NILPOTENT LENGTH OF A FINITE SOLVABLE GROUP
WITH A FROBENIUS GROUP OF AUTOMORPHISMS

GÜLİN ERCAN
ISMAIL Ş. GULOĞLU
ELİF ÖĞUT

Abstract We prove that a finite solvable group $G$ admitting a Frobenius group $FH$ of automorphisms of coprime order with kernel $F$ and complement $H$ such that $[G, F] = G$ and $C_{C_G(F)}(h) = 1$ for all nonidentity elements $h \in H$, is of nilpotent length equal to the nilpotent length of the subgroup of fixed points of $H$.

1. Introduction

Let $A$ be a finite group that acts on the finite solvable group $G$ by automorphisms. There have been a lot of research to obtain information about certain group theoretical invariants of $G$ in terms of the action of $A$ on $G$. A particular major problem is to bound the nilpotent length $f(G)$ of $G$ in terms of information about the structure of $A$ alone when $C_G(A) = 1$, that is, the action of $A$ is fixed point free. One of the recent results in this framework is [2] in which Khukhro handled the case where $A = FH$ is a Frobenius group with kernel $F$ and complement $H$. He proved that the nilpotent lengths of $G$ and $C_G(H)$ are the same if $C_G(F) = 1$ and $(|G|, |H|) = 1$ and later in [3], he removed the coprimeness assumption of the theorem in [2]. In the present paper, we keep the coprimeness condition but weaken the fixed point freeness of $F$ on $G$ slightly, and obtain the same conclusion about the nilpotent length of $G$. More precisely, we prove the following:

**Theorem.** Let $G$ be a finite solvable group admitting a Frobenius group of automorphisms $FH$ of coprime order with kernel $F$ and complement $H$ such that $C_{C_G(F)}(h) = 1$ for all nonidentity elements $h \in H$. Then $f([G, F]) = f(C_{[G,F]}(H))$ and $f(G) \leq f([G, F]) + 1$.

We obtained the following proposition which is crucial in proving the theorem above and is of independent interest, too.

**Proposition.** Let $Q$ be a normal $q$-subgroup of a group having a complement $FH$ which is a Frobenius group with kernel $F$ and complement $H$ such that $C_{C_Q(F)}(h) = 1$ for all nonidentity elements $h \in H$. Assume further that $|FH|$ is not divisible by $q$ and $Q$ is of class at most 2. Let $V$ be a $kFQH$-module where $k$ is a field with characteristic not dividing $|QFH|$. Then we have

$$\text{Ker}(C_{[Q,F]}(H) \text{ on } C_V(H)) = \text{Ker}(C_{[Q,F]}(H) \text{ on } V).$$

2000 Mathematics Subject Classification. 20D10, 20D15, 20D45.

Key words and phrases. solvable group, nilpotent length, Frobenius group, automorphisms.
Throughout the article all groups are finite. The notation and terminology are mostly standard.

2. Proof of the Proposition

In this section we establish the key result in proving the main theorem of this paper.

Proposition 2.1. Let $Q$ be a normal $q$-subgroup of a group having a complement $FH$ which is a Frobenius group with kernel $F$ and complement $H$ such that $C_{C_{Q,F}}(h) = 1$ for all nonidentity elements $h \in H$. Assume further that $|FH|$ is not divisible by $q$ and $Q$ is of class at most 2. Let $V$ be a $kQFH$-module where $k$ is a field with characteristic not dividing $|QFH|$. Then we have

$$\text{Ker}(C_{Q,F}(H) \text{ on } C_V(H)) = \text{Ker}(C_{Q,F}(H) \text{ on } V).$$

Proof. Suppose the proposition is false and choose a counterexample with minimum $\dim_k V + |QFH|$. We split the proof into a sequence of steps. To simplify the notation we set $K = \text{Ker}(C_{Q,F}(H) \text{ on } C_V(H)).$

Claim 1. We may assume that $k$ is a splitting field for all subgroups of $QFH$.

