Lectures 5-6

Determinants. Properties of Determinants.

In this and in the following lecture we study real functions defined on square matrices. As an example of such a function we consider firstly the trace:

$$tr(A) = A_{11} + A_{22} + \dots + A_{nn}$$

Theorem 1 (Properties of the trace function)

a) $tr(A+B) = tr(A) + tr(B)$ and $tr(aA) = a \cdot tr(A)$

b) $tr(A^T) = tr(A)$

c) $tr(AB) = tr(BA)$

d) $tr(I) = n$ where I is the $n \times n$-identity matrix

Proof: Properties a), b), and d) are obvious. For c):

$$tr(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} A_{ik} B_{ki} =$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n} B_{ki} A_{ik} = \sum_{k=1}^{n} (BA)_{kk} = tr(BA) \quad \blacksquare$$

Another important function of square matrices is the <u>determinant</u> function. There are many equivalent definitions of the determinant. Let us use the <u>following one</u>

<u>Definition</u> (inductive)

1) $D([A_{11}]) = \det([A_{11}]) = |A_{11}| = A_{11}$

2) $D(A) = \sum_{j=1}^{n} (-1)^{1+j} \cdot A_{1j} \cdot D(\tilde{A}_{1j})$ for an $n\times n$-matrix A, where $\tilde{A}_{ij}$ is the $(n-1)\times(n-1)$-matrix obtained from A by <u>deleting</u> of <u>i-th row</u> and <u>j-th column</u>. $\tilde{A}_{ij}$ is called the ij-th <u>minor</u> of A.

<u>Examples</u> $\underset{3\times3}{A}$, $D(A) = \det(A) = |A| =$

$$= (-1)^{1+1} A_{11} |\tilde{A}_{11}| + (-1)^{1+2} A_{12} |\tilde{A}_{12}| + (-1)^{1+3} |\tilde{A}_{13}| =$$

$$= A_{11} \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{12} \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13} \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} =$$

$$= A_{11}\left((-1)^{1+1} A_{22}A_{33} + (-1)^{1+2} A_{23}A_{32}\right) - A_{12}\left(A_{21}A_{33} - A_{23}A_{31}\right) + A_{13}\left(A_{21}A_{32} - A_{22}A_{32}\right)$$

$$\begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 2\cdot(4-3) - (6-4) = 2-2 = 0$$

$$\begin{vmatrix} 2 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{vmatrix} = 2\begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 5 & 7 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 2(21-24) - 2(14-20) + 3(12-15) = -6+12-9 = -3$$

Theorem 2 For any $1 \le j_0, i_0 \le n$

a) $$\det(A) = \sum_{i=1}^{n} (-1)^{i+j_0} A_{ij_0} \det(\tilde{A}_{ij_0})$$

This sum is called the expansion of $\det(A)$ by the j_0-th column

b) $$\det(A) = \sum_{j=1}^{n} (-1)^{i_0+j} A_{i_0 j} \det(\tilde{A}_{i_0 j})$$

This sum is called the expansion of $\det(A)$ by the i_0-th row.

Examples

$$\begin{vmatrix} 3 & 2 & 0 & 1 & 3 \\ -2 & 4 & 1 & 2 & 1 \\ 0 & -1 & 0 & 1 & 8 \\ -1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} \overset{\text{Expanding by 5th row}}{=} (-1)^{5+5} \cdot 2 \begin{vmatrix} 3 & 2 & 0 & 1 \\ -2 & 4 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ -1 & 2 & 0 & -1 \end{vmatrix} \overset{\text{Expanding by 3rd column}}{=}$$

$$= 2 \cdot (-1)^{2+3} \cdot 1 \cdot \begin{vmatrix} 3 & 2 & 1 \\ 0 & -1 & 1 \\ -1 & 2 & -1 \end{vmatrix} \overset{\text{Expanding by 1st column}}{=} -2\left(3\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}\right) =$$

$$= -6\left((-1)^2 - 1\cdot 2\right) + 2\left(2\cdot 1 - 1(-1)\right) = 6 + 6 = 12$$

$$\begin{vmatrix} 7 & 0 & 2 & 4 \\ 5 & 1 & 10 & 15 \\ 0 & 0 & 3 & 0 \\ 2 & 0 & 6 & 1 \end{vmatrix} = 1 \begin{vmatrix} 7 & 2 & 4 \\ 0 & 3 & 0 \\ 2 & 6 & 1 \end{vmatrix} = 3 \begin{vmatrix} 7 & 4 \\ 2 & 1 \end{vmatrix} = 3\cdot(7-8) = -3$$

The number $a_{ij} = (-1)^{i+j} |\tilde{A}_{ij}|$ is called <u>cofactor</u> of A_{ij}

<u>Definition</u> Let $A = [A_{ij}]_{i=1\, j=1}^{n\ \ n}$ be an <u>$n\times n$-matrix</u>

