ON THE INFLUENCE OF FIXED POINT FREE NILPOTENT AUTOMORPHISM GROUPS

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ABSTRACT. A finite group FH is said to be Frobenius-like if it has a nontrivial nilpotent normal subgroup F with a nontrivial complement H such that $[F, h] = F$ for all nonidentity elements $h \in H$. Let FH be a Frobenius-like group with complement H of prime order such that $C_F(H)$ is of prime order. Suppose that FH acts on a finite group G by automorphisms where $(|G|, |H|) = 1$ in such a way that $C_G(F) = 1$. In the present paper we prove that the Fitting series of $C_G(H)$ coincides with the intersections of $C_G(H)$ with the Fitting series of G, and the nilpotent length of G exceeds the nilpotent length of $C_G(H)$ by at most one. As a corollary, we also prove that for any set of primes π, the upper π-series of $C_G(H)$ coincides with the intersections of $C_G(H)$ with the upper π-series of G, and the π- length of G exceeds the π-length of $C_G(H)$ by at most one.

1. INTRODUCTION

All groups mentioned are assumed to be finite. Let G be a group. A subgroup A of $AutG$ is said to be *fixed point free* if the only element of G fixed by every element of A is the identity, that is, $C_G(A) = \{g \in G \mid g^a = g \text{ for all } a \in A\} = 1$. By a celebrated theorem due to Thompson, the group G is nilpotent in case where A is of prime order. This result is known as the starting point of the research on the structure of groups admitting a fixed point free group of automorphisms. A long-standing conjecture which has been extensively studied over the years states that the nilpotent length of a group G admitting a fixed point free automorphism group A such that $(|G|, |A|) = 1$ is bounded above by the length of the longest chain of subgroups of A. Turull settled the conjecture for almost all A. [16] contains a detailed survey of the problem and a complete list of related papers then actual. When A acts fixed point freely and noncoprimely, a result of Bell and Hartley [2] shows that this conjecture is not true if A is a nonnilpotent group. Therefore one is naturally led to impose the restriction that A is nilpotent. However, the noncoprime problem has turned out to be a very difficult question due to the lack of nice techniques which are valid in the coprime case.

Within the past few years some authors (see [11], [12], [13], [14], [15]) studied a similar problem which is not directly related to the above conjecture, but involves the fixed point free action of a nilpotent group. More precisely they investigated the structure of groups admitting Frobenius groups of automorphisms with fixed point free kernel. Generalizing these in a sequence of papers ([5], [6], [7], [8], [9]) we

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studied the action of Frobenius-like groups with fixed point free kernel under some additional assumptions. (Recall that a finite group FH is said to be Frobenius-like if it has a nontrivial nilpotent normal subgroup F with a nontrivial complement H such that $[F, h] = F$ for all nonidentity elements $h \in H$.)

In the present paper we will be calling attention not to all conclusions which can be derived but only to the one that the Fitting series of $C_G(H)$ coincides with the intersections of $C_G(H)$ with the Fitting series of G. In [15] (see also [12]) Khukhro obtained this conclusion under the hypothesis that FH is a Frobenius group with fixed point free kernel F. Later in [5] we extended his result to the case where the group FH is a Frobenius-like group with fixed point free kernel F under the additional hypothesis that $[F, F]$ is of prime order and is centralized by H. In [3] Collins and Flavell has resolved the special case for which F is an extra-special group with automorphism group H of prime order fixing $[F, F]$ elementwise. Recently a theorem of similar nature with the same conclusion is proved by de Melo in [10] by assuming that the group FH has normal abelian subgroup F which has a unique subgroup of order p so that every element in FH outside F is of order p for a prime p.

