1. Introduction

Let $G$ be a finite group. For a prime number $p$, $H_p(G) = \langle x \in G | x^p \neq 1 \rangle$ is called the Hughes subgroup of $G$. It is well known that $[G : H_p(G)] = p$ and $H_p(G)$ is nilpotent if $G$ is not a $p$-group and $H_p(G)$ is properly contained in $G$. In this case $G = H_p(G) \langle \alpha \rangle$ for some element $\alpha$ of order $p$ and $\alpha$ induces on $H_p(G)$ a so called splitting automorphism of order $p$, that is, $\alpha$ acts on $H_p(G)$ so that $xx^\alpha x^{\alpha^2} \cdots x^{\alpha^{p-1}} = 1$ for any $x \in H_p(G)$.

The information about the structure of $H_p(G)$ is then a consequence of

Thompson-Hughes-Kegel Theorem. [7,9] If $G$ is a finite group admitting a splitting automorphism of prime order then $G$ is nilpotent.

This is clearly a generalization of a well known result due to Thompson [11] about finite groups admitting a fixed point free automorphism of prime order. In this sense the concept of a splitting automorphism can be considered as a generalization of fixed point free action. If $\alpha$ is an automorphism of prime order $p$ of the finite group $G$ then

$$(xx^\alpha x^{\alpha^2} \cdots x^{\alpha^{p-1}} = 1 \text{ for any } x \in G) \Leftrightarrow (|y| = p \text{ for any } y \in G \langle \alpha \rangle - G).$$

Therefore one can study extensions of the fixed point free action by putting conditions on the orders of elements outside a proper subgroup. The papers [1,2,3] of the authors are such examples. We call the action of an automorphism $\alpha$ of $G$ a Hughes type action if it is described by conditions on the orders of elements of $G \langle \alpha \rangle - G$.

In the present paper we study the structure of finite groups $G$ admitting an automorphism $\alpha$ of prime order $p$ so that the orders of elements in $G \langle \alpha \rangle - G$ are not divisible by $p^2$. This is of course a very weak condition as it is trivially satisfied in the case that $G$ is a $p'$-group independent of the action of $\alpha$ on $G$. So one cannot hope to get much information under such a condition. We have mainly proven the following:

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Let $G$ be a finite group admitting an automorphism $\alpha$ of prime order $p$ so that $x^N = 1$ for every element $x$ in $G(\alpha) = G$ where $N$ is a positive integer not divisible by $p^2$. Suppose that $G$ is $\pi$-separable where $\pi = \pi(N)$. Then $G$ is of $\pi'$-length 1 and has a nilpotent Hall $\pi'$-subgroup. Furthermore $p$ does not divide the index of $O_{\pi,\pi'}(G)$.

We would like to thank Jürgen Müller for pointing out the following example which shows that $\pi$-separability of $G$ in the hypothesis is indispensable:

Example 1.1. Let $G = Sz(8)$ be the smallest Suzuki group. Then $|G| = 2^6 \cdot 5 \cdot 7 \cdot 13$, and $G$ has an automorphism $\alpha$ of order 3. Set $H = G(\alpha)$. Since $|G|$ is not divisible by 3, there are no elements of order $3^2$ in $H$. Moreover, the elements of order 7 and 13 are self-centralizing in $G$, and each of them are distributed into 3 conjugacy classes which are fused under $\alpha$. Hence there cannot possibly be elements of order 21 or 39 in $H$. As there exist elements of order 15 in $H - G$ we can take $N = 2^6 \cdot 3 \cdot 5$ and hence $\pi = \{2, 3, 5\}$. It follows that $G$ is not $\pi$-separable.

2. Main Result

In this section we shall present a proof of the main result of this paper.

Theorem 2.1. Let $p$ be a prime number and $N$ be a positive integer not divisible by $p^2$. Let $G$ be a finite group admitting an automorphism $\alpha$ of prime order $p$, and $H$ denote the semidirect product of $G$ by $\langle \alpha \rangle$. Assume that $x^N = 1$ for every element $x$ in $H - G$. If $G$ is $\pi$-separable, where $\pi = \pi(N)$, then the following hold:

(a) $G$ has $\pi'$-length 1,

(b) $O_{\pi,\pi'}(G)/O_{\pi}(G)$ is nilpotent and $G/O_{\pi,\pi'}(G)$ is a $p'$-group,

(c) If $2 \not\in \pi$, then $[G, \alpha] = O_{\pi,\pi'}([G, \alpha]),$

(d) If $G$ is a solvable group and 4 does not divide $N$, then $G$ has $p$-length 1.

