Lecture 9

SUMMARY OF BASES AND DIMENSION

Exercise 1 Find the conditions on a, b, c, d, under which:

$$(a, b, c, d) \in \langle (3, -4, 1, 0), (0, 6, 1, -3) \rangle.$$

Solution

The condition is consistancy of the equation in variables t, u:

$$(a, b, c, d) = t(3, -4, 1, 0) + u(0, 6, 1, -3).$$

This yields the system with augmented matrix:

$$\left[\begin{array}{cc|c} 3 & 0 & a \\ -4 & 6 & b \\ 1 & 1 & c \\ 0 & -3 & d \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_4]{R_1 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & 1 & c \\ 0 & -3 & d \\ 3 & 0 & a \\ -4 & 6 & b \end{array}\right] \xrightarrow[4 \cdot R_1 + R_4]{(-3) R_1 + R_3}$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & c \\ 0 & -3 & d \\ 0 & -3 & a + (-3)c \\ 0 & 10 & b + 4c \end{array}\right] \xrightarrow[(\frac{10}{3}) R_2 + R_4]{(-1) R_2 + R_3} \left[\begin{array}{cc|c} 1 & 1 & c \\ 0 & -3 & d \\ 0 & 0 & a + (-3)c + (-1)d \\ 0 & 0 & b + 4c + \frac{10}{3} d \end{array}\right].$$

We conclude that

$a - 3c - d = 0$ and $b + 4c + \frac{10}{3} d = 0$ are the conditions on (a, b, c, d) to be an element of $\mathrm{span}((3, -4, 1, 0), (0, 6, 1, -3))$. That is

$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is a solution of the homogeneous system:

$$\begin{cases} a - 3c - d = 0 \\ b + 4c + \frac{10}{3} d = 0 \end{cases} \Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3c - d \\ -4 - \frac{10}{3} d \\ c \\ d \end{bmatrix}.$$

Exercise 2 Show that the set

$B = \{(1,2,-1), (-3,1,0), (2,-3,2)\}$ is a basis for $\mathbb{R}^3$

Solution 1 Construct the matrix whose rows are vectors given above.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & 1 & 0 \\ 2 & -3 & 2 \end{bmatrix}$$

This matrix is invertible since

$|A| = 1(2-0) - 2(-6-0) - 1(9-2) = 2 + 12 - 9 = 5 \neq 0.$

Thus the set B is a **basis** for $\mathbb{R}^3$.

Solution 2 Consider the linear system

$$(a, b, c) = x_1(1,2,-1) + x_2(-3,1,0) + x_3(2,-3,2).$$

Its augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & a \\ 2 & 1 & -3 & b \\ -1 & 0 & 2 & c \end{array}\right] \xrightarrow[R_1+R_3]{(-2)R_1+R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & a \\ 0 & 7 & -7 & b-2a \\ 0 & -3 & 4 & c+a \end{array}\right] \xrightarrow[\frac{1}{3}R_3]{\frac{1}{7}R_2}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 2 & a \\ 0 & 1 & -1 & \frac{b-2a}{7} \\ 0 & -1 & \frac{4}{3} & \frac{c+a}{3} \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & a \\ 0 & 1 & -1 & \frac{b-2a}{7} \\ 0 & 0 & \frac{1}{3} & \frac{c+a}{3} + \frac{b-2a}{7} \end{array}\right]$$

Thus the system is consistent and hence every vector $(a,b,c) \in \mathbb{R}$ belongs to $\langle B \rangle$. Since B has three elements then B is a basis for $\mathbb{R}^3$.