Proof. We consider the $QFH$-module $\bar{V} = V \otimes_k \bar{k}$ where $\bar{k}$ is the algebraic closure of $k$. Notice that $\dim_k \bar{V} = \dim_k V$ and $C_{\bar{V}}(H) = C_V(H) \otimes_k \bar{k}$. Therefore once the proposition has been proven for the group $QFH$ on $\bar{V}$, it becomes true for $QFH$ on $V$ also. \hfill \square

Claim 2. We have $Q = [Q,F]$ and hence $C_Q(F) \leq Q' \leq Z(Q)$.

Proof. We may assume that $[Q,F]$ acts nontrivially on $V$. If $[Q,F] \neq Q$, then the proposition holds by induction for the group $[Q,F]FH$ on $V$. Since $[Q,F,F] = [Q,F]$ due to the coprime action of $F$ on $Q$, the conclusion of the proposition is true. This contradiction shows that $[Q,F] = Q$ and hence $C_Q(F) \leq Q'$. \hfill \square

Claim 3. $V$ is an irreducible $QFH$-module on which $Q$ acts faithfully.

Proof. Since $V$ is completely reducible as a $QFH$-module, $V = \bigoplus_{i=1}^s W_i$ for irreducible $QFH$-modules $W_i$. Suppose $s > 1$. Then we have

$$\text{Ker}(C_Q(H) \text{ on } C_{W_i}(H)) = \text{Ker}(C_Q(H) \text{ on } W_i)$$

for each $W_i$ on which $Q$ acts nontrivially by induction. This equality holds obviously also for each $W_i$ on which $Q$ acts trivially. Hence

$$\text{Ker}(C_Q(H) \text{ on } V) = \bigcap_{i=1}^s \text{Ker}(C_Q(H) \text{ on } C_{W_i}(H)) = K,$$

which is nothing but the claim of the theorem. Therefore we can regard $V$ as an irreducible $QFH$-module.

We set next $\bar{Q} = Q/\text{Ker}(Q \text{ on } V)$ and consider the action of the group $\bar{QFH}$ on $V$ assuming $\text{Ker}(\bar{Q} \text{ on } V) \neq 1$. An induction argument gives

$$\text{Ker}(C_{\bar{Q}}(H) \text{ on } C_V(H)) = \text{Ker}(C_{\bar{Q}}(H) \text{ on } V).$$

This leads to a contradiction as $C_{\bar{Q}}(H) = C_{\bar{Q}}(H)$ due to the coprime action of $H$ on $Q$. Thus we may assume that $Q$ acts faithfully on $V$. \hfill \square
It should be noted that we need only to prove $K = 1$ due to the faithful action of $Q$ on $V$. So we assume this to be false.

Claim 4. Let $\Omega$ denote the set of $Q$-homogeneous components of $V$, and let $\Omega_i$ be an $F$-orbit on $\Omega$. Set $H_1 = \text{Stab}_H(\Omega_1)$. Then $H_1$ is a nontrivial subgroup of $H$ stabilizing exactly one element $W$ of $\Omega_1$ and all the remaining orbits of $H_1$ on $\Omega_1$ are of length $|H_1|$. Furthermore $K$ acts trivially on each member of $\Omega_1$ except $W$.

Proof. Suppose that $H_1 = 1$. Pick an element $W$ from $\Omega_1$. Clearly, we have $\text{Stab}_H(W) \leq H_1 = 1$ and hence the sum $X = \sum_{h \in H} W^h$ is direct. It is straightforward to verify that $C_X(H) = \{\sum_{h \in H} v^h \mid v \in W\}$. By definition, $K$ acts trivially on $C_X(H)$. Note also that $K$ normalizes each $W^h$ as $K \leq Q$. It follows now that $K$ is trivial on $X$. Notice that the action of $H$ on the set of $F$-orbits on $\Omega$ is transitive, and hence $K$ is trivial on the whole of $V$ contrary to Claim 3. Thus $H_1 \neq 1$.

