follows.

Pick a characteristic abelian subgroup \( S \) of \( F \). By the above claim \( V_S \) is homogeneous and hence \( S \) is cyclic. Applying [[2], page 360, Aufgabe 33] to the action of \( H \) on \( F \) we see that the group \( F \) is either cyclic or extraspecial. Recall that \( F \) is nonabelian by (3). Then the group \( F \) is extraspecial as desired. As \( V_{Z(F)} \) is homogeneous we also see that \( Z(F) \leq Z(FH) \). Now Theorem B applied to the group \( FH \) on \( V \) shows that \( V_H \) contains the regular \( kH \)-module.

This completes the proof of the theorem. \( \square \)

Remark. Notice that if \( FH \) is not of odd order, then the theorem above is not true due to the following observation:

For a prime \( p \), let \( F \) be an extraspecial group of order \( p^{2m+1} \) and \( H \) be the cyclic group of order \( p^m + 1 \). There is a regular action of \( H \) on \( F/\Phi(F) \) so that \( \Phi(F), H \) \( = 1 \). Therefore \( FH \) is Frobenius-like. By [[2], V.17.13] there exists an irreducible and faithful \( kFH \)-module \( V \) over an algebraically closed field of characteristic coprime to the order of \( FH \) such that \( V_H \oplus U \cong kH \) where \( U \) is the irreducible trivial \( H \)-module. In particular \( V \) does not contain any submodule isomorphic to the regular \( H \)-module, more precisely \( CV(H) = 0 \).
3. Applications

**Corollary C.** Let $G$ be a finite solvable group acted on coprimely by a Frobenius-like group $FH$ of odd order so that $[G,F] \neq 1$. Then $C_G(H) \neq 1$.

**Proof.** We proceed by induction on the order of $G$. Then $F$ acts trivially on every proper $FH$-invariant subgroup of $G$ and hence $G$ is a $q$-group for some prime $q$. Theorem A applied to the action of $FH$ on $V = G/\Phi(G)$ gives the result. □

**Corollary D.** Let $P$ be a $p$-group acted on coprimely by a Frobenius-like group $FH$ of odd order so that $[P,F] = P$. Then

(i) the nilpotency class of $P$ is at most $2\log_p |C_P(H)|$,
(ii) $|P|$ is bounded in terms of $|F|$ and $|C_P(H)|$,
(iii) the rank of $P$ is bounded in terms of $|F|$ and the rank of $C_P(H)$.

**Proof.** (i) Notice that by theorem the group $H$ fixes a point in each $FH$-invariant section of $P$ on which $F$ acts nontrivially. Then the proof goes similarly as in the proof of Theorem 1(a) in [6].

(ii) It suffices to bound $|P/P'|$ in the required form since the nilpotency class of $P$ is bounded in terms of $|C_P(H)|$ by (i). We consider now a series

$$P_0 = P' \leq P_1 \leq P_2 \leq \ldots \leq P_m = P$$

of $FH$-invariant normal subgroups of $P$ such that $E_i = P_i/P_{i-1}$ is an irreducible $FH$-module for each $i = 1, \ldots, m$. Due to coprime action of $F$ on $P$ and the fact $[P,F] = P$ we see that $C_{E_i}(F) = 0$. Then Theorem A applied to the action of $FH$ on $E_i$ yields $C_{E_i}(H) \neq 0$ for every $i = 1, \ldots, m$. As $\dim E_i \leq |FH|$ we get $|E_i| \leq |C_{E_i}(H)||^{|FH|}$ and hence

$$|P/P'| = \prod_{i=1}^m |E_i| \leq (\prod_{i=1}^m |C_{E_i}(H)||^{|FH|} = (C_{P/P'}(H))^{|FH|}. $$

That is $|P/P'|$ is bounded in terms of $|F|$ and $|C_P(H)|$ since clearly $|H|$ is bounded in terms of $|F|$.

(iii) This can be proven in a similar fashion as in [6] with obvious changes that is using Corollary 4.1 instead of Lemma 1.2 in [6]. □
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**References**