Our goal in this article is to study the case where FH is a Frobenius-like group with complement H of prime order which is coprime to the order of G under the hypotheses that $C_F(H)$ is of prime order. We mainly prove the following:

Theorem *Let FH be a Frobenius-like group with kernel F and complement H of order p for a prime p where $C_F(H)$ is of prime order. Suppose that FH acts on a p'-group G via automorphisms in such a way that $C_G(F) = 1$. Then*
(i) the Fitting series of $C_G(H)$ coincides with the intersections of $C_G(H)$ with the Fitting series of G;
(ii) the nilpotent length of G exceeds the nilpotent length of $C_G(H)$ by at most one; and the equality holds if the group FH is of odd order.

We would like to call attention to the Example in [6] which shows that we are required to assume that $C_F(H)$ is of prime order. It should be noted that the present paper extends [3] to a more general context such as Frobenius-like groups without the restriction that $C_F(H) = [F, F]$. It also generalizes our first result [5] in this context as well by replacing the condition that $[F, F]$ is of prime order by $C_F(H)$ is of prime order at least in case H is of prime order.

It is also obtained as a corollary of the theorem above that for any set of primes π, the π-length of G may exceed the π-length of $C_G(H)$ by at most one, and the upper π-series of $C_G(H)$ coincides with the intersections of $C_G(H)$ with the upper π-series of G. More precisely we prove

Corollary *Let FH be a Frobenius-like group with kernel F and complement H of order p for a prime p where $C_F(H)$ is of prime order. Suppose that FH acts on a p'-group G via automorphisms in such a way that $C_G(F) = 1$. Then we have*
(i) $O_\pi(C_G(H)) = O_\pi(G) \cap C_G(H)$ for any set of primes π;
(ii) the π-length of G may exceed the π-length of $C_G(H)$ by at most one, and the equality holds if FH is of odd order;

(iii) $O_{\pi_1,\pi_2,\ldots,\pi_k}(C_G(H)) = O_{\pi_1,\pi_2,\ldots,\pi_k}(G) \cap C_G(H)$ *where* π_i *is a set of primes for each* $i = 1, \ldots, k$.

The notation and terminology are standard with few exceptions.

2. THE KEY PROPOSITION AND ITS PROOF

This section is devoted to the proof of the following proposition from which our theorem is deduced.

Proposition 2.1. *Let* FH *be a Frobenius-like group with kernel* F *and complement* $H = \langle h \rangle$ *of order* p *for a prime* p. *Suppose that* $C_F(H)$ *is of prime order. Let* FH *act on a* q*-group* Q *for some prime* $q \neq p$. *If* V *is a* $kQFH$*-module for a field* k *of characteristic not dividing* q *such that* F *acts fixed point freely on the semidirect product* VQ *then we have* $Ker(C_Q(H)\, on\, C_V(H)) = Ker(C_Q(H)\, on\, V)$.

Proof. Here we use alternative notation for the kernel of an action of a group A by automorphisms on a group B denoting $Ker(A\, on\, B) := C_A(B)$ in order to avoid cumbersome subscripts. We shall proceed over several steps. Set $K = Ker(C_Q(H)\, on\, C_V(H))$.

(1) We may assume that $char k \neq p$.

Proof. Suppose that $char k = p$. Then $q \neq p$. Set $A = K$ and $B = H$. Applying Thompson $A \times B$-lemma to the action of $A \times B$ on V, we get the result. Therefore we may assume that $char k \neq p$. ☐

(2) We may assume that k *is a splitting field for all subgroups of* QFH.

Proof. We consider the QFH-module $\bar{V} = V \otimes_k \bar{k}$ where $\bar{k}$ is the algebraic closure of k. Notice that $dim_k V = dim_{\bar{k}} \bar{V}$ and $C_{\bar{V}}(H) = C_V(H) \otimes_k \bar{k}$. Therefore once the proposition has been proven for the group QFH on $\bar{V}$, it becomes true for QFH on V also. ☐

Suppose that the proposition is false and choose a counterexample with minimum $\dim_k V + |QFH|$. To ease the notation we set $K = Ker(C_Q(H)\, on\, C_V(H))$.

(3) Q *acts faithfully on* V.

