Lecture 8

LINEAR INDEPENDENCE. LINEAR DEPENDENCE. BASES. DIMENSION.

Df Let X be a linear space and let $Y \subseteq X$. We say that Y is <u>linearly dependent</u> if there exist <u>non-zero</u> scalars $c_1, c_2, \dots, c_n$ and non-zero vectors $x_1, x_2, \dots, x_n \in Y$ satisfying

$$c_1 \cdot x_1 + c_2 \cdot x_2 + \dots + c_n \cdot x_n = 0 .$$

Y is <u>linearly independent</u> if Y is not linearly dependent.

<u>Example 1</u> The homogeneous system $AX=0$ or

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ -3x_1 + 2x_2 + 5x_3 = 0 \\ 2x_1 - x_2 - 3x_3 = 0 \end{cases}$$

has a non-zero solution since $A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & 2 & 5 \\ 2 & -1 & -3 \end{bmatrix}$ is not invertible because of $|A| = 0$. Therefore vectors $(1, -3, 2)$, $(2, 2, -1)$, and $(1, 5, -3)$ of $\mathbb{R}^3$ are linearly dependent. Indeed, let $C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ be such that $AC = 0$.

Then $c_1 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} = 0.$

Ex 2 The functions $1, \cos x, \cos 2x, \cos^2 x$ are linearly dependent. Indeed $1+\cos 2x - 2\cos^2 x = 0$.

Ex 3 Empty set is linearly independent. The set $\{0\}$ which consists of zero is linearly dependent.

Ex 4 The set $\{1, x, x^2, \dots x^n, \dots\}$ is linearly independent.

Theorem 1. Let $V = \langle v_1, v_2, \dots, v_m \rangle$. Any linearly independent $S \subseteq V$ cannot contain more than m vectors.

Proof: Let $S = \{w_1, \dots w_n\}$, $n > m$. Then

$$w_1 = A_{11} v_1 + A_{21} v_2 + \dots + A_{m1} v_m$$
$$w_2 = A_{12} v_1 + A_{22} v_2 + \dots + A_{m2} v_m$$
$$\dots$$
$$w_n = A_{1n} v_1 + A_{2n} v_2 + \dots + A_{mn} v_m$$

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & & & \dots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix}.$$

In matrix form $[w_1 w_2 \dots w_n] = [v_1 v_2 \dots v_m] A$.

Since $n > m$ the homogeneous system $AX = 0$ has a nonzero solution, say $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, that is $AC = 0$.

Hence

$$c_1 w_1 + c_2 w_2 + \dots + c_n w_n = [w_1 w_2 \dots w_n] \cdot C = [v_1 v_2 \dots v_m] \cdot A \cdot C = 0.$$

Thus $S = \{w_1, w_2, \dots, w_n\}$ is linearly dependent. ■

Ex 5 The five polynomials $-2+3x-x^3$, $x+x^2$, $2x-3x^3$, $7+4x^3$, and x^2+6x^3 are linearly dependent since they belong to the span $\langle 1, x, x^2, x^3 \rangle$ of four vectors.

Ex 6 $\{(1,-2,4),(0,1,2),(0,0,-3)\}$ is linearly independent since the matrix $A=\begin{bmatrix}1&0&0\\-2&1&0\\4&2&-3\end{bmatrix}$ is invertible. Therefore there is no set of two vectors in $\mathbb{R}^3$ such that $\mathbb{R}^3=\langle y_1,y_2\rangle$.

Def Let V be a vector space. A subset $B\subseteq V$ is called a basis for V if

1) $V=\operatorname{span}(B)$, and
2) B is linearly independent.

Theorem 2. In the vector space $\mathbb{R}^n$ n vectors $S=\{(x_{11},\dots x_{1n}),(x_{21},\dots x_{2n}),\dots,(x_{n1},\dots x_{nn})\}$ form a basis iff the matrix

$$G=\begin{bmatrix}x_{11}&x_{21}&\dots&x_{n1}\\x_{12}&x_{22}&\dots&x_{n2}\\\vdots&&&\\x_{1n}&\dots&&x_{nn}\end{bmatrix}$$

is invertible.