Proof. We should note first that the following three facts are immediate consequences of the Hughes type action of $\alpha$ on $G$ (see [1, 2, 3])

(1) If $x \in H - G$, for any $x$-invariant section $S$ of $G$, the exponent of $C_S(x)$ divides $N$.

(2) Every element $x$ in $H - G$ of order $p$ acts trivially or exceptionally on every elementary abelian $x$-invariant $p$-section $S$ of $G$, that is, the degree of the minimum polynomial of $x$ on $S$ is less than the order of the linear operator induced by $x$ on $S$. 
(3) For any subgroup $Q$ of $G$ with $N_H(Q) \not\subseteq G$, there exists $\beta \in N_H(Q) - G$ of order $p$ so that $N_H(Q) = N_G(Q)\langle \beta \rangle$.

Let now $G$ be a minimal counterexample to the Claim(a) of the theorem. We shall proceed towards a contradiction in a series of steps. First three of them can be easily deduced from the minimality of $G$.

(4) Every $\alpha$-invariant proper subgroup of $G$ and every factor group of $G$ by a nontrivial $\alpha$-invariant normal subgroup satisfy Claim(a).

(5) $O_\pi(G) = 1$ and $O^{\pi}(G) = G$.

(6) $G$ has a unique minimal normal $\alpha$-invariant subgroup $M$ which is contained in $O^\pi(G)$.

(7) $G = O_{\pi',\pi,\pi'}(G)$ and $G/O_{\pi',\pi}(G)$ is an elementary abelian $r$-group for some prime $r$ not in $\pi$, on which $\alpha$ acts fixed point freely and nontrivially.

By (4), we have the equality $G = O_{\pi',\pi,\pi'}(G)$. We may choose an $\alpha$-invariant Sylow $r$-subgroup $\overline{R} = R/O_{\pi',\pi}(G)$ of $G/O_{\pi',\pi}(G)$ by [6, 6.2.2]. Set $\overline{X} = \Omega_1(Z(\overline{R}))$. It follows by (4) that $G = O_{\pi',\pi}(G)X$ where $C_{\overline{X}}(\alpha) = 1$.

(8) $M = O_{\pi'}(G)$ is a self-centralizing elementary abelian $s$-group for some prime $s$, on which $\alpha$ acts fixed point freely and nontrivially.

We observe that $O_{\pi'}(G)$ is nilpotent as $\alpha$ acts fixed point freely on $O_{\pi'}(G)$ by a well known result due to Thompson [11, Theorem 1]. In fact, $O_{\pi'}(G)$ is an $s$-group for some prime $s$ by (6). We may also assume that it is elementary abelian. Set $\overline{G} = G/M$. By (4), Claim(a) of the theorem holds in the factor group $\overline{G}$ and hence we have the equality $\overline{G} = O_{\pi',\pi}(\overline{G})$. It follows by (5) that $\overline{G} = O_{\pi',\pi'}(\overline{G})$. Let now $H$ be a Hall $\pi$-subgroup of $G$. Then $O_{\pi'}(G)H = O_{\pi',\pi}(G)$ and hence $C_{O_{\pi'}(G)}(H) = Z(O_{\pi',\pi}(G)) = Z$ is a normal subgroup of $G$. If $Z \neq 1$, then $M \leq Z$ by (6). This leads to the contradiction that $H \leq C_G(O_{\pi'}(G)) \leq O_{\pi'}(G)$ since $G$ is $\pi$-separable. Therefore $Z = 1$, and hence $O_{\pi'}(G) = [O_{\pi'}(G), H]$ by [6, 5.3.5]. Set $K/M = O_{\pi'}(\overline{G})$. As $H \leq O_{\pi',\pi}(G) \cap K$, we have $[H,O_{\pi'}(G)] \leq M$. Thus $M = O_{\pi'}(G)$ and $C_G(M) = M$, as desired.

(9) $G = MQR$ where $Q$ and $R$ are $\alpha$-invariant Sylow $q$ and $r$-subgroups of $G$, respectively, for a prime $q$ in $\pi$.

