Lecture 7

Vector spaces

Motivation. The idea of the concept of a vector space is the closedness under the linear operations which are addition and scalar multiplication. More precisely:

Definition A vector space (= linear space) is a 4-tuple $(X, +, \cdot, 0)$ where X is a set whose elements are called vectors, „+" is a function $X \times X \xrightarrow{+} X$ which is called the addition of two vectors, „$\cdot$" is a function $\mathbb{R} \times X \xrightarrow{\cdot} X$ which is called the scalar multiplication, and $0 \in X$ which is called the zero vector of X. The 4-tuple $(X, +, \cdot, 0)$ must satisfy the following properties

1) $x+(y+z)=(x+y)+z$ for all $x,y,z \in X$ [the associativity of addition].
2) $x+y = y+x$ for all $x,y \in X$ [the commutativity].
3) $c(x+y) = cx+cy$ for all $x,y \in X$; $c \in R$.
4) $(c_1+c_2)x = c_1x + c_2x$
5) $(c_1 \cdot c_2)x = c_1(c_2x)$ $\Big\}$ for all $x \in X$; $c_1, c_2 \in R$.
6) $1 \cdot x = x$ and $x+0$ for all $x \in R$.
7) For each $x \in X$, there exists $(-x) \in X$ s.t. $(-x)+x = 0$.

Elementary properties of vector spaces

a) A vector $(-x)$ which is called the <u>negative</u> of x is <u>uniquely determined by x</u>.

<u>Proof</u>: Suppose $x' + x = 0$. Add $(-x)$:

$(-x) + (x' + x) = (-x) + 0$. By 6) and 2):

$(x' + x) + (-x) = (-x)$. By 1) $x' + (x + (-x)) = (-x)$.

By 7) and 6): $x' = x' + 0 = (-x)$. ▨

b) $\boxed{c \cdot 0 = 0 \cdot x = 0}$

<u>Proof</u>: $c \cdot 0 \overset{\text{by } 6)}{=} c(0+0) \overset{\text{by } 3)}{=}$

$= c \cdot 0 + c \cdot 0$. Add $(-c0)$:

$(-c0) + c0 = (-c0) + c0 + c \cdot 0$. Hence

$0 = c \cdot 0$.

$0 \cdot x = (0+0)x \overset{\text{by } 4)}{=} 0 \cdot x + 0 \cdot x$. Add $(-(0 \cdot x))$:

$(-(0 \cdot x)) + 0 \cdot x = (-(0 \cdot x)) + 0 \cdot x + 0 \cdot x$. Hence $0 = 0 \cdot x$. ▨

c) $\boxed{(-1)x = -x}$

<u>Proof</u>: Accordingly to a) we must show that $(-1)x$ is the negative of x. Indeed:

$(-1)x + x \overset{\text{by } 6)}{=} (-1)x + 1 \cdot x \overset{\text{by } 4)}{=} (-1+1)x = 0 \cdot x \overset{\text{by } b)}{=} 0$. ■

d) $\boxed{(-c)\cdot x = c(-x) = -(c\cdot x)}$

Proof: Accordingly to a) we must prove that $(-c)\cdot x + c\cdot x = 0 = c\cdot(-x) + c\cdot x$. Indeed:

$$(-c)\cdot x + c\cdot x \overset{\text{by } 4)}{=} (-c+c)x = 0\cdot x \overset{\text{by } 8)}{=} 0, \text{ and}$$

$$c(-x) + c\cdot x \overset{\text{by } 3)}{=} c\cdot(-x+x) \overset{\text{by } 7)}{=} c\cdot 0 \overset{\text{by } 8)}{=} 0. \blacksquare$$

Examples of vector spaces

Ex 1 The vector space $\mathbb{R}^{m\times n}$ of all $m\times n$ matrices with natural matrix addition and scalar multiplication

Ex 2 The vector space $\mathbb{R}^n$ of all n-tuples $(x_1, x_2, \ldots x_n)$ of real scalars with the coordinatewise addition and scalar multiplication.