Proof: Assume that S is a basis for $\mathbb{R}^n$. Then the homogeneous system $GX=0$ has no non-zero solutions. Thus G is invertible.

Assume that G is invertible. Then for any vector $[a_1\ a_2\dots a_n]\in\mathbb{R}^n$ there exists $[b_1,\dots b_n]$ such that $[a_1\ a_2\dots a_n]=[b_1\ b_2,\dots b_n]G=$
$=b_1[x_{11}\ x_{12}\dots x_{1n}]+b_2[x_{21}\ x_{22}\dots x_{2n}]+\dots+b_n[x_{n1}\ x_{n2}\dots x_{nn}]$

That is $\mathbb{R}^n=\langle S\rangle$. Suppose S is not linearly independent, then we may find a proper subset $S_1\subseteq S$ containing $<n$ vectors such that $\mathbb{R}^n=\langle S\rangle=\langle S_1\rangle$ which is impossible by Theorem 1 since $\mathbb{R}^n$ contain n linearly independent vectors. Thus S must be linearly independent. Hence S is a basis for $\mathbb{R}^n$. ▨

Standard Bases for some vector Spaces.

1) $X = \mathbb{R}^n$.

$$e_1 = (1, 0, \dots, 0)$$
$$e_2 = (0, 1, \dots, 0)$$
$$\dots$$
$$e_n = (0, 0, \dots, 1)$$

is a basis of rows of the identity matrix.

2) $X = M^{m\times n}$.

The space $M^{m\times n}$ is linearly isomorphic to $\mathbb{R}^{m\cdot n}$, say $f : M^{m\times n} \to \mathbb{R}^{m\cdot n}$.
Take the basis $e_1, e_2, \dots, e_{mn}$ from 1).
The set $f^{-1}(e_1), f^{-1}(e_2), \dots, f^{-1}(e_{mn})$ is called the standard basis in $M^{m\times n}$.

3) Polinomials.
The infinite set $\{1, x, x^2, \dots, x^n, \dots\}$ is the standard basis for $\mathcal{P}(\mathbb{R})$.

4) Let A be an $m\times n$-matrix. Then the set S of all solutions of the homogeneous system $AX = 0$ (if non-empty) is a vector subspace of $\mathbb{R}^n$.
Consider the fundamental solutions
$\xi_1 = (\xi_{11}, \xi_{12}, \dots, \xi_{1n}), \dots, \xi_r = (\xi_{r1}, \xi_{r2}, \dots, \xi_{rn})$.
Then $(\xi_1, \xi_2, \dots, \xi_r)$ is a basis of S.

5) By the Axiome of Choice, it can be proved that <u>every</u> vector space possesses a basis. In this course we study mostly vector spaces which are linearly isomorphic to $\mathbb{R}^n$ for some n and have finite bases.

Theorem 3. Let X be a vector space spanned by m vectors $x_1, x_2, \ldots, x_m$. Then

a) X has a basis and it can be obtained by deleting vectors from $\{x_1, x_2, \ldots, x_m\}$ which are linear combination of their predecessors.

b) Any two bases for X have the same number of elements that is $\leq m$. ▨

(Df) The number of elements of any basis of X is called the **dimension** of X and is denoted by $\dim(X)$. If there is no finite basis then X is called infinite dimensional ($\dim(X) = \infty$ for short).

Corollary Let X be a finite dimensional vector space, then:

a) for any subspace $Y \subseteq X$, $\dim(Y) \leq \dim(X)$.

b) If $\dim(X) = n$ then every set of n linearly independent vectors of X is a basis.

c) If $\dim(X) = n$ and

$$X = \mathrm{span}(x_1, x_2, \ldots, x_n)$$

then $\{x_1, x_2, \ldots, x_n\}$ is a basis. ▨

Ex Calculate $\dim(L^{2\times 2})$. The matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis in $L^{2\times 2} = \left\{ \begin{bmatrix} a & 0 \\ c & b \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$. Hence $\dim(L^{2\times 2}) = 3$.