Let $Q$ be a Sylow $q$-subgroup of $G$ for a prime $q$ in $\pi$. Then $Q \leq O_{\pi',\pi}(G)$ and hence $H = O_{\pi',\pi}(G)N_H(Q)$ by the Frattini argument. It follows by (3) that $N_H(Q) = N_G(Q)\langle \beta \rangle$ for some $\beta \in H - G$ of order $p$. Set $L = MN_G(Q)$. It is straightforward to verify that the action of $\beta$ on $L$ satisfies the hypothesis and hence $L$ is of $\pi'$-length 1 by induction. That is
\(L = O_{\pi,\pi'}(L)\). Notice that \(O_{\pi}(L) \leq C_L(M) = M\) and hence \(O_{\pi}(L) = 1\). Then \(L = O_{\pi,\pi'}(L)\).

Let now \(S\) be a Sylow \(r\)-subgroup of \(N_G(Q)\). It follows that \(G = O_{\pi',\pi'}(G)S\) and \(S \leq O_{\pi'}(L)\), whence \([S, Q] \leq Q \cap O_{\pi'}(L) = 1\). As a consequence, we get \(G = O_{\pi',\pi'}(G)C_G(Q)\) for each \(q \in \pi\) and for every Sylow \(q\)-subgroup \(Q\) of \(G\).

Set \(\overline{G} = G/M\) and pick a Sylow \(r\)-subgroup \(R\) of \(C_G(Q)\). Then \(G = O_{\pi',\pi'}(G)R\) by (7) and hence \(\overline{R}\) is a Sylow \(r\)-subgroup of \(\overline{G}\). In fact, \(\overline{G} = O_{\pi}(\overline{G})\overline{R} = O_{\pi}(\overline{G})C_{\overline{G}}(\overline{Q})\). Let \(\overline{Q}\) be a Sylow \(r\)-subgroup of \(C_{\overline{G}}(\overline{Q})\).

Then \(\overline{G} = O_{\pi}(\overline{G})\overline{RQ}\), implying that \(|\overline{RQ}| = |\overline{R}|\). Hence \(\overline{R}\) is a Sylow \(r\)-subgroup of \(C_{\overline{G}}(\overline{Q})\), and so \(\overline{Q} \leq C_{\overline{G}}(\overline{R})\). As \(q\) and \(Q\) are arbitrary, we see that \(|O_{\pi}(\overline{G})|\) divides \(|C_{\overline{G}}(\overline{R})|\), leading to the contradiction that \(\overline{G} = C_{\overline{G}}(\overline{R})\). Thus we have \(L = G\); more precisely, \(H = MN_H(Q)\).

Recall that \(Q\) is \(\beta\)-invariant for some \(\beta \in H - G\) of order \(p\) where \(N_H(Q) = N_G(Q)\langle \beta \rangle\), and that \(R\) is a Sylow \(r\)-subgroup of \(N_G(Q)\). Now \(K = N_G(Q)\langle \beta \rangle = N_G(Q)NR(R)\) by the Frattini argument. Then, by (3), there exists \(\gamma \in N_G(Q)\langle \beta \rangle - N_G(Q)\) of order \(p\) so that \(R\) is \(\gamma\)-invariant and hence \(Q\) is also \(\gamma\)-invariant. Without loss of generality we may assume that \(Q\) and \(R\) are both \(\alpha\)-invariant. By the minimality of \(G\), it follows that \(G = MQR\), as desired.

\((10)\) Claim(a) follows.

We observe first that \(s = r\), because otherwise \([M, R] = 1\) as \(C_{MR}(\alpha) = 1\). In case where \(G\) is a \(p^\prime\)-group we obtain \(C_M(\alpha) \neq 1\) by [12, Theorem 2.1.A] applied to the action of \(QR(\alpha)\) on \(M\), which is not the case. Thus \(|G|\) is divisible by \(p\), whence \(q = p\). Applying [4, Lemma 1] to the action of \(QR(\alpha)\) on \(M\) we obtain again \(C_M(\alpha) \neq 1\). This contradiction completes the proof of Claim(a). As a consequence we have \(G = O_{\pi,\pi'}(G)\).

