FIXED-POINT FREE ACTION OF AN ABELIAN GROUP OF ODD NON-SQUAREFREE EXPONENT

GÜLİN ERCAN\textsuperscript{1}, İSMAIL Ş. GÜLOĞLU\textsuperscript{2} AND ÖZNUR MUT SAĞDIÇOĞLU\textsuperscript{1}

\textsuperscript{1}Department of Mathematics, Middle East Technical University, Ankara 06531, Turkey (ercan@metu.edu.tr; oznur@uekae.tubitak.gov.tr)
\textsuperscript{2}Department of Mathematics, Doğuş University, Istanbul, Turkey (iguloglu@dogus.edu.tr)

(Received 4 May 2009)

Abstract
Let $A$ be a finite group acting fixed-point freely on a finite (solvable) group $G$. A longstanding conjecture is that if $(|G|, |A|) = 1$, then the Fitting length of $G$ is bounded by the length of the longest chain of subgroups of $A$. It is expected that the conjecture is true when the coprimeness condition is replaced by the assumption that $A$ is nilpotent. We establish the conjecture without the coprimeness condition in the case where $A$ is an abelian group whose order is a product of three odd primes and where the Sylow 2-subgroups of $G$ are abelian.

Keywords: automorphisms of solvable groups; non-coprime action; fixed-point free action; Carter subgroup

2010 Mathematics subject classification: Primary 20D10
Secondary 20D15; 20D30; 20D45

1. Introduction

A well-known conjecture of Thompson states that if $A$ is a finite group acting fixed-point freely on a finite solvable group $G$ of order coprime to $|A|$, then the Fitting length $f(G)$ of $G$ is bounded by $\ell(A)$, the length of the longest chain of subgroups of $A$. By an elegant result due to Bell and Hartley [1], it is known that any finite non-nilpotent group $A$ can act fixed-point freely on a solvable group $G$ of arbitrarily large Fitting length with $(|G|, |A|) \neq 1$. If $A$ is nilpotent, it is expected that the conjecture is still true without the coprimeness condition. This question is settled for cyclic groups $A$ of order $pq$ [2] and $pqr$ [4] for pairwise distinct primes $p$, $q$ and $r$. We also establish the conjecture without the coprimeness condition in the case where $A$ is abelian of squarefree exponent coprime to 6 and $|G|$ is odd [5]. The main theorems of [2] and [4] have the advantage of providing an answer in the absence of any restriction on the nature of the primes involved. In comparison with the former results mentioned above, the novelty of the present article consists in the possibility of allowing non-squarefree exponents. To be precise, the main result of this paper is the following theorem, which settles the conjecture without the
coprimeness condition when $A$ is an abelian group whose order is a product of three odd primes that are not necessarily distinct under some additional hypothesis on $G$.

**Theorem C.** Let $G \triangleleft GA$, where $A$ is an abelian group whose order is a product of three odd primes. Assume that Sylow 2-subgroups of $G$ are abelian. If $A$ acts fixed-point freely on $G$, then $f(G) \leq 3$.

This statement is new only in the case $|A| = p^2q$ for distinct primes $p$ and $q$, since the case $|A| = pqr$ is treated in [4] and if $|A| = p^3$ for a prime $p$, the action is coprime and the result is well known. When $|A|$ is divisible by the primes 2 or 3, the study of such problems needs much more effort. Our Theorem C allows $|A|$ to be odd and divisible by 3, in contrast to the assumption in [5] that $|A|$ is coprime to 6.

If $A$ is a finite nilpotent group acting fixed-point freely on a finite group $G$, then $A$ is a self-normalizing nilpotent subgroup of the semidirect product $GA$ of $G$ by $A$: that is, it is a Carter subgroup of $GA$. This observation allows us to exploit the results and methods of Dade’s fundamental paper [3]. However, by employing the ideas of both [3] and [9], we have been able to obtain Theorem A in [5]. In the present paper, we take this result further by slightly modifying the techniques used in [5]; namely, we obtain the following theorem.

**Theorem A.** Let $A$ be an abelian group acting fixed-point freely on a group $G$ whose Sylow 2-subgroups are abelian. If $A$ has a squarefree exponent coprime to 6, then $f(G) \leq \ell(A)$.

**Theorem B.** Let $A$ be an abelian group acting fixed-point freely on a group $G$. If $A$ has squarefree odd exponent coprime to all Fermat primes, then $f(G) \leq \ell(A)$.

Theorem A is a generalization of Theorem A in [5] that replaces the hypothesis that $|G|$ is odd by the weaker one that the Sylow 2-subgroups of $G$ are abelian. While this is a theorem that was designated as a partial special case in the survey article [10] by Turull, Theorem B is a full special case theorem in the sense of the same article.*

All the groups considered in this paper are finite and solvable. The notation and terminology are as in [5].

2. Proofs of Theorems A and B

The proofs of Theorems A and B depend on a few technical results presented below.

**Theorem 2.1.** Let $S(\alpha)$ be a group such that $S \triangleleft S(\alpha)$, $S$ is an $s$-group with $\Phi(\Phi(S)) = 1$, $\Phi(S) \leq Z(S)$ and $\langle \alpha \rangle$ is cyclic of order $p$ for primes $s$ and $p$. Assume that either $s = p \geq 5$ or $s \neq p$, $p$ is odd. Assume further that if $s = 2$, either $S$ is abelian or $p$ is not a Fermat prime. Let $V$ be a $kS(\alpha)$-module, where $k$ is a field of characteristic not dividing $ps$ such that $[S, \alpha]^{p-1}$ acts non-trivially on each irreducible submodule of $V|_S$. Let $\Omega$ be an $S(\alpha)$-stable subset of $V^*$ that linearly spans $V^*$ and set $V_0 = \bigcap\{\text{Ker} f \ |

* We are indebted to the anonymous referee who suggested that we include Theorem B.
$f \in \Omega - C_\Omega(\alpha)$. Then $C_V(\alpha) \nsubseteq V_0$ and $(C_D(\alpha) \text{ on } C_V(\alpha)/C_{V_0}(\alpha)) \equiv_w (C_D(\alpha) \text{ on } V)$, where

$$D = \begin{cases} [S, \alpha]^{p-1} & \text{when } s = p, \\ S & \text{otherwise}. \end{cases}$$

**Proof.** One may observe that [5, Theorem 1] can be amended using the weakened conditions ‘either $s = p \geq 5$ or $s \neq p$, $p$ is any odd prime’ and ‘if $s = 2$, either $S$ is abelian or $p$ is not a Fermat prime’. 

**Theorem 2.2.** Let $S(\alpha)$ be a group such that $S \triangleleft S(\alpha)$, $S$ is an $s$-group, $\langle \alpha \rangle$ is cyclic of order $p$ for distinct primes $s$ and $p$, $\Phi(\Phi(S)) = 1$ and $\Phi(S) \leq Z(S)$. Assume that if $s = 2$, either $S$ is abelian or $p$ is not a Fermat prime. Let $V$ be an irreducible $kS(\alpha)$-module on which $[S, \alpha]$ acts non-trivially, where $k$ is a field of characteristic different from $s$. Then $[V, \alpha]^{p-1} \neq 0$ and $(C_S(\alpha) \text{ on } V) \equiv_w (C_S(\alpha) \text{ on } [V, \alpha]^{p-1})$.