Examples

1. Find a basis for the space spanned by $x,\ 1+x,\ 2+x,\ 2+x^2$.

The sets $\{x\}$ and $\{x, 1+x\}$ are linearly independent, but the set $\{x, 1+x, 2+x\}$ is linearly dependent since $2+x = (2)(1+x) + (-1)\cdot x$. So we delete $2+x$.

The set $\{x, 1+x, 2+x^2\}$ is linearly independent, therefore this set is a basis for $\mathrm{Span}(x, 1+x, 2+x, 2+x^2)$.

2. Show that the set $\{(1,1,1), (1,-1,1), (2,0,1)\}$ is a basis in $\mathbb{R}^3$.

Form a matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$.

$\det(A) = 2(1+1) + 1(-1-1) = 2 \neq 0$

Therefore this set is a basis in $\mathbb{R}^3$

3. Whether or not $S = \{(1,1,1), (1,1,0), (0,0,1)\}$ is a basis for $\mathbb{R}^3$?

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \det(A) = (-1)^{3+3}(1-1) = 0$.

Therefore S is not a basis for $\mathbb{R}^3$. Indeed,

$1\cdot(1,1,1) + (-1)\cdot(1,1,0) + (-1)\cdot(0,0,1) = 0$.

4. Prove that $\{(1,1,1), (1,1,0), (1,0,0)\}$ is a basis for $\mathbb{R}^3$.

Indeed, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad \det(A) = 1 \neq 0$

Theorem 4. Let X be a vector space and let $B = \{x_1, x_2, \ldots, x_n\}$. Then B is a basis for X iff every element of X can be written **uniquely** in the form

$$x = c_1 \cdot x_1 + c_2 \cdot x_2 + \cdots + c_n \cdot x_n. \quad \blacksquare$$

(Df) If $B = \{x_1, \ldots, x_n\}$ is a basis in X then

$$[x]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad x \in X.$$

is called the **coordinate** matrix of x in the basis B if $x = c_1x_1 + c_2x_2 + \ldots + c_nx_n$.

Theorem 5. Let $B = \{v_1, v_2, \ldots, v_n\}$ and $C = \{w_1, w_2, \ldots, w_n\}$ be two bases for a vector space X. Then there is a unique invertible matrix P of the size $n \times n$ such that $[x]_C = P[x]_B$ $(x \in X)$. The matrix $P = \left[[v_1]_C, [v_2]_C, \ldots [v_n]_C\right]$. $\blacksquare$

(Df) The matrix P is called the **transition matrix from B to C**.

Example. Find the transition matrix from

$$B = \{\underset{(v_1)}{(1,1,1)}, \underset{(v_2)}{(1,1,0)}, \underset{(v_3)}{(1,0,0)}\} \text{ to}$$

$$C = \{(1,0,0), (0,1,0), (0,0,1)\}.$$

$$P = \left[[v_1]_C \ [v_2]_C \ \ldots \ [v_n]_C\right] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Example Find the transition matrix from $B = \{(1,1,1), (1,1,0), (1,0,0)\}$ to $C = \{(1,1,1), (1,-1,1), (1,0,0)\}$.

$$P = \left[[v_1]_C \ [v_2]_C \ [v_3]_C \right]$$

$$[v_1]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[v_2]_C = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$(1,1,0) = a(1,1,1) + b(1,-1,1), + c(1,0,0)$$

$$\begin{cases} a + b + c = 1 \\ a - b = 1 \\ a + b = 0 \end{cases} \Rightarrow 2a = 1, \quad a = \frac{1}{2}$$

$$b = -a = -\frac{1}{2}$$

$$\frac{1}{2} - \frac{1}{2} + c = 1, \quad c = 1$$

$$[v_3]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore
$$P = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Ex 4.4.1 (a), (b) p 168

Solve exercises on p 185–195