The transpose $(A')^T$ of the matrix $A' = [(-1)^{i+j}|\tilde{A}_{ij}|]$ of cofactors is called the <u>adjugate</u> or the <u>adjoint</u> of A and is denoted by $\mathrm{adj}(A)$:

$$\mathrm{adj}(A) = \left[(-1)^{i+j}|\tilde{A}_{ij}|\right]^T = [a_{ij}]^T$$

Examples

a) $A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$ $\quad A' = \begin{bmatrix} (-1)^{1+1}\cdot(-1) & (-1)^{1+2}\cdot 1 \\ (-1)^{2+1}\cdot 0 & (-1)^{2+2}\cdot 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$

$\mathrm{adj}(A) = \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 5 & 2 & 0 \end{bmatrix}$ $\quad A' = \begin{bmatrix} (-1)^{1+1}\cdot(8) & (-1)^{1+2}\cdot(-20) & (-1)^{1+3}(-15) \\ (-1)^{2+1}(-6) & (-1)^{2+2}\cdot(-15) & (-1)^{2+3}(-8) \\ (-1)^{3+1}(-1) & (-1)^{3+2}\cdot(4) & (-1)^{3+3}(3) \end{bmatrix} =$

$= \begin{bmatrix} 8 & 20 & -15 \\ 6 & -15 & 8 \\ -1 & -4 & 3 \end{bmatrix}$ $\quad \mathrm{adj}(A) = (A')^T = \begin{bmatrix} 8 & 6 & -15 \\ 20 & -15 & 4 \\ -15 & 8 & 3 \end{bmatrix}$

Our theorem 2 becomes now:

Theorem 3 The adjugate of A satisfies:

$$adj(A)\cdot A = A\cdot adj(A) = \det(A)\cdot I$$

For the proof see the textbook.

Corollary 1 Let $|A| \neq 0$. Then A is invertible and $$A^{-1} = \frac{1}{\det(A)}\cdot adj(A).$$

Examples Find the inverse of A if

a) $A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

Solution: by the example (a) befor Theorem 3

$adj(A) = \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}$. Thus $A^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & -1 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 5 & 2 & 0 \end{bmatrix}$

Solution $adj(A) = \begin{bmatrix} 8 & 6 & -15 \\ 20 & -15 & 4 \\ -15 & 8 & 3 \end{bmatrix}$

$$|A| = 5\cdot\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = 5\cdot(-1) - 8 = -13$$

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{13}\begin{bmatrix} -8 & -6 & 15 \\ -20 & 15 & -4 \\ 15 & -8 & -3 \end{bmatrix}$$

Ex $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad |A| = ad - bc \neq 0$

$\mathrm{adj}(A) = (A')^T \quad A' = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \quad \mathrm{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{4-2} \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow A^{-1} = (-1) \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow A^{-1} = (-1) \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Corollary 2 (Cramer's Rule) Consider the linear system $AX = B$ with an invertible matrix A. Then the system has a unique solution given by

$$X = A^{-1}B = \frac{\mathrm{adj}(A) \cdot B}{\det A} = \left[\frac{\det(C_k)}{\det(A)} \right]_{k=1}^{n}, \text{ where}$$

C_k is the matrix obtained from A by replacing the k-th column by B.

Proof: $AX = B \Rightarrow X = A^{-1}B = \dfrac{\mathrm{adj}(A)}{\det(A)} \cdot B \Rightarrow \det(A) X = \mathrm{adj}(A) \cdot B \Rightarrow$

$\Rightarrow \det(A) x_k = a_{1k} b_1 + a_{2k} b_2 + \ldots + a_{nk} b_k$, for $k = 1, \ldots, n$.

k-th column expansion of $\det(C_k)$

Hence $x_k = \det(C_k) / \det(A)$ as required ∎

Ex Solve the system by using Cramer's rule

$$5x_1 - 2x_2 + x_3 = 1$$
$$3x_1 + 2x_2 = 3$$
$$x_1 + x_2 - x_3 = 0$$

Solution
$$A = \begin{bmatrix} 5 & -2 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad C_2 = \begin{bmatrix} 5 & 1 & 1 \\ 3 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix} \quad C_3 = \begin{bmatrix} 5 & -2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

$$|A| = -15 \quad |C_1| = -5 \quad |C_2| = -15 \quad |C_3| = -20$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} |C_1| \\ |C_2| \\ |C_3| \end{bmatrix} = -\frac{1}{15} \begin{bmatrix} -5 \\ -15 \\ -20 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \\ \frac{4}{3} \end{bmatrix}$$

Exercise Solve for z:

$$\begin{cases} x - 2y + z = 1 \\ 2x + y - 2z = 0 \\ -x + 3y - z = 0 \end{cases}$$

Solution: $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -2 \\ -1 & 3 & -1 \end{bmatrix}$, $C_3 = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ -1 & 3 & 0 \end{bmatrix}$.

$$|A| = 1 \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} + 2 \begin{vmatrix} 2 & -2 \\ -1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} = 5 + 2(-4) + 7 = 4$$

$$|C_3| = 1 \cdot \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} = 7$$

$$z = \frac{|C_3|}{|A|} = \frac{7}{4}$$

Properties of determinants

a) The determinant is n-linear function of rows and n-linear function of columns:

$$\left| \begin{bmatrix} A_{11} \\ \vdots \\ A_{n1} \end{bmatrix} \cdots \left[\alpha \cdot \begin{bmatrix} A'_{1j} \\ \vdots \\