Proof. We set $\overline{Q} = Q/Ker(Q\, on\, V)$ and consider the action of the group $\overline{Q}FH$ on V assuming $Ker(Q\, on\, V) \neq 1$. An induction argument gives $Ker(C_{\overline{Q}}(H)\, on\, C_V(H)) = Ker(C_{\overline{Q}}(H)\, on\, V)$. This leads to a contradiction as $C_{\overline{Q}}(H) \geqslant \overline{C_Q(H)}$. Thus we may assume that Q acts faithfully on V. ☐

(4) V *is an irreducible* QFH*-module.*

Proof. As $char$(k) is coprime to the order of Q and $K \neq 1$, there is a QFH-composition factor W of V on which K acts nontrivially. If $W \neq V$, then the proposition is true for the group QFH on W by induction. That is,

$$Ker(C_Q(H)\, on\, C_W(H)) = Ker(C_Q(H)\, on\, W)$$

and hence

$$K = Ker(K\, on\, C_W(H)) = Ker(K\, on\, W)$$

as $char k \neq q$. This contradicts the fact that K acts nontrivially on W. Hence $V = W$. $\square$

By Clifford's theorem the restriction of the QFH-module V to the normal subgroup Q is a direct sum of Q-homogeneous components. Let Ω denote the set of Q-homogeneous components of V.

(5) K acts trivially on the sum of components in any regular $|H|$-orbit in Ω.

Proof. Let W be an element in Ω such that $\{W^y : y \in H\}$ is a regular $|H|$-orbit in Ω and let X be the sum of components. Then K acts trivially on $C_X(H) = \left\{\sum_{y\in H} v^y : v \in W\right\}$ and hence trivially on X. $\square$

(6) F acts transitively on Ω and H fixes an element of Ω.

Proof. By (5) it is not possible that every H-orbit in Ω is regular. So there exists $W \in \Omega$ such that $Stab_H(W) \neq 1$. In this case we have $Stab_H(W) = H$. Let now Ω_1 be the F-orbit on Ω containing W. Then Ω_1 is stabilized by FH. As FH acts transitively on Ω we see that $\Omega = \Omega_1$ and hence F acts transitively on Ω. $\square$

From now on W will denote an H-invariant element in Ω the existence of which is established by (6). It should be noted that the group $Z(Q/Ker(Q\, on\, W))$ acts by scalars on the homogeneous Q-module W, and so $[Z(Q), F_1H] \leqslant Ker(Q\, on\, W)$ where $F_1 = Stab_F(W)$ as W is stabilized by H.

Let T be a transversal for F_1 in F. Then $F = \bigcup_{t\in T} F_1 t$ and so $V = \bigoplus_{t\in T} W^t$. An H-orbit on $\Omega = \{W^t : t \in T\}$ is of length 1 or p. Let $\{W^{t_1}, \dots, W^{t_s}\}$ with $t_1 = 1$ be the set of all H-invariant elements of Ω and set $U = \bigoplus_{i=1}^{s} W^{t_i}$. Now $V = U \oplus Y$ where Y is the sum of the components of all regular H-orbits on Ω. By (5) K acts trivially on Y. Set $L = K \cap Z(C_Q(H))$. Since $1 \neq K \trianglelefteq C_Q(H)$, the group L is nontrivial. Then there exists $1 \neq z \in L$ acting nontrivially on at least one H-invariant element of Ω. Without loss of generality we may assume that z acts nontrivially on W.

(7) We may assume that $T \cap C_F(H) = \{t_1, \dots, t_s\}$. Then $s = |C_F(H) : C_{F_1}(H)|$. Now $s = 1$ if and only if $C_F(H) \leqslant F_1$. We also observe that $K^x \leqslant C_Q(U)$ for every $x \in F - F_2$ where $F_2 = Stab_F(U)$.