Let now $S = \text{Stab}_{FH_1}(W)$ and $F_1 = F \cap S$. Then $|F : F_1| = |\Omega_1| = |FH_1 : S|$ and so $|S:F_1| = |H_1|$. Notice next that as $(|F_1|,|H_1|) = 1$ there exists a complement, say $S_1$, of $F_1$ in $S$ with $|H_1| = |S_1|$ by Schur-Zassenhaus theorem. Therefore by passing, if necessary, to a conjugate of $W$ in $\Omega_1$, we may assume that $S = F_1H_1$, that is, $W$ is $H_1$-invariant.

It remains to show that $W$ is the only member of $\Omega_1$ which is stabilized by $H_1$, and all the remaining orbits are of length $|H_1|$: Let $x \in F$ and $1 \neq h \in H_1$ such that $(W^x)^h = W^x$ holds. Then $[h,x^{-1}] \in F_1$ and so $F_1x = F_1x^h = (F_1x)^h$ implying the existence of an element $g \in F_1x \cap C_F(h)$ by Theorem 3.27 in [1]. Now the Frobenius action of $H$ on $F$ gives that $C_F(h) = 1$ and so $x \in F_1$. This means that $\text{Stab}_H(W^x) = 1$ for each $x \in F - F_1$. Then, as a consequence of the argument in the first paragraph, $K$ acts trivially on $W^x$ for every $x \in F - F_1$. □

Claim 5. $F$ acts transitively on $\Omega$ and hence we have $H = H_1$.

Proof. The group $H$ acts transitively on $\{\Omega_i \mid i = 1,2,\ldots,s\}$, the collection of $F$-orbits on $\Omega$. Let now $V_i = \bigoplus_{W \in \Omega_i} W$ for $i = 1,2,\ldots,s$. Suppose that $H_1 = \text{Stab}_H(\Omega_1)$ is a proper subgroup of $H$. Equivalently, $s > 1$. By induction the proposition holds for the group $QFH_1$ on $V_i$, that is,

$$\text{Ker}(C_Q(\Omega_1) \mid C_{V_i}(H_1)) = \text{Ker}(C_Q(H_1) \mid V_i).$$

In particular, we have $\text{Ker}(C_Q(H) \mid C_{V_i}(H_1)) = \text{Ker}(C_Q(H) \mid V_i)$. On the other hand we observe that $C_V(H) = \{u^{x_1} + u^{x_2} + \ldots + u^{x_s} \mid u \in C_{V_i}(H_1)\}$ where $x_1,\ldots,x_s$ is a complete set of right coset representatives of $H_1$ in $H$. By definition, $K$ acts trivially on $C_V(H)$ and normalizes each $V_i$. Then $K$ is trivial on $C_{V_i}(H_1)$ and hence on $V_i$. As $K$ is normalized by $H$ we see that $K$ is trivial on each $V_i$ and hence on $V$ contrary to Claim 3. Therefore $H_1 = H$ and $F$ acts transitively on $\Omega$ as desired. □

From now on the unique $H$-invariant element of $\Omega$ the existence of which is established by Claim 4 and Claim 5 will be denoted by $W$.

Claim 6. $C_Q(F) = 1$.

Proof. Due to the coprime action of $H$ on $C_Q(F)$ and the fact that $C_Q(FH) = 1$, we have $C_Q(F) = [C_Q(F),H]$. Since $Z(Q/C_Q(W))$ acts by scalars on the homogeneous $Q$-module $W$, $Z(Q/C_Q(W))$ and $H$ commute. In particular as $C_Q(F) \leq...
$Z(Q)$ and $C_Q(F) = [C_Q(F), H]$ we see that $C_Q(F) \leq [Z(Q), H] \leq C_Q(W)$. Then
\[ C_Q(F) \leq \bigcap_{x \in F} C_Q(W)^x = C_Q(V) = 1, \]
as desired, since $F$ acts transitively on $\Omega$ by Claim 5.