Exercise 3 Find $\dim(\langle (10201), (21412), (11211) \rangle)$

Solution
$$\begin{bmatrix} 1&0&2&0&1 \\ 2&1&4&1&2 \\ 1&1&2&1&1 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1&0&2&0&1 \\ 1&1&2&1&1 \\ 1&1&2&1&1 \end{bmatrix} \rightarrow \begin{bmatrix} 1&0&2&0&1 \\ 1&1&2&1&1 \\ 0&0&0&0&0 \end{bmatrix} \rightarrow \begin{bmatrix} 1&0&2&0&1 \\ 0&1&0&1&0 \\ 0&0&0&0&0 \end{bmatrix}$$

The system has two basic variables hence the dimension of rows of the coefficient matrix is two. (row operations do not change the dimension of rows)

Consider an $m \times n$-matrix A. It consists of m rows $\{R_1, R_2, \dots R_m\} \subseteq \mathbb{R}^n$ and of n colums $\{C_1, C_2, \dots, C_n\} \subseteq \mathbb{R}^m$

<u>Theorem 1.</u> $\dim\{R_1, R_2, \dots R_m\} = \dim\{C_1, C_2, \dots C_n\}$ or $\dim(\text{row}(A)) = \dim(\text{column}(A))$. ▨

This number is called the <u>rank</u> of A.

<u>Ex 4</u> (Example 4.5.2 on p 176)

Show that the vectors $(1,-1,2), (2,1,1), (0,3,-3)$ are linearly dependent.

<u>Solution</u> It is enough to show that $\text{rank}(A) < 3$ where

$$A = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 0 & 3 & -3 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{vmatrix}$$

$\text{rank}(A) = 2 < 3$.

<u>Ex 5</u> Find a basis for $\langle 1+x^2, 1+x+x^3, 1+x^2+x^3+x^4, -1+x^2+x^4, x^3+x^4 \rangle$

<u>Solution</u> Consider the correspondence $c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \to (c_0, c_1, c_2, c_3, c_4)$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \\ \end{matrix} \Big\} \text{ A basis}$$

A basis is $\{1+x^2, x - x^2 + x^3, 2x^2 + x^3 + 2x^4, x^3 + x^4\}$

Exercise 6 Describe the subspace

$$Y = \left\langle \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \right\rangle$$

of $M^{2\times 2}$ by means of homogeneous system.

Solution A matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Y$ iff the system (*) with augmented matrix A is consistent:

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 2 & b \\ 1 & 2 & 2 & c \\ -1 & -2 & 3 & d \end{array}\right] \xrightarrow[1\cdot R_1 + R_4]{(-1)R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 2 & b \\ 0 & 0 & 1 & c-a \\ 0 & 0 & 4 & d+a \end{array}\right] \longrightarrow$$

$$\xrightarrow{(-4)R_3 + R_4} \left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 2 & b \\ 0 & 0 & 1 & c-a \\ 0 & 0 & 0 & d + a - \frac{c-a}{4} \end{array}\right].$$

It is consistent iff $d + a - \frac{c-a}{4} = 0$, ie $\frac{3}{4}a - \frac{1}{4}c + d = 0$.

Thus the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in Y iff $3a - c + 4d = 0$.

Ex 7 (Ex 4.5.3 p 178) Find a basis for the space $\mathcal{P}^3$ containing x and $x + x^2$.

Solution We know that $\{1, x, x^2, x^3\}$ is a basis for $\mathcal{P}^3$. Consider the generating set $\{x, x+x^2, 1, x^2, x^3\}$. Its matrix is

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A

Thus 1^{st}, 2^{nd}, 3^{d}, and 5^{th} columns of A form a basis. That is

$$\{x, x+x^2, 1, x^3\}$$

Exercise 8 Find a basis for a subspace spanned by 4×1 - matrices:

$$X_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 3 \end{bmatrix}, \quad X_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Solution

We shall form the 4×5 - matrix

$$A = \begin{bmatrix} -1 & 2 & 1 & 3 & 1 \\ 2 & 1 & 3 & 4 & 0 \\ 1 & -1 & 0 & -1 & 0 \\ 3 & 0 & 3 & 3 & 1 \end{bmatrix}$$

and construct the echelon matrix:

$$A \xrightarrow[3R_1 + R_4]{\substack{2R_1 + R_2 \\ R_1 + R_3}} \begin{bmatrix} -1 & 2 & 1 & 3 & 1 \\ 0 & 5 & 5 & 10 & 2 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 6 & 6 & 12 & 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} -1 & 2 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 5 & 5 & 10 & 2 \\ 0 & 6 & 6 & 12 & 4 \end{bmatrix} \xrightarrow[(-6)R_2 + R_4]{(-5)R_2 + R_3}$$

$$\rightarrow \begin{bmatrix} -1 & 2 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{\frac{2}{3}R_3 + R_4} \begin{bmatrix} -1 & 2 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{1}{\uparrow} \quad \underset{2}{\uparrow} \qquad\qquad \underset{5}{\uparrow}$$