**Proof.** One may observe that the conclusion of [5, Theorem 2] is true under the assumption that ‘if $s = 2$, either $S$ is abelian or $p$ is not a Fermat prime’. 

**Theorem 2.3.** Let $G \triangleleft GA$, let $\langle a \rangle \triangleleft A$ be of prime order $p$ for distinct primes $s$ and $p$, $\Phi(\Phi(S)) = 1$ and $\Phi(S) \leq Z(S)$. Assume that if $p_i = 2$, either $P_i$ is abelian or $p$ is not a Fermat prime. There are then sections $D_{i_0}, \ldots, D_i$ of $P_{i_0}, \ldots, P_i$, respectively, forming an $A$-Fitting chain of $G$ such that $a$ centralizes each $D_j$ for $j = i_0, \ldots, i$ where

$$i_0 = \begin{cases} 2 & \text{if } p_1 \neq p, \\ 3 & \text{if } p_1 = p. \end{cases}$$

**Proof.** It will be sufficient to demonstrate Claim 1 and Claim 2 appearing in the proof of [5, Theorem 3] with the hypothesis as revised above. One can observe that these claims can be restated and proven as by-products of Theorems 2.1 and 2.2, which are slightly altered versions of Theorems 1 and 2 in [5], by an analogous reasoning.

One may easily check that Theorems A and B below can be proven by essentially the same reasoning as in the proof of Theorem A in [5], with the only difference being the replacement of Theorem 3 in [5] by Theorem 2.3.

**Theorem A.** Let $A$ be an abelian group acting fixed-point freely on a group $G$ whose Sylow 2-subgroups are abelian. If $A$ has a squarefree exponent coprime to 6, then $f(G) \leq \ell(A)$.

**Theorem B.** Let $A$ be an abelian group acting fixed-point freely on a group $G$. If $A$ has squarefree odd exponent coprime to all Fermat primes, then $f(G) \leq \ell(A)$.
3. Concerning the presence of the factor 3

In this section we present two results that allow $|A|$ to be divisible by 3 in Theorem C. The following result, which is essentially due to Dade [3], is offered here for the sake of completeness in a formulation and with a notation that is suitable for our purposes.

**Theorem 3.1.** Let $S(\alpha)$ be a 3-group where $S < S(\alpha)$, $\Phi(S) \leq Z(S)$, $\Phi(\Phi(S)) = 1$ and $(\alpha)$ is cyclic of order 3. Assume that there exists an $\alpha$-invariant $t$-group $T$ acting on $S$ for some prime $t \neq 3$, $[S, T] = S$, $[\Phi(S), T] = 1$ and $\tilde{S}$ is completely reducible as a $T(\alpha)$-module. Let $V$ be an irreducible $kS(\alpha)$-module for a field $k$ with characteristic different from 3 and let $\Omega$ be an $S(\alpha)$-stable subset of $V^*$ which linearly spans $V^*$. Set $V_0 = \bigcap \{\ker f \mid f \in \Omega - C_{\Omega}(\alpha)\}$ and $D = [[S, \alpha], C_T(\alpha)]$. Assume that $(S \text{ on } V)$ is weakly $T$-invariant. If $D$ is non-trivial on $V$, then $C_V(\alpha) \not\subseteq V_0$ and

$$\ker(C_T(\alpha) \text{ on } D/\ker(D \text{ on } C_V(\alpha))) = \ker(C_T(\alpha) \text{ on } D/\ker(D \text{ on } C_{V/V_0}(\alpha))).$$

**Proof.** It is easy to observe that $D \leq C_S(\alpha)$. We also have that $\ker(S \text{ on } V)$ is $T$-invariant as $(S \text{ on } V)$ is weakly $T$-invariant. It follows by [5, Lemma 1] that $\ker(D \text{ on } C_V(\alpha)) = \ker(D \text{ on } V)$ and hence $\ker(D \text{ on } C_V(\alpha))$ is $C_T(\alpha)$-invariant.

We use induction on $|ST| + \dim_k V$.

(1) $S$ acts faithfully on $V$.

Set $\bar{S} = S/\ker(S \text{ on } V)$. We observe that $[[\bar{S}, \alpha], C_T(\alpha)] = \bar{D}$. If $|\bar{S}| < |S|$, an induction argument applied to $(V, \bar{S}, T, (\alpha))$ shows that $C_V(\alpha) \not\subseteq V_0$ and

$$\ker(C_T(\alpha) \text{ on } \bar{D}/\ker(\bar{D} \text{ on } C_V(\alpha))) = \ker(C_T(\alpha) \text{ on } \bar{D}/\ker(\bar{D} \text{ on } C_{V/V_0}(\alpha))).$$

It follows that the conclusion of Theorem 3.1 holds. Hence $|\bar{S}| = |S|$: that is, $S$ is faithful on $V$.

(2) $[Z(S), \alpha, \alpha] = 1$.

This is obtained by the same argument as in the proof of Claim 2 appearing in the proof of [5, Theorem 1], just by replacing $C$ there with $D$.

(3) We may assume that $T(\alpha)$ acts irreducibly on $\bar{S}$.

As $\bar{S}$ is a completely reducible $T(\alpha)$-module, there is a collection $\{W_1, \ldots, W_l\}$ of irreducible $T(\alpha)$-submodules such that $\bar{S} = \bigoplus_{i=1}^{l} W_i$. For each $i$, $[W_i, T]$ is either $W_i$ or trivial. Let $[W_i, T] = W_i$ for $i = 1, \ldots, s$ and $[W_i, T] = 0$ for $i = s + 1, \ldots, l$. Set $\bar{D} = \Phi(S)/\Phi(S)$. Then $\bar{D} = \bigoplus_{i=1}^{s} [[W_i, \alpha], C_T(\alpha)]$. Since $t \neq 3$, $\bar{D} = [\bar{D}, C_T(\alpha)]$: that is, $C_D(\bar{C}(\alpha)) = 1$.