Remark that $\mathbb{R}^n$ can be naturaly identified with $\mathbb{R}^{1\times n}$ by

$$\mathbb{R}^n \ni (x_1, x_2, \ldots x_n) \longleftrightarrow [x_1, x_2, \ldots x_n] \in \mathbb{R}^{1\times n}$$

(Such an identification of vector spaces that preserves linear operations is called a linear isomorphism)

Ex 3 The vector space $L^{2\times 2}$ of all low-triangular 2x2-matrices $\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}$ is linear isomorphic with R^3. An isomorphism can be defined as follows $L^{2\times 2} \xrightarrow{T} \mathbb{R}^3$:

$$T\left(\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}\right) = (a, b, c)$$

In certain sence there are no difference between $L^{2\times 2}$ and $\mathbb{R}^3$. Note that the property of vector spaces to be linearly isomorphic is obviously an equivalence relation.

Ex 4 The space $\mathcal{F}(D)$ of all real-valued functions with the domain $D \neq \emptyset$ is a vector space under linear operations:

$$(f+g)(t) = f(t) + g(t) \qquad (f, g \in \mathcal{F}(D), t \in D)$$

$$(c \cdot f)(t) = c \cdot f(t) \qquad (f \in \mathcal{F}(D), c \in \mathbb{R}, t \in D)$$

Remark that when $D = \{1, 2, \dots, n\}$ then $\mathcal{F}(D)$ is obviously linear isomorphic to $\mathbb{R}^n$.

Ex 5 The space $\mathcal{P}(D)$ of all real polynomials

$$p(t) = a_0 + a_1 x + \dots + a_n x^n \quad (n \geq 0, x \in D)$$

with the domain $\emptyset \neq D \subseteq \mathbb{R}$ and the same vector addition and the scalar multiplication as in the previous example.

Ex 6 The space $\mathcal{P}_n(D)$ of all real polynomials on $\emptyset \neq D \subseteq \mathbb{R}$ of degree $\leq n$.
Remark that $\mathcal{P}_n(\mathbb{R})$ is linear isomorphic to $\mathbb{R}^{n+1}$ by $\mathcal{P}_n(\mathbb{R}) \xrightarrow{T} \mathbb{R}^{n+1}$ defined by

$$T(a_0 + a_1 x + \dots + a_n x^n) = (a_0, a_1, \dots a_n).$$

Ex 7 The set $Inv^{n\times n} \subseteq \mathbb{R}^{n\times n}$ of all invertible $n\times n$ matrices with the operation $\oplus$ of matrix multiplication, usual scalar multiplication and the „zero" element I, is not a vector space. That is the 4-tuple $(Inv^{n\times n}, \oplus, \cdot, I)$ is not a vector space. Indeed the commutativity axiome 2) is violated for instance.

Ex 8 The set of all square matrices is not a vector space under the matrix operations. Indeed the sum of two matrices of different size is undefined

Ex 9 The space $C[0,1]$ of all continuous real-valued functions on $[0,1]$ is a vector space under usual pointwise linear operations.

Ex 10 The set S_A of all solutions X of a homogeneous system $AX=0$ a vector space under natural addition and scalar multiplication of solutions.

Subspaces

Df Let X be a vector space and let $Y \subseteq X$ be a nonempty subset of X. Then Y is called a **subspace** of X if: for any elements $y_1, y_2 \in Y$ and for any scalars $c_1, c_2 \in \mathbb{R}$, we have

$$\boxed{c_1 \cdot y_1 + c_2 \cdot y_2 \in Y}.$$

Example 11 The set $L^{n\times n}$ of all low-triangular $n\times n$-matrices is a subspace of $\mathbb{R}^{n\times n}$.

Example 12 The set $D^{n\times n}$ of all diagonal $n\times n$-matrices is a subspace of $L^{n\times n}$.

Example 13 The set of all 2×2-matrices with nonnegative entries is closed under the addition, but is not closed under the scalar multiplication. Hence it is not a subspace of $\mathbb{R}^{2\times 2}$.

Theorem 1 Let V be a vector space and $W \subseteq V$. Then W is a subspace of V iff $W \neq \emptyset$, $W+W \subseteq W$, and $\mathbb{R}\cdot W \subseteq W$.

Proof: It is just reformulation of the definition above. ∎

Example 14 The set $\mathcal{P}_n[0,1]$, of all polynomials of degree less than or equal to n, is a subspace of the space $\mathcal{P}[0,1]$ of all polynomials on $[0,1]$. It can be shown that $\mathcal{P}_n[0,1]$ is linear isomorphic to $\mathbb{R}^{n+1}$.