Proof. Notice that $W^{t_i h} = W^{t_i}$ implies $[t_i, h] \in F_1$ for any $i \in \{1, \dots, s\}$. That is, $t_i F_1$ is a coset of F_1 in F which is fixed by H. Since the orders of F and H are coprime we may choose $t_i \in C_F(H)$. Conversely we see that for each $t \in C_F(H)$, W^t is H-invariant. Hence we may assume that $T \cap C_F(H) = \{t_1, \dots, t_s\}$. Then $s = |C_F(H) : C_{F_1}(H)|$. Notice also that for every $x \in F - F_2$ and for every $i = 1, \dots, s$, $W^{t_i x} \in Y$ and hence $K^{x^{-1}} \leqslant C_Q(W^{t^i})$ for every $i = 1, \dots, s$ by (5). This means that $K^x \in C_Q(U)$ for every $x \in F - F_2$. $\square$

(8) $F_1 C_F(H) = F_2$.

Proof. By (7), $C_F(H)$ acts transitively on the set of fixed points of H on Ω and hence $C_F(H) \leqslant F_2$. Clearly we also have $F_1 \leqslant F_2$. Therefore $F_2 = F_1 C_F(H)$. $\square$

(9) $Q = \langle z^F \rangle$ is abelian with $[Q, F_1H] \leqslant C_Q(U)$. Furthermore we observe that $F_2 \neq F_1$.

Proof. Clearly $Q = \langle z^F \rangle$ by induction. By (7) we have $Q = \langle z^{F_2} \rangle C_Q(U)$. Set $\bar{Q} = Q/C_Q(U)$. Suppose first that $C_F(H) \neq 1$. We observe that $[\bar{L}, H, Z_2(\bar{Q})] = 1$. Due to the scalar action of also $Z(\bar{Q})$ on each W^{t_i} for each $i = 1, \ldots, s$ we also have $[\bar{L}, Z_2(\bar{Q}), H] \leqslant [Z(\bar{Q}), H] = 1$. It follows by the three subgroups lemma that $[Z_2(\bar{Q}), H, \bar{L}] = 1$. Notice that $Z_2(\bar{Q}) = [Z_2(\bar{Q}), H]C_{Z_2(\bar{Q})}(H)$ as $q \neq p$. Since $\bar{L} \leqslant Z(C_{\bar{Q}}(H))$ we get $[\bar{L}, Z_2(\bar{Q})] = 1$ whence $[\bar{Q}, Z_2(\bar{Q})] = 1$. That is, $\bar{Q}$ is abelian. Now $Q' \leqslant C_Q(U)$ implies $Q' \leqslant C_Q(V) = 1$. Therefore Q is abelian as claimed. Hence $Q/C_Q(W)$ acts by scalars on W and so $[Q, F_1H] \leqslant C_Q(W)$. Since $|F_2 : F_1|$ is at most a prime, $F_1 \lhd F_2$ whence $[Q, F_1H] \leqslant C_Q(U)$. Set $X = F_{q'}$. As $C_Q(F) = 1$ we have

$$1 = \prod_{f \in X} z^f = (\prod_{f \in X - F_1} z^f)(\prod_{f \in X \cap F_1} z^f) \equiv (\prod_{f \in X - F_1} z^f)(z^{|X \cap F_1|})C_Q(U).$$

In case $F_1 = F_2$ we have $\prod_{f \in X - F_1} z^f \in C_Q(U)$ by (7) and hence $z^{|X \cap F_1|} \in C_Q(U)$. This leads to the contradiction that $z \in C_Q(U)$. Therefore $F_1 \neq F_2$ as claimed. $\square$

(10) Final contradiction.

Proof. By (8) and (9) we have $C_F(H) \nleqslant F_1$. Then $F_1 \cap C_F(H) = 1$ whence the group F_1H is Frobenius. It follows now by Lemma 1.3 in [15] that $C_W(H) \neq 0$. On the other hand $KC_Q(W)/C_Q(W)$ acts by scalars and nontrivially on W and hence $C_W(H) = 0$. This contradiction completes the proof. $\square$

3. Proof of Theorem

In this section we present a proof of the theorem. We firstly gather together some certain facts which will be particularly useful.