Let $W_i = Y_i/\Phi(S)$ for $i = 1, \ldots, s$. $Y_i$ is a normal $T(\alpha)$-invariant subgroup of $S$. Now $C_Y(T) = C_Y(T)/\Phi(S)/\Phi(S) = 1$, giving $\Phi(S) = C_Y(T)$. Set $S_i = [Y_i, T]$ and $D_i = [[[S_i, \alpha], C_T(\alpha)]$ for $i = 1, \ldots, s$. Now $S_i \triangleleft S_i T(\alpha)$, $[S_i, T] = S_i$, $\Phi(S_i) \leq \Phi(Y_i) \leq \Phi(S)$. Furthermore, $\Phi(S) \cap S_i = \Phi(S_i)$, since $\Phi(S) \cap S_i \leq C_{S_i}(T) \leq \Phi(S_i)$. Then

$$S_i/\Phi(S_i) = S_i/([\Phi(S) \cap S_i] \cong S_i/\Phi(S_i) = [Y_i, T]\Phi(S)/\Phi(S) = W_i$$

and so $S_i/\Phi(S_i)$ is an irreducible $T(\alpha)$-module.
Fixed-point free action

Assume that $D_i \neq 1$ for $i = 1, \ldots, u$ and apply induction to $(V, S, T, (\alpha))$ for $i = 1, \ldots, u$. It follows that $C_V(\alpha) \not\subseteq V_0$ and

$$\text{Ker}(C_T(\alpha) \mid D_i \mid \text{Ker}(D_i \mid C_V(\alpha))) = \text{Ker}(C_T(\alpha) \mid D_i \mid \text{Ker}(D_i \mid C_{V,V_0}(\alpha))).$$

$D_i\Phi(S)/\Phi(S) \cong [[W_i, \alpha]^2, C_T(\alpha)]$, giving $\prod_{i=1}^s D_i\Phi(S)/\Phi(S) \cong \hat{D}$.

Let $x \in \text{Ker}(C_T(\alpha) \mid D \mid \text{Ker}(D \mid C_{V,V_0}(\alpha)))$ and observe that

$$[\hat{D}, x] = \prod_{i=1}^s [D_i, x]\Phi(S)/\Phi(S) \leq \prod_{i=1}^s \text{Ker}(D_i \mid C_V(\alpha))\Phi(S)/\Phi(S).$$

Then

$$[D, x] = \prod_{i=1}^s \text{Ker}(D_i \mid C_V(\alpha))\Phi(S) \leq \text{Ker}(D \mid C_V(\alpha))\Phi(S).$$

This shows that

$$\text{Ker}(C_T(\alpha) \mid D \mid \text{Ker}(D \mid C_{V,V_0}(\alpha))) = \text{Ker}(C_T(\alpha) \mid D \mid \text{Ker}(D \mid C_V(\alpha))),$$

which is not the case.

(4) $S$ is a non-abelian special group such that $D \cap \Phi(S) = 1$, $V |_{C_{\Phi(S)}(\alpha)}$ is homogeneous and $|C_{\Phi(S)}(\alpha)| = 3 = |\Phi(S)/[\Phi(S), \alpha]|$.

This is [3, Lemma 6.23] by letting $B = S$ and $P = \langle \alpha \rangle$.

(5) Let $1 \neq \delta \in D$ and let $C = C_{\Phi(S)}(\alpha)\langle \delta \rangle$. For each $1 \neq \tau \in C$, there is an irreducible component of $V|_C$ on which both $\delta$ and $\tau$ act non-trivially.

This is [3, Lemma 6.24] by letting $B = S$ and $P = \langle \alpha \rangle$.

(6) For each $1 \neq \delta \in D$, there exists an element $\tilde{\tau} \in \hat{S}$ such that (i) $[\tilde{\tau}, \alpha, \alpha] = \delta$ and (ii) $[[\tau, \alpha]^i, [\tau, \alpha]^j] \in C_{\Phi(S)}(\alpha)$ for any $i, j = 0, 1, 2$.

Since $\delta \in [S, \alpha]^2$, there exists $\tilde{\tau} \in \hat{S}$ such that (i) holds. If $[\Phi(S), \alpha] = 1$, then (ii) trivially holds. Hence we may assume that $[\Phi(S), \alpha] \neq 1$.

Lemmas 6.30 and 6.32 in [3] with $B = S$ and $P = \langle \alpha \rangle$ imply that $\hat{S}$ is a free $\mathbb{Z}_3\langle \alpha \rangle$-module and that the function $h : \hat{S} \times \hat{S} \rightarrow \Phi(S)/[\Phi(S), \alpha]$ defined by $h(\hat{x}, \hat{y}) = [x, [y, \alpha]]/[[y, [x, \alpha]]\Phi(S), \alpha]$ is a symmetric bilinear map with radical $C_{\hat{S}}(\alpha)$. Also, the subspace $[S, \alpha]$ is $h$-isotropic.

Repeating the argument in the proof of [3, Lemma 6.34], we may obtain $\tilde{\tau} \in \hat{S}$ such that (i) and (ii) hold.

(7) The theorem follows.

It will be sufficient to show that each non-trivial element of $D$ acts non-trivially on $C_{V,V_0}(\alpha)$. Let $1 \neq \delta \in D$. We may choose $\tilde{\tau} \in \hat{S}$ satisfying (6). Let $S_1$ be the inverse image in $S$ of the $\mathbb{Z}_3\langle \alpha \rangle$-module $[\tilde{\tau}, [\tilde{\tau}, \alpha], [\tilde{\tau}, \alpha, \alpha] = \delta]$ in $\hat{S}$. Then $S_1$ is a non-trivial $\alpha$-invariant subgroup of $S$ containing $\delta$. One can observe that $\Phi(S) \times \langle \delta \rangle \subseteq Z(S_1)$. (See the proof of the inequality 6.27 in [3, Lemma 6.25] by letting $B = S$ and $B_1 = S_1$.)
Now $[\tau, \alpha, \alpha] \equiv \delta \pmod{\Phi(S)}$, where $\hat{\tau} = \tau \Phi(S)$. Recall that $C = \langle C_{\Phi(S)}(\alpha), \delta \rangle$. It can be observed that there exists $\sigma \in S_1$ such that $1 \neq [\sigma, \alpha, \alpha] = [\sigma, \alpha, \alpha]^t \in C$ for $i = 1, 2$. (See the proof of the equation 6.28 in [3, Lemma 6.25] by letting $B = S$ and $B_1 = S_1$.)

By (5), there is an irreducible component $W$ of $V|_{\mathcal{C}}$ such that neither $[\sigma, \alpha, \alpha]$ nor $\delta$ acts trivially on $W$. As $W$ is completely reducible as an $S_1(\alpha)$-module, there is a collection $\{V_1, \ldots, V_i\}$ of irreducible $S_1(\alpha)$-modules $V_i$, $i = 1, \ldots, l$, such that $W$ is contained in one of them, say $V_1$. Since $\langle \delta \rangle \leq Z(S_1(\alpha))$, $V_1|_{\langle \delta \rangle}$ is homogeneous. Now $[S_1(\alpha), \alpha]^2$ acts non-trivially on $V_1$, because $[\sigma, \alpha, \alpha] \in [S_1(\alpha), \alpha]^2$. That is, $V_1$ is an ample $S_1(\alpha)$-module. It follows that $C_{V_1}(\alpha) \neq 0$ by [5, Lemma 1] applied to $S_1(\alpha)$ on $V_1$.