Example 15 $\mathcal{P}[0,1]$ is a subspace of the space $C[0,1]$ of all continuous real-valued functions on $[0,1]$.

Example 16 $C[0,1]$ is a subspace of the space $\mathcal{F}[0,1]$ of all $\mathbb{R}$-valued functions defined on $[0,1]$.

Example 17 The set of all solutions of a homogeneous system $AX=0$ is a subspace of $\mathbb{R}^{n\times 1}$, where A is an $m\times n$-matrix.

Df For a nonempty set $S\subseteq X$, the collection of all linear combinations:

$$c_1 x_1 + c_2 x_2 + \dots + c_r x_r$$

for all $x_1,\dots,x_r\in S$ and $c_1,\dots,c_r\in\mathbb{R}$ is called the **linear span** (or just the **span** of S). It is denoted by $\langle S\rangle$ (or $\mathrm{Span}(S)$, or $\mathrm{span}(S)$). S is called a set of generators of $\langle S\rangle$.

Theorem 2 For any $\emptyset\neq S\subseteq X$, the span $\langle S\rangle$ is a subspace of X.

Proof: Apply Theorem 1. $\blacksquare$

Ex 18 Let $S \subseteq M^{n\times n}$ be a set of all $n\times n$-matrices with only a one non-zero entry A_{ij} for $i \le j$. Then $\langle S \rangle = U^{n\times n}$, that is the space of all $n\times n$ **upper-triangular** matrices.

Ex 19 Find the conditions satisfied by a, b, and c when (a, b, c) is a linear combination of vectors $(1,-2,1), (2,1,-2)$, and $(4,-2,1)$ in $\mathbb{R}^3$.

Solution To be a linear combination, (a,b,c) have to satisfy:

$$(a,b,c) = x(1,-2,1) + y(2,1,-2) + z(4,-2,1)$$

Thus the system

$$\begin{cases} x + 2y + 4z = a \\ -2x + y - 2z = b \\ x - 2y + z = c \end{cases}$$

must be consistent

$$\det(A) = \begin{vmatrix} 1 & 2 & 4 \\ -2 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix} = 1\cdot(1-4) - 2(-2+2) + 4(4-1) = 9 \neq 0$$

Therefore the system $AX = B$ is consistent for all $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Thus (a,b,c) is <u>always</u> in $\langle (1,-2,1), (2,1,-2), (4,-2,1) \rangle$.

In other wards $\langle (1,-2,1), (2,1,-2), (4,-2,1) \rangle = \mathbb{R}^3$.

Ex 20 Describe the subspace S of $\mathbb{R}^{2\times 2}$:

$$S = \left\langle \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 6 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 9 & 5 \end{bmatrix} \right\rangle$$

Solution

$\begin{bmatrix} a & b \\ d & c \end{bmatrix} \in S$ iff the linear system

$$\begin{aligned} x + 4y + 3z &= a \\ 2x + 4y + 6z &= b \\ 5z &= c \\ 3x + 6y + 9z &= d \end{aligned}$$

is consistent. Consider the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & a \\ 2 & 4 & 6 & b \\ 0 & 0 & 5 & c \\ 3 & 6 & 9 & d \end{array}\right] \xrightarrow{-\frac{3}{2}R_2 + R_4} \left[\begin{array}{ccc|c} 1 & 4 & 3 & a \\ 2 & 4 & 6 & b \\ 0 & 0 & 5 & c \\ 0 & 0 & 0 & d - \frac{3}{2}b \end{array}\right] \xrightarrow{-2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 4 & 3 & a \\ 0 & -4 & 0 & b - 2a \\ 0 & 0 & 5 & c \\ 0 & 0 & 0 & d - \frac{3}{2}b \end{array}\right]$$

Thus $d = \frac{3}{2}b$ and hence S can be described by three parameters: a, b, and c:

$$S = \left\{ \begin{bmatrix} a & b \\ \frac{3}{2}b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Solve exercises on pages 144 – 147:

For example: Ex 6, Ex 10(b), Ex 11, Ex 12, Ex 13 (b) and Ex 14.