Thus the columns with leading entries are 1-st, 2nd, and 5-th columns, therefore a basis can be obtained by <u>deleting</u> vectors X_3 and X_4.

Thus, a basis is $\{X_1, X_2, X_5\}$.

Exercise 9 Find a basis for

$Y = \text{span}\big((1,-1,1,0,1),\ (2,1,-1,1,1),\ (0,3,-3,1,-1),$
$(1,0,1,0,1)\big).$

Solution Form the matrix as in Ex 8 and construct the echelon matrix:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 3 & 0 \\ 1 & -1 & -3 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix} \xrightarrow[(-1)R_1+R_5]{\substack{R_1+R_2 \\ (-1)R_1+R_3}} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 3 & 1 \\ 0 & -3 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \xrightarrow[R_4+R_5]{\substack{(-3)R_4+R_2 \\ 3R_4+R_3}} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, the only columns with leading entries are 1-st, 2nt, and 4-th.

Thus, a basis is:

$\{(1,-1,1,0,1),\ (2,1,-1,1,1),\ (1,0,1,0,1)\}$

Ex 10 Find a basis for the solution space of

$\begin{cases} x+y+z-t=0 \\ 2x+2y-z+t=0 \end{cases}$. Solution: $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 2 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & -3 & 3 \end{bmatrix}$ y and t are free

Fundamental solutions are 1) $y=1, t=0$ $\begin{cases} x+1+z-0=0 \\ z=0 \end{cases}$ $X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

2) $y=0, t=1$ $\begin{cases} x+0+z-1=0 \\ z=1 \end{cases}$ $X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

Hence a basis for solution space is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Exercise 11.

Show that if u, v, and w are linearly independent vectors then $u+v$, $u-v$, and $u+v+w$ are linearly independent.

Solution

The set $B=\{u, v, w\}$ is a basis for $X=\text{span}(u, v, w)$.

Coordinates of $x_1=u+v$, $x_2=u-v$, $x_3=u+v+w$ are $(1,1,0)$, $(1,-1,0)$, and $(1,1,1)$.

The matrix $A=\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is invertible

Since $\det(A)=(-1)^{3+3}\cdot 1\cdot \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0.$

Thus $\{x_1, x_2, x_3\}=C\subseteq X$ is a basis for X.

Exercise 12.

Let $A=\begin{bmatrix} a_{11}, \dots, a_{1n} \\ \dots\dots\dots \\ a_{n1}, \dots a_{nn} \end{bmatrix}$ be an $n\times n$-matrix such that $\sum_{j=1}^{n} a_{kj}=0$ for every k. Show that $|A|=0$.

Solution Take the sum of columns vectors $\begin{bmatrix} a_{1k} \\ a_{2k} \\ \dots \\ a_{nk} \end{bmatrix}$ and get zero vector $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. These columns are linearly dependent. Hence $\det(A)=0$.

Exercise 13.

Show that
$$\begin{vmatrix} 0 & * & 0 & * & \dots & * & 0 \\ * & 0 & * & 0 & \dots & 0 & * \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & * & 0 & * & \dots & * & 0 \end{vmatrix} = 0,$$
where we denote by $*$'s arbitrary numbers.

Solution:

This is a $(2k+1)\times(2k+1)$-matrix.

The collection of all its odd rows is linearly dependent (as any collection of $k+1$ vectors in any k-dimensional space).

Then the collection of all rows of our matrix is linearly dependent.

Hence, the determinant is zero.