Assume that $C_{V_1}(\alpha) \subseteq V_0$. Set $\Omega_1 = \Omega|_{V_1}$. Then $\Omega_1$ is an $S_1(\alpha)$-stable subset of $V_1$ and there exists $f \in \Omega_1$ such that $f(V_1 \cap V_0) \neq 0$: that is, $f \in \Omega_1 - C_{\Omega_1}(\alpha)$. By [5, Lemma 2], we have $C_{S_1}(f) \cap C = \text{Ker}(C|_{V_1})$ because $C \leq Z(S_1)$. As $[\sigma, \alpha, \alpha] \notin \text{Ker}(C|_W)$, we also have $[\sigma, \alpha, \alpha] \notin \text{Ker}(C|_{V_1}) = C_{S_1}(f) \cap C$. Now Lemma 3 in [5] applied to the action of $S_1(\alpha)$ on $V_1$ with $\Omega_1$ leads to a contradiction. Therefore, we have obtained that $C_{V_1}(\alpha) \not\subseteq V_0$.

We also observe that $\delta$ acts non-trivially on $C_{V_1/\Omega_1 V_0}(\alpha)$, since $V_1|_{\langle \delta \rangle}$ is homogeneous and $\delta$ acts non-trivially on $W$. So $\delta$ acts non-trivially on $C_{V/V_0}(\alpha)$: that is, $\text{Ker}(\delta)$ on $C_{V/V_0}(\alpha)) = 1$. It follows that $\text{Ker}(\sigma|_{C_{V/V_0}(\alpha)}) = 1$ and so

$$\text{Ker}(\sigma|_{C_{V/V_0}(\alpha)}) = \text{Ker}(\sigma|_{C_{V/V_0}(\alpha)})$$

This forces that $C_{V}(\alpha) \subseteq V_0$, which means that $\text{Ker}(\delta|_{C_{V/V_0}(\alpha)}) = D = 1$, which is a contradiction. □

**Theorem 3.2.** Let $G \lhd GA$ and $\langle a \rangle \leq A$ be of prime order $p$ for an odd prime $p$. Suppose that $P_1, \ldots, P_t$ is an $A$-Fitting chain of $G$ such that $[P_i, a] \neq 1$, $P_i$ is a $p_i$-group for a prime $p_i$, with $p_1 \neq p$, $i = 1, \ldots, t$, and $P_i$ is abelian when $p_i = 2$. If $p = 3$, suppose that $p_{i+2} \neq p_j$ for any $j \in \{2, \ldots, t-2\}$ with $p_j = p$. There are then $A$-invariant non-trivial sections $D_2, \ldots, D_t$ of $P_2, \ldots, P_t$, respectively, each of which is centralized by $a$ such that $D_i$ normalizes $D_{i+1}$ with $\text{Ker}(D_i|_{D_{i+1}}) = 1$ for $i = 2, \ldots, t-1$.

**Proof.** By Theorem 2.3, we may assume that $p = 3$ and $p \in \{p_1, \ldots, p_t\}$. Let $q$ be a prime different from $p$, let $p_{t+1} = q$ and let $P_{t+1}$ stand for the regular $\mathbb{Z}_q[P_{t-1}A]$-module. We shall add $P_{t+1}$ to the given chain and define subspaces $E_i$ of $P_i$ for each $i = 1, \ldots, t+1$ as follows: $E_1 = P_1$, $E_i = [X_i, E_{i-1}]$ for $i = 2, \ldots, t+1$, where $X_{i}/\Phi(P_i)$ is the sum of all ample irreducible $E_{i-1}(a)$-submodules of $P_i$. It is easy to observe that for each $i = 2, \ldots, t+1$, the $E_i$ are all $E_{i-1}A$-invariant subgroups of $P_i$ and $E_i$ is a direct sum of ample irreducible $E_{i-1}(a)$-submodules.

We now define subgroups $F_i$ of $E_i$ for $i = 1, \ldots, t+1$ as follows:

- $F_1 = \{1\}$,
- $F_1 = C_{E_1}(a)$ if $p_1 \neq p$ and $i \geq 2$,
- $F_2 = C_{[E_2,a]^2}(a)$ if $p_2 = p = 3$,
- $F_2 = \langle [E_i,a]^2, F_{i-1} \rangle$ if $p_i = p = 3$ and $i \geq 3$. 


It can also be easily seen that for each \(i = 2, \ldots, t + 1\), \(F_i\) is \(F_{i-1}\)-invariant and is centralised by \(a\).

We next define the sections \(D_i\) by \(D_i = F_i / \ker(F_i)\) on \(\tilde{E}_{i+1}\) if \(p_i \neq p\) for \(i = 2, \ldots, t\) and \(D_i = F_i / \ker(F_i)\) on \(\tilde{D}_{i+1}\) if \(p_i = p\) for \(i = 2, \ldots, t - 1\), and \(D_i = F_i\) if \(p_i = p\).

**Claim A.** Assume that \(i \geq 2\) and that \(p_i \neq p\). If \(E_{i+1} \neq 1\), then \(D_i\) is a non-trivial \(F_{i-1}\)-invariant section such that \((F_{i-1} \mid \tilde{E}_i) \equiv_w (F_{i-1} \mid \tilde{D}_i)\).

Notice that Claim A can be proved by imitating the proof of Claim 1 in [5, Theorem 3] by replacing every occurrence of [5, Theorem 1] with Theorem 2.1 of this paper.

We shall proceed by proving the following claim.

**Claim B.** Assume that \(i \geq 3\) and that \(p_i = p = 3\). If \(D_{i-1} \neq 1\), then \(\ker(F_i \mid \tilde{E}_{i+1}) = 1\), \(D_i \neq 1\), \(\ker(F_{i-1} \mid \tilde{E}_i) = \ker(F_{i-1} \mid \tilde{D}_i)\).

By our hypothesis, \(p_{i+2} \neq 3\). As \(D_{i-1} = F_{i-1} / \ker(F_{i-1})\) on \(\tilde{E}_i\) we see that \(E_i \neq 1\). Now, \(\ker(F_{i-1} \mid \tilde{D}_i) = \ker(F_{i-1} \mid D_i) = \ker(F_{i-1} \mid F_i / \ker(F_i \mid \tilde{D}_{i+1}))\).

We know that \(\tilde{E}_{i+1} = \bigoplus_{j=1}^l W_j\), where \(W_j = U_j / \Phi(E_{j+1})\) are irreducible ample \(E_i(a)\)-submodules for \(j = 1, \ldots, l\). Since \(\tilde{P}_{i+2} |_{E_{i+1}}\) is completely reducible and \(E_{i+1}\) is faithful on \(\tilde{P}_{i+2}\), there exists at least one irreducible component of \(\tilde{P}_{i+2} |_{E_{i+1}}\) on which \(U_j\) acts non-trivially. Let \(\mathfrak{N}_j\) denote the set of all such components of \(\tilde{P}_{i+2} |_{E_{i+1}}\). There are two cases: either (I) there is at least one \(N\) in \(\mathfrak{N}_j\) on which \(\Phi(E_{i+1})\) acts trivially; or (II) there is no \(N\) in \(\mathfrak{N}_j\) on which \(\Phi(E_{i+1})\) acts trivially.