**Claim 7. Final Contradiction.**

**Proof.** Since $1 \neq K \leq C_Q(H)$, the group $L = K \cap Z(C_Q(H))$ is nontrivial. Pick $1 \neq z \in L$ and consider the group $Q_0 = \langle z^F \rangle$. As $C_Q(F) = 1$ by Claim 6, we have $[Q_0, F] = Q_0$. If $Q_0 \neq Q$, the proposition holds by induction for the group $Q_0F^H$ on $V$, that is,
\[ Ker(C_{Q_0}(H) \text{ on } C_V(H)) = Ker(C_{Q_0}(H) \text{ on } V) = 1. \]
This leads to a contradiction since $z \in Ker(C_{Q_0}(H) \text{ on } C_V(H))$. Therefore $Q = Q_0$. Note that $Q = [Q, H]C_Q(H)$ as $(|Q|, |H|) = 1$. We have
\[ [Q, L, H] \leq [Q', H] \leq [Z(Q), H] \leq C_Q(W) \]
and also $[L, H, Q] = 1$ as $[L, H] = 1$. It follows now by the three subgroup lemma that $[H, Q, L] \leq C_Q(W)$. On the other hand $[C_Q(H), L] = 1$ by the definition of $L$. Thus $LC_Q(W)/C_Q(W) \leq Z(Q/C_Q(W))$ and hence $z^f \in zC_Q(W)$ for any $f \in F_1$ due to the scalar action of $Z(Q/C_Q(W))$ on $W$. Recall that $K$ acts trivially on $Wg^{-1}$ and hence $z^g \in C_Q(W)$ for any $g \in F - F_1$ by Claim 4. So we have $Q = (z)C_Q(W)$ implying that $Q' \leq C_Q(W)$. This forces that
\[ Q' \leq \bigcap_{x \in F} C_Q(W)^x = C_Q(V) = 1, \]
as $F$ acts transitively on $\Omega$ by Claim 5, that is, $Q$ is abelian.

We consider now $\prod_{f \in F} z^f$. It is a well defined element of $Q$ which lies in $C_Q(F) = 1$. Thus we have
\[ 1 = \prod_{f \in F} z^f = (\prod_{f \in F_1} z^f)(\prod_{f \in F - F_1} z^f) C_Q(W) = z^{[F_1]} C_Q(W) \]
leading to the contradiction $z \in C_Q(W)$ as $|F_1|$ is coprime to $|z|$. This completes the proof of Proposition 2.1. \qed

**Remark** Our proof uses an inductive argument in which we need to know that every subgroup of $FH$ containing $F$ satisfies the same hypothesis. The assumption $C_{C_Q(F)}(h) = 1$ for all nonidentity elements $h \in H$ is valid for any subgroup of $FH$ containing $F$ so that induction becomes possible. This property is heavily used in Claim 5. It is very natural to ask whether the proposition is true under the weaker condition $C_Q(FH) = 1$. We don’t yet know the answer.

3. The Main Theorem

In this section we prove our main result which gives a bound for the nilpotent length of solvable groups admitting a coprime Frobenius group of automorphisms under some additional hypothesis.
Theorem 3.1. Let $G$ be a finite solvable group admitting a Frobenius group of automorphisms $FH$ of coprime order with kernel $F$ and complement $H$ such that $C_{C_G(F)}(h) = 1$ for all nonidentity elements $h \in H$. Then $f([G,F]) = f(C_{[G,F]}(H))$ and $f(G) \leq f([G,F]) + 1$.

Proof. $C_G(F)$ is nilpotent by a well known result of Thompson as any element of prime order in $H$ acts fixed point freely on $C_G(F)$. Note also that $G/[G,F]$ is covered by the image of $C_G(F)$ due to the coprime action of $F$ on $G$ and so $f(G) \leq f([G,F]) + 1$. Therefore we may assume $G = [G,F]$ and prove that $f(G) = f(C_G(H))$. Let $f(G) = n$. We proceed by induction on the order of $G$.