Lemma 3.1. *Suppose that a Frobenius-like group FH acts on the finite group G by automorphisms so that $C_G(F) = 1$. Then the following hold:*

(i) There is a unique FH-invariant Sylow p-subgroup of G for each prime p dividing the order of G.
(ii) $C_{G/N}(F) = 1$ for every FH-invariant subgroup N of G.

Proof. The proof of Lemm 2.2 and Lemma 2.6 in [13] applies also to this statement. $\square$

Proof of Theorem We already know by [1] that G is solvable due to the nilpotency of F and the assumption $C_G(F) = 1$.

Firstly we will prove that the equality $F(C_G(H)) = F(G) \cap C_G(H)$ is true under the hypothesis of the theorem. It is straightforward to verify that $F(G) \cap C_G(H) \leqslant F(C_G(H))$. To prove the reversed inclusion $F(C_G(H)) \leqslant F(G)$ we shall proceed by induction on the order of G. Consider now the nontrivial group $\overline{G} = G/F(G)$. By Lemma 3.1 (ii) above $C_{\overline{G}}(F)$ is trivial. Then, an induction argument yields that $F(C_{\overline{G}}(H)) \leqslant F(\overline{G}) = \overline{F_2(G)}$ whence $F(C_G(H)) \leqslant F_2(G)$. Notice that $\overline{C_G(H)} = C_{\overline{G}}(H)$ since G is a p'-group. If $F_2(G) \neq G$, another induction argument applied to the action of FH on $F_2(G)$ implies that $F(C_G(H)) = F(C_{F_2(G)}(H)) \leqslant F(F_2(G)) = F(G)$. Thus we may assume that $F_2(G) = G$. It is clear that there exist distinct primes r and q such that $[O_q(C_G(H)), O_r(G)]$ is nontrivial. The group $O_{r,q}(G/O_{r'}(G))$ is a counterexample, whence $F(G) = O_r(G)$ and $\overline{G}$ is a q-group. By Lemma 3.1 (i) there is a unique FH-invariant Sylow q-subgroup Q of G. Then $\overline{G} = \overline{Q}$, that is $G = F(G)Q$. Note that $C_Q(H)$ is nontrivial.

On the other hand, applying the above Proposition to the action of the group QFH on $V = F(G)/\Phi(G)$ we get

$$Ker(C_Q(H)\, on\, C_V(H)) = Ker(C_Q(H)\, on\, V) = 1$$

establishing the desired equality.

To prove (i) is equivalent to showing that $F_k(C_G(H)) = F_k(G) \cap C_G(H)$ for each natural number k. This is true for $k = 1$ by the preceding paragraph. Assume that $F_k(C_G(H)) = F_k(G) \cap C_G(H)$ holds for a fixed but arbitrary $k > 1$. Due to the coprime action of H on G we have $C_{G/F_k(G)}(H) = C_G(H)F_k(G)/F_k(G)$ and hence

$F_{k+1}(C_G(H))F_k(G)/F_k(G) \leqslant F(C_{G/F_k(G)}(H)) \leqslant F(G/F_k(G))$,

This forces $F_{k+1}(C_G(H)) \leqslant F_{k+1}(G) \cap C_G(H)$, as desired.

Let now n denote the nilpotent length of $C_G(H)$. Then $C_G(H) = F_n(C_G(H)) \leqslant F_n(G)$ whence H acts fixed point freely on $G/F_n(G)$ by the coprime action of H on G. It follows that the nilpotent length of G exceeds the nilpotent length of $C_G(H)$ by at most one as claimed. Notice that if FH is of odd order then $C_{G/F_n(G)}(H)$ is nontrivial by in Theorem A in [4], that is, $C_G(H)$ is not contained in $F_n(G)$. Therefore the nilpotent length of G is equal to the nilpotent length of $C_G(H)$ when FH is of odd order. $\square$

Proof of Corollary It can be proven using the same argument as in the proof of Corollary 4.1 of [15] and in the proof of the theorem above. $\square$ $\square$

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