In the latter case, for each irreducible component \(N\) of \(\tilde{E}_{i+2} |_{E_{i+1}}, N \in \mathfrak{N}_j\) if and only if \(\Phi(E_{i+1})\) acts non-trivially on \(N\). (This is obvious when we replace \(i\) by \(i + 1\) in [5, Lemma 4].) Thus \(U_j\) is trivial on each irreducible component \(N\) of \(\tilde{P}_{i+2} |_{E_{i+1}}\), lying outside \(\tilde{E}_{i+2}\), because otherwise \(N \in \mathfrak{N}_j\), implying that \(\Phi(E_{i+1})\) and hence \([E_{i+1}, a]\) is non-trivial on \(N\), which is a contradiction. It follows that \(1 = \ker(U_j \mid \tilde{P}_{i+2}) = \ker(U_j \mid \tilde{E}_{i+2})\) when (II) holds.

Now suppose that \(\ker(U_j \mid \tilde{E}_{i+2}) = 1\) for each \(j = 1, \ldots, l\) and \(\ker(U_j \mid \tilde{E}_{i+2}) \neq 1\) for each \(j = s + 1, \ldots, l\). One may observe that for each \(j = s + 1, \ldots, l\),

\[
\Omega_j = \{ f \in W_j^+ \mid \text{there exists } N \in \mathfrak{N}_j \text{ on which } \Phi(E_{i+1}) \text{ acts trivially} \}
\]

\[
\text{and } \ker(U_j \mid N) / \Phi(E_{i+1}) \subseteq \ker(f)
\]

is an \(E_i(a)\)-invariant subset of \(W_j^+\) such that \(\langle \Omega_j \rangle = W_j^+\). For \(j = 1, \ldots, s\), let \(\Omega_j\) denote the whole of \(W_j^+\). Now for each \(j = 1, \ldots, l\), we appeal to Theorem 3.1 by letting \(V = W_j, S = E_t, T = E_{i-1}\) and \(\Omega = \Omega_j\) and get \(C_{W_j^+}(a) \nsubseteq (W_j)^0\) and

\[
\ker(F_{i-1} \mid F_i / \ker(F_i) | C_{W_j}(a)) = \ker(F_{i-1} \mid F_i / \ker(F_i) | C_{W_j^+}(a)) = \ker(U_j \mid \tilde{E}_{i+2})
\]

We may observe that for each \(j = 1, \ldots, l\), \(K_j \Phi(E_{i+1}) / \Phi(E_{i+1}) \subseteq (W_j)^0\), where \(K_j = \ker(U_j \mid \tilde{E}_{i+2})\). If not, then \(j \in \{s + 1, \ldots, l\}\) and there exist \(x \in K_j, f \in \Omega_j - C_{\Omega_j}(a) \subseteq (W_j)^0\) such that \(f(x \Phi(E_{i+1})) \neq 0\). By the definition of \(\Omega_j\), we can find an irreducible submodule \(N\) of \(\tilde{P}_{i+2} \mid E_{i+1}\) on which \(U_j\) is non-trivial, \(\Phi(E_{i+1})\) is trivial and \(\ker(U_j \mid N) / \Phi(E_{i+1}) \subseteq (W_j)^0\)
Ker \( f \). Then \( x \notin \text{Ker}(U_j \text{ on } N) \). As \( x \in \text{Ker}(U_j \text{ on } \tilde{E}_{i+2}) \), \( N \) lies outside \( \tilde{E}_{i+2}|_{E_{i+1}} \); that is, \([E_{i+1}, a] \) acts trivially on \( N \). Thus \([U_j, a] \) is trivial on \( N \) and so \( f \in \text{C}_{\mathcal{O}_{j}}(a) \), which is a contradiction.

Hence \( L_j = \text{Ker}(C_{U_j}(a) \text{ on } \tilde{E}_{i+2}) \subseteq Y_j \) where \((W_j)_0 = Y_j/\Phi(E_{i+1})\) for \( j = 1, \ldots, l \), implying that \( L_j = \text{Ker}(C_{U_j}(a) \text{ on } \tilde{E}_{i+2}) = \text{Ker}(C_{Y_j}(a) \text{ on } \tilde{E}_{i+2}) \) for \( j = 1, \ldots, l \). We observe that

\[
C_{W_j}(a)/C_{(W_j)_0}(a) \cong C_{U_j}(a)\Phi(E_{i+1})/C_{Y_j}(a)\Phi(E_{i+1})
\]

and so \( \text{Ker}(F_i \text{ on } C_{W_j}(a)/C_{(W_j)_0}(a)) \supseteq \text{Ker}(F_i \text{ on } C_{U_j}(a)/L_j) \) as \([\Phi(E_{i+1}), F_i] = 1\). Set \((\tilde{E}_{i+1})_0 = \bigoplus_{j=1}^{l}(W_j)_0\). Notice that

\[
C_{\tilde{E}_{i+1}}(a) = \bigoplus_{j=1}^{l} C_{W_j}(a) = \bigoplus_{j=1}^{l} C_{U_j}(a)\Phi(E_{i+1})/\Phi(E_{i+1})
\]

and

\[
C_{(\tilde{E}_{i+1})_0}(a) = \bigoplus_{j=1}^{l} C_{(W_j)_0}(a)
\]

and

\[
\tilde{E}_{i+1}/(\tilde{E}_{i+1})_0 \cong \bigoplus_{j=1}^{l} W_j/(W_j)_0.
\]

We observe that

\[
\text{Ker}(F_{i-1} \text{ on } D_i) = \text{Ker}(F_{i-1} \text{ on } F_i/\text{Ker}(F_i \text{ on } D_{i+1}))
\]

\[
\subseteq \text{Ker}(F_{i-1} \text{ on } F_i/\text{Ker}(F_i \text{ on } C_{\tilde{E}_{i+1}}/(\tilde{E}_{i+1})_0(a)))
\]

\[
\subseteq \text{Ker}(F_{i-1} \text{ on } F_i/\text{Ker}(F_i \text{ on } C_{W_j}/(W_j)_0(a)))
\]

\[
= \text{Ker}(F_{i-1} \text{ on } F_i/\text{Ker}(F_i \text{ on } C_{W_j}(a)))
\]

for each \( j = 1, \ldots, l \). Thus we have

\[
\text{Ker}(F_{i-1} \text{ on } D_i) \subseteq \text{Ker}(F_{i-1} \text{ on } F_i/\text{Ker}(F_i \text{ on } \tilde{E}_{i+1})).
\]