The theorem is trivially true when $G = 1$. We assume now that the theorem is true for every group satisfying the hypothesis and of order smaller than the order of $G$. As $G = [G,F]$ and $([G],[FH]) = 1$, there exists an irreducible $FH$-tower $P_1, \ldots, P_n$ in the sense of [4] where

(a) $P_i$ is an $FH$-invariant $p_i$-subgroup, $p_i$ is a prime, $p_i \neq p_{i+1}$, for $i = 1, \ldots, n-1$;
(b) $P_i \leq N_G(P_j)$ whenever $i \leq j$;
(c) $P_n = P_0$ and $P_i = P_i/C_P(P_{i+1})$ for $i = 1, \ldots, n-1$ and $P_i \neq 1$ for $i = 1, \ldots, n$;
(d) $\Phi(\Phi(P_i)) = 1$, $\Phi(P_i) \leq Z(P_i)$, and $\exp(P_i) = p_i$ when $p_i$ is odd for $i = 1, \ldots, n$;
(e) $|\Phi(P_{i+1}), P_i| = 1$ and $[P_{i+1}, P_i] = P_{i+1}$ for $i = 1, \ldots, n-1$;
(f) $(\Pi_{j<i} P_j)FH$ acts irreducibly on $P_i/\Phi(P_i)$ for $i = 1, \ldots, n$;
(g) $P_1 = [P_1,F]$.

Set now $X = \prod_{i=1}^n \hat{P}_i$. As $P_1 = [P_1,F]$ by (g), we observe that $X = [X,F]$ and so $F$ is not contained in $Ker(FH)$ on $X$. Therefore $FH/Ker(FH)$ on $X$ is a Frobenius group of automorphisms of the group $X$. If $X$ is proper in $G$, by induction we have $f(X) = f(C_X(H))$ and so the theorem follows. Hence $X = G$. Lemma 1.3 in [2] shows that $C_G(H) \neq 1$, that is $f(C_G(H)) \geq 1$. Therefore the theorem is true if $G = F(G)$. We set next $\overline{G} = G/F(G)$. As $\overline{G}$ is a nontrivial group such that $\overline{G} = [\overline{G},F]$, it follows by induction that $f(\overline{G}) = n - 1 = f(C_G(H))$.

That is, $Y = [C_{\hat{P}_{n-1}}(H), \ldots, C_{\hat{P}_1}(H)] \not\subseteq F(G) \cap \hat{P}_{n-1} = C_{\hat{P}_{n-1}}(\hat{P}_n)$.

Note that $C_{\hat{P}_{n-1}}(H) = C_{\hat{P}_{n-1}}(H)$ as $C_{\hat{P}_{n-1}}(F(H)) = 1$. Also $[\hat{P}_{n-1}, F] \neq 1$ because otherwise $F$ centralizes $P_i$ for each $i \leq n - 1$ contradicting the fact that $P_1 = [P_1,F]$. By Proposition 2.1 applied to the action of the group $\hat{P}_{n-1}FH$ on the module $P_n/\Phi(\hat{P}_n)$ we get

$$Ker(C_{\hat{P}_{n-1}}(H) on C_{\hat{P}_n/\Phi(\hat{P}_n)}(H)) = Ker(C_{\hat{P}_{n-1}}(H) on \hat{P}_n/\Phi(\hat{P}_n)).$$

It follows now that $Y$ does not centralize $C_{\hat{P}_n}(H)$ and hence $f(C_G(H)) = n = f(G)$. This completes the proof. \hfill \square

Acknowledgement The authors would like to thank Evgeny Khukhro for pointing out an error in an earlier version of the article, and the anonymous referee for his/her valuable suggestions which made the text more readable.
References


Gülin Ercan
Department of Mathematics
Middle East Technical University
Ankara, Turkey
e-mail: ercan@metu.edu.tr

İsmail Ş. Güloğlu
Department of Mathematics
Doğuș University
İstanbul, Turkey
e-mail: iguloglu@dogus.edu.tr

Elif Öğüt
Department of Mathematics
Middle East Technical University
Ankara, Turkey
e-mail: elif.ogut@yahoo.com