As \(\alpha\) acts fixed point freely on \(O_{\pi,\pi'}(G)/O_{\pi}(G)\), it is a nilpotent group, and hence the first part of Claim (b) follows. We prove next that \(G/O_{\pi,\pi'}(G)\) is a \(p^\prime\)-group: Set \(\overline{G} = G/O_{\pi}(G)\). Let \(P\) be a Sylow \(p\)-subgroup of \(H\) containing \(\alpha\). Then \(P \cap G\) is an \(\alpha\)-invariant Sylow \(p\)-subgroup of \(G\). Pick now a nontrivial element \(x\) from \(C_{\overline{P}/\overline{G}}(\alpha)\) of order \(p\) and form the elementary abelian \(p\)-group \(X = \langle x, \alpha \rangle\). Due to the fact that \(C_{\overline{G}}(O_{\pi'}(\overline{G})) \leq O_{\pi'}(\overline{G})\) we can choose an \(\alpha\)-invariant Sylow \(s\)-subgroup \(V\) of the Frattini factor group of \(O_{\pi'}(\overline{G})\) for a prime \(s\) not in \(\pi\) so that \([V, x] \neq 1\). It follows by [6, 5.3.16] that \(V = \langle C_V(t) \mid 1 \neq t \in X \rangle\). On the other hand we have \(C_V(t) = 1\) for each \(t \in X - \langle x \rangle\) by (1). This contradiction completes the proof of Claim(b).

We are now ready to establish Claim(c). Suppose that \(2 \notin \pi\). The proof of the equality \([G, \alpha] = O_{\pi,\pi'}([G, \alpha])\) goes by induction on \(|G|\). We may assume that \([G, \alpha] = O_{\pi,\pi'}([G, \alpha])\). By similar reduction arguments as in the proof of Claim (a), it can be shown that \([G, \alpha] = MQ\) where \(M\) is an elementary abelian \(\alpha\)-invariant \(s\)-group for some prime \(s\) which is not in \(\pi\),
and $Q$ is an $\alpha$-invariant $q$-group for and odd prime $q$ in $\pi$ so that $[Q, \alpha] = Q$ and $[M, Q] = M$. As $C_M(\alpha) = 1$, it follows by [5, Lemma 2.2] that $Q$ is a 2-group, which is impossible. Hence we have $[G, \alpha] = O_{\pi, m'\{[G, \alpha]\}}$ as claimed.

Finally we observe that Claim(d) holds: Recall that $G/O_{\pi, m'}(G)$ is a $p'$-group by Claim(b). Hence it suffices to show that $O_{\pi}(G)$ has $p$-length 1. Assume the contrary. As $O_{\pi}(G)$ is solvable in this case, there exist subgroups $S, T$ and $U$ of $O_{\pi}(G)$ and an element $\beta \in H - G$ of order $p$ by (3) satisfying the following (see also [3, Lemma 1]):

(i) $S$ and $U$ are $p$-groups, $T$ is a $q$-group for some prime $q$ in $\pi - \{p\}$;
(ii) $\langle \beta \rangle$ normalizes $S, T$ and $U$; $U$ normalizes $S$ and $T$ and $T$ normalizes $S$;
(iii) $STU$ is a group of nilpotent length 3.

We may also assume that $T = T/C_T(S)$ is a special $q$-group so that $U/\langle \beta \rangle$ acts irreducibly on its Frattini factor group, $TU/\langle \beta \rangle$ acts irreducibly on $S/\Phi(S)$ and $U/C_T(T)$ is an elementary abelian $p$-group on which $\langle \beta \rangle$ acts irreducibly. It is straightforward to verify that $\beta$ is trivial on $U/C_T(T)$. It should also be noted that $[S/\Phi(S), C_T(T)] = 1$ because $C_{S/\Phi(S)}(C_T(T)) \neq 1$ as $S$ and $U$ are both $p$-groups. Thus we have $C_{U(T)} = C_{C_T(T)}(S/\Phi(S))$.

To simplify the notation we set $X = STU/\Phi(S)C_T(S)C_{U(T)}$. If $[T, \beta] = 1$, then $[S/\phi(S), \beta] = 1$ holds, and so the group $X$ is centralized by $\beta$. It follows by (1) that a Sylow $p$-subgroup of $X$ has exponent $p$ and hence $X$ has $p$-length 1 by [8, IX.4.3]. This contradiction shows that $[T, \beta] = T$. Applying now [5, Lemma 2.2] to the action of $T/\langle \beta \rangle$ on $S/\Phi(S)$, we see that $T$ is a nonabelian special group. Using [5] and [8, IX.3.2] and with some effort one gets $q = 2$. Set $Y = U/C_{U(T)}(\beta)$. Using [6, 5.3.16] we see that $T/\phi(T) = \langle C_{T/\phi(T)}(\alpha) \mid 1 \neq \alpha \in Y \rangle$. Then there exists $\alpha \in Y(\beta) - Y$ such that $[T/\phi(T), \alpha] = 1$. By the hypothesis of Claim(d), we see that $T/\phi(T)$ is of exponent 2 and hence abelian. This contradiction completes the proof.

\[\square\]

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