By [5, Lemma 1] applied to the action \( E_i(a) \) on \( \tilde{E}_{i+1} \), we see that \( \text{Ker}(F_i \text{ on } \tilde{E}_{i+1}) = \text{Ker}(F_i \text{ on } \tilde{E}_{i+1}) \). Notice that \([|E_i, a|^2 \text{ on } \tilde{E}_{i+1}] = |E_i, a|^2 \text{ on } \tilde{E}_{i+1} = 1\) and so \( \text{Ker}(F_i \text{ on } \tilde{E}_{i+1}) = 1\). Consequently, \( \text{Ker}(F_{i-1} \text{ on } D_i) \subseteq \text{Ker}(F_{i-1} \text{ on } F_i) \). As the reversed inclusion also holds, we have

\[
\text{Ker}(F_{i-1} \text{ on } \tilde{D}_i) = \text{Ker}(F_{i-1} \text{ on } \tilde{F}_i) = \text{Ker}(F_{i-1} \text{ on } [\tilde{E}_i, a]^2).
\]

Applying Theorem 2.2 to \( E_{i-1}(a) \) on \( E_i \) gives \( \text{Ker}(F_{i-1} \text{ on } \tilde{E}_i) = \text{Ker}(F_{i-1} \text{ on } [\tilde{E}_i, a]^2) \)
and so \( \text{Ker}(F_{i-1} \text{ on } \tilde{E}_i) = \text{Ker}(F_{i-1} \text{ on } \tilde{D}_i) \). We observe that \( D_i \neq 1 \) as \( 1 \neq D_{i-1} = F_{i-1}/\text{Ker}(F_{i-1} \text{ on } \tilde{E}_i) = F_{i-1}/\text{Ker}(F_{i-1} \text{ on } \tilde{D}_i) \), completing the proof of Claim B.
We are now ready to complete the proof of Theorem 3.2. Firstly, we shall observe that 
$D_2 \neq 1$, for if $p_2 = p$, then the fact that $D_2 \neq 1$ is a consequence of [3, Proposition 7.13]
by observing that the critical index $j$ appearing in that proposition can actually be taken
as 1, which means that $E_2 \neq 1$ in our case. And if $p_2 \neq p$, Claim A applied to the action
of $E_1$ on $E_2$ gives that $D_2 \neq 1$.

Suppose that $D_{i-1} \neq 1$ for some $i \geq 3$. If $p_{i-1} = p$, then $D_{i-1} = F_{i-1}/\ker(F_{i-1}$ on $\hat{D}_i)$
and so $D_i \neq 1$ holds. Assume that $p_{i-1} \neq p$. Appealing to Claims A and B, respectively,
when $p_i \neq p$ and $p_i = p$, we see that $D_i \neq 1$ always holds.

Therefore, we have obtained a chain of non-trivial $A$-invariant sections $D_2, \ldots, D_t$ that
are all centralized by $a$ and $\ker(D_{i-1}$ on $D_i) = 1$ for each $i = 3, \ldots, t$. □

4. Some technical results pertaining to the proof of Theorem C

Lemma 4.1. Let $S \triangleleft SA$, where $A$ is an abelian group and $S$ is an s-group for some
prime $s$ which is coprime to $|A|$. Assume that $S$ is abelian when $s = 2$. Let $V$ be an
irreducible $kSA$-module, where $k$ is a splitting field for all subgroups of $SA$ and is of
characteristic not dividing $|SA|$. Suppose that $S$ acts non-trivially and $A$ acts fixed-point
freely on $V$. Then there is a non-trivial subgroup $D$ of $A$ such that $[S, D]$ acts trivially
on $V$.

Proof. If $S$ is abelian, $D$ can be taken as the stabilizer of an $S$-homogeneous com-
ponent of $V$, and the result follows. Therefore, we may assume that $S$ is a non-abelian
group of odd order. By induction on $|S|$, we may also assume that $S$ is faithful on $V$.
Now the result follows directly from [8, Theorem 4.1]. □

Lemma 4.2. Let $A$ be an abelian group whose order is a product of two odd primes
and let $G$ be a group on which $A$ acts. If $P_i$, $i = 1, \ldots, t$, are $A$-invariant sections of $G$
such that $P_1$ is a $p_i$-group, $P_i$ is abelian when $p_i = 2$, $C_{P_i}(A) = 1$ for $i = 1, \ldots, t$ and $P_i$
regularizes $P_{i+1}$ with $\ker(P_i$ on $P_{i+1}) = 1$ for $i = 1, \ldots, t-1$, then $t \leq 2$.

Proof. We may assume that $t = 3$ and that $P_1$ is an elementary abelian group on
which $A$ acts irreducibly. If every element $a \in A$ of prime order centralizes $P_1$, then $A$
must be cyclic of order $p^2$ for an odd prime $p$. Let $A = \langle a \rangle$ and let $z = a^p$. Now $[P_2, z] \neq 1$,
because otherwise $A/\langle z \rangle$ acts fixed-point freely on $P_2P_1$. This leads to a contradiction
by Lemma 4.1. Thus there exists $a \in A$ of prime order $p$ such that $[P_1, a] \neq 1$. Now $[P_1, a] = P_1$ and so $p_1 \neq p$. We first consider the case $p_2 = p$. Theorem 2.2 applied to
the action of $P_1\langle a \rangle$ on $P_2$ then tells us that $[P_2, a]^{p-1} = 1$. This enables us to apply [5,
Lemma 1] to the action of $P_2\langle a \rangle$ on $P_3$. It follows that $[C_{P_3}(a), C_{[P_2, a]^{p-1}}(a)] \neq 1$, which is
impossible as $A/\langle a \rangle$ is fixed-point free on $C_{P_3}P_2(a)$. Hence $p_2 \neq p$. Notice that $C_{P_2}(a) \neq 1$.
Applying Theorems 2.1 and 2.2 to the action of $P_2\langle a \rangle$ on $P_3$, respectively, when $p_3 \neq p$ and $p_3 = p$, we see that $[C_{P_3}(a), C_{P_3}(a)] \neq 1$, which is not the case. □

Lemma 4.3. Let $A = \langle a \rangle$ be a cyclic group of order $p^n$ for some prime $p$, and let $G$ be
a group acted on by $A$. Suppose that $S \triangleleft GA$ is an s-group and that $T$ is an $A$-invariant
$T$-subgroup of $G$ for distinct primes $s$ and $t$ that are both different from $p$ such that
Let $\alpha$ with $\prod_{i=1}^{k} G. Ercan, İ. Ş. Güloğlu and Ö. M. Sağdıçoğlu

centralized by $S$, $T$, $\alpha$ is a $G$-invariant $h$-subgroup $H$ of $G$ for a prime $h$ different from $p$

such that $H \trianglelefteq C_{G}(\Phi(T/T_{0}))$, $H/C_{G}(T/T_{0})$ is elementary abelian and $[T/T_{0}, H] = T/T_{0}$.

Let $V$ be a $kG$-module on which $S$ acts non-trivially and let $k$ be a field of characteristic not dividing $|STHA|$. Then $[C_{V}(A), C_{S}(A)] \neq 1$.

Proof. We use induction on $|SA| + \dim_{k} V$. Set $\bar{S} = S/\text{Ker}(S$ on $V)$. Let $\bar{S}_{1}$ be a minimal $T\langle \alpha \rangle$-invariant subgroup of $\bar{S}$ on which $T$ acts non-trivially. Then $[\bar{S}_{1}, T] = \bar{S}_{1}$, $\bar{S}_{1}/\Phi(\bar{S}_{1})$ is an irreducible $T\langle \alpha \rangle$-module, $[\Phi(\bar{S}_{1}), T] = 1$ and $\bar{S}_{1}$ is a special group. If $|\bar{S}_{1}| < |\bar{S}|$, an induction argument gives that $[C_{V}(A), C_{\bar{S}_{1}}(A)] = [C_{V}(A), C_{\bar{S}_{1}}(A)] \neq 1$:

that is, $[C_{V}(A), C_{S}(A)] \neq 1$. Thus $\bar{S}_{1} = S$.

We may also assume that $G = STH$. Notice that $[T, z] = T$ acts non-trivially on each irreducible component of $\bar{S}|_{T}$. It is easy to see that

$$
\Omega = \left\{ f \in (\bar{S})^{*} \mid \text{there exists an irreducible component } N \text{ of } V|_{S} \right. \text{such that } \text{Ker}(S \text{ on } N)\Phi(S)/\Phi(S) \subseteq \text{Ker } f \left\}
$$

is a $GA$-invariant subset that linearly spans the dual space $(\bar{S})^{*}$. Applying [9, Theorem 2.1.1] to the action of $(T/T_{0})A$ on $\bar{S}$ with $\Omega$, we see that $C_{S}(A) \notin \bigcap \{ \text{Ker } f \mid f \in \Omega - C_{\Omega}(z) \}$. This gives an $\tilde{x} \in C_{S}(A)$ and an $f \in \Omega - C_{\Omega}(z)$ such that $f(\tilde{x}) \neq 0$. Now $\tilde{x} \notin \text{Ker}(S$ on $N)$ for some irreducible component $N$ of $V|_{S}$ by the definition of $\Omega$.

On the other hand, one more application of [9, Theorem 2.1.1] to the action of $GA$ on $V$ gives that $C_{S}(A) = C_{C_{S}(A)}(C_{V}(A))$ is contained in the kernel of each irreducible component of $V|_{S}$ on which $[S, z]$ acts non-trivially. It follows that $[S, z]$ is trivial on $N$: that is, $[S, z] \subseteq \text{Ker } f$ and so $f \in C_{\Omega}(z)$, which is a contradiction.

Lemma 4.4. Let $G = ST$, where $S \triangleleft G$, $S$ is an $s$-group with $\Phi(S) \subseteq Z(S)$, $\Phi(\Phi(S)) = 1$, $T/T_{0}$ is a $t$-group for distinct primes $s$ and $t$ with $[\Phi(S), T] = 1$, $\Phi(T/T_{0}) \subseteq Z(T/T_{0})$, $\Phi(\Phi(T/T_{0})) = 1$, where $T_{0} = C_{T}(S)$. Assume that Sylow 2-subgroups of $G$ are abelian.

Let $\alpha$ be an automorphism of $G$ of order $p$ for an odd prime $p$ which leaves $T$ invariant with $[T/T_{0}, \alpha] = T/T_{0}$. Let $V$ be a $kG(\alpha)$-module on which $S$ acts faithfully for a field $k$ of characteristic not dividing $sp$. If $[C_{V}(\alpha), C_{S}(\alpha)] = 1$, then $[S, T] = 1$.

Proof. We first observe that $s \neq p$. If this is not the case, appealing to Theorem 2.2 with the action of $T/T_{0}\langle \alpha \rangle$ on $\bar{S}$, we get $[\bar{S}, \alpha]^{p-1} \neq 0$. Now [5, Lemma 1] applies to $S\langle \alpha \rangle$ on $V$ and gives that $[C_{V}(\alpha), C_{S}(\alpha)] \neq 1$. Thus we have obtained that $s \neq p$. We may consider $V, S, T/T_{0}$ as an $\alpha$-Fitting chain and by Theorem 2.3 we get sections $D_{2}, D_{3}$ centralized by $\alpha$, of $S$ and $V$, respectively, such that $D_{2}$ acts non-trivially on $D_{3}$. This contradiction completes the proof.

Lemma 4.5. Let $SA$ be a group where $S \triangleleft SA$, $S$ is a $q$-group for an odd prime $q$, $\Phi(S) \subseteq Z(S)$, and $A$ is cyclic of order $pq$ for some prime $p$. Suppose that $[S, A_{q}]^{q-1} \neq \Phi(S)$ and $[S, A_{p}] = S$, where $A_{p}$ and $A_{q}$ denote the Sylow $p$- and $q$-subgroups of $A$, respectively.

Let $V$ be a $\mathbb{C}SA$-module on which $[S, A_{q}]^{q-1}$ acts non-trivially. Then $C_{V}(A) \neq 0$. 


Fixed-point free action

Proof. Assume the contrary. Set $S = S/\ker(S\text{ on } V)$. By [5, Lemma 1] applied to the action of $SA_q$ on $V$, we see that $C_V(A_q) \neq 0$ and $\ker(C_D(A_q)\text{ on } C_V(A_q)) = \ker(C_D(A_q)\text{ on } V)$, where $D = [S, A_q]^{q-1}$. This gives

$$\ker([C_D(A_q), A_p]\text{ on } C_V(A_q)) = \ker([C_D(A_q), A_p]\text{ on } V)$$

also. If $[C_D(A_q), A_p] \neq 1$, then we apply Lemma 4.1 to the action of $C_D(A_q), A_p$ on $C_V(A_q)$ and get a contradiction. Hence $[C_D(A_q), A_p] = 1$, forcing $[D, A_p] = 1$ by Thompson’s $A \times B$ Lemma. But then $D \leq \Phi(S) = \Phi(S)$, which is not the case. □

5. Proof of Theorem C

Theorem C. Let $G \triangleleft GA$, where $A$ is an abelian group whose order is a product of three odd primes. Assume that Sylow $2$-subgroups of $G$ are abelian. If $A$ acts fixed-point freely on $G$, then $f(G) \leq 3$.

Proof. By Lemmas 8.1 and 8.2 in [3], we may assume the existence of an $A$-Fitting chain $P_1, P_2, P_3, P_4$ of length $4$ in $G$, where $P_i = S_i/T_i$ are $A$-invariant sections of $G$ satisfying the following.

(a) $P_i$ is a $p_i$-group for some prime $p_i$, $\Phi(P_i) \leq Z(P_i)$, $\Phi(\Phi(P_i)) = 1$, and $P_i$ has exponent $p_i$ when $p_i$ is odd, for $i = 1, 2, 3, 4$.

(b) $p_i \neq p_{i+1}$ for $i = 1, 2, 3$.

(c) $S_i$ normalizes $P_j$ for $i, j$ with $i < j$, $T_i = \ker(S_i\text{ on } P_{i+1})$ for $i = 1, 2, 3$ and $T_4 = 1$.

(d) $[\Phi(P_{i+1}), S_i] = 1$ for $i = 1, 2, 3$.

(e) $P_4 \leq F(G)$ and $P_4$ is irreducible as an $S_3S_2S_1A$-module.

We may also assume that $P_1$ is an elementary abelian group on which $A$ acts irreducibly. Since $A$ is nilpotent, it is a Carter subgroup of any semidirect product of it with a section of $G$. Thus $A$ acts fixed-point freely on any section of this chain. As $P_i$ is an $HA$-module for $H = P_3S_2S_1$, we may assume that $|\langle P_i, H A \rangle| = 1$ by the Fong–Swan Theorem.

If $[P_i, a] \neq 1$ for some element $a$ in $A$ of prime order, then $[P_i, a] = P_i$ and so $p_i \neq p$. Now Theorem 3.2 gives an $A$-chain $D_1, D_2, D_3$ whose terms are all centralized by $a$ such that $\ker(D_i\text{ on } D_{i+1}) = 1$ for $i = 1, 2$. As $A/(a)$ acts fixed-point freely on $D_i$ for each $i = 1, 2, 3$, we get a contradiction by Lemma 4.2. Thus $[P_i, a] = 1$ for every element $a$ in $A$ of prime order. This forces $A$ to be cyclic of order $p^2q$, for two distinct odd primes $p$ and $q$. Let $A = A_p \times A_q$, where $|A_p| = p^2$ and $|A_q| = q$, and let $z$ denote the unique element of order $p$ in $A_p$. We have $[P_i, (z)A_q] = 1$ and so $p_i \neq p$, because otherwise $C_{P_i}(A_q) = 1$. As $p_i \neq p$, we may assume that $[S_1, z] = 1$.

As $A$ is cyclic, we may assume that the $S_i$ are $p_i$-groups for $i = 1, 2, 3$. Passing to an irreducible $A$-tower in the sense of [9], we may also assume that $P_i$ is a special group on the Frattini factor group of which $(\prod_{i>j} S_j)A$ acts irreducibly for each $i = 1, 2, 3, 4$ and $[S_{i+1}, S_i] = S_{i+1}$ for $i = 1, 2, 3$. 


As $A$ is nilpotent, $HA$ is a relative $M$-group with respect to $H$ by [7, 6.22]: that is, there is a subgroup $HB$ of $HA$ and an irreducible $HB$-module $U$ such that $U^{HA} = P_4$ and $U|_B$ is irreducible. If $P_1|_B$ is irreducible, by [6, Theorem] we see that $C_{Y}(A) \neq 1$. Thus $B \neq A$. On the other hand, by Mackey's Theorem, $P_4|_A \cong U|_B \otimes k(A/B)$, and so $C_{P_4}(B) \cong C_U(B) \otimes k(A/B)$ and $C_{P_4}(A) \cong C_U(B) \otimes C_{k(A/B)}(A/B)$. It follows that $C_U(B) = 1$ and so $C_{P_4}(B) = 1$. This argument shows the existence of a non-trivial proper subgroup $B$ of $A$ such that $C_{P_4}(B) = 1$.

If $|B| = q$, then we set $Y = P_4C_H(A_p)$. As $C_Y(B) = 1$, $Y$ must be nilpotent and so $C_Y(A_p) = 1$. This forces $f(H) \leq 2$, which is not the case. Therefore, $|B|$ is either $pq$ or a divisor of $p^2$.

We shall observe that $P_3$ is centralized by neither $z$ nor $A_q$; assume that $[P_3, a] = 1$ for an element $a$ in $A$ of prime order. Then both $P_2$ and $P_1$ are centralized by a. Now $P_1$, $P_2$, $P_3$ are $A$-invariant sections on each of which $A/\langle a \rangle$ acts fixed-point freely, which is impossible by Lemma 4.2.

We first assume that $|B|$ divides $p^2$. Then $C_{P_4}(A_p) = 1$. Also, if $P_3$ is a $p'$-group, Lemma 4.1 applies to the action of $P_3A_p$ on $P_4$ and gives that $[P_3, z] = 1$, which is not the case. Hence $P_3$ is a $p$-group. It follows that $[P_2, z] = P_2$, because otherwise we would have $[P_3, z] = 1$. Lemma 4.1 applied to the action of $S_2A_p$ on $P_4$ leads to a contradiction.

Thus we have $B = \langle z \rangle A_q$. If $P_3$ is a $p$-group, then $A_q$ is fixed-point free on $C_{P_4}(z)$ implying that $C_{P_4}(z)$ and $C_{P_3}(z)$ commute. As $[P_3, z] \neq 1$, we have $[P_2, z] \neq 1$. Appealing to Lemma 4.4 with the action of $P_3[S_2, z]\langle z \rangle$ on $P_4$, we get $[C_{P_4}(z), C_{P_3}(z)] \neq 1$, which is not the case. Thus $P_3 \neq p$.

If, in addition, $p_3 \neq q$, then we consider an irreducible $[P_3, z]B$-submodule $Y$ of $P_4$ on which $[P_3, z]$ acts non-trivially. It follows by Lemma 4.1 that $[P_3, z, A_q]$ is trivial on $Y$. We now have $[P_3, z, A_q] = 1$ by the faithful action of $P_3$ on $P_4$, forcing that $[P_3, z] \neq P_3$. Hence $[P_2, z] \neq 1$. If $p_2 \neq p$, we apply Lemma 4.3 by letting $V = P_4$, $S = P_3$, $T = [S_2, z]$ and $H = S_1$ and get $[C_{P_4}(A_p), C_{P_3}(A_p)] \neq 1$, which is a contradiction as $A_q$ acts fixed-point freely on $C_{P_4P_3}(A_p)$. Thus we have $p_2 = p$ and so $[P_2, z] = 1$, which is not the case. Therefore, $P_3$ must be a $q$-group.

Notice that $[P_2, A_q] \neq 1$, because otherwise $[P_3, A_q] = 1$, which is not the case. Applying Theorem 2.2 to the action of $P_2A_q$ on $P_3$, we see that $[P_3, A_q]^{q-1} \neq \Phi(P_3)$. If $[P_3, z] = P_3$, then Lemma 4.5 applies to the action of $P_3B$ on an irreducible $P_3B$-submodule of $P_4$ on which $[P_3, A_q]^{q-1}$ acts non-trivially, and it tells us that $C_{P_4}(B) \neq 1$, which is a contradiction. Hence $[P_3, z] \neq P_3$. This forces $P_2$ to be a $p'$-group and $[P_2, z] = P_2$, as $[P_1, z] = 1$. Now Lemma 4.3 applied to the action of $P_3S_2A_p$ on $P_4$ supplies that $C_{P_4P_3}(A_p)$ is not nilpotent, which is a contradiction that completes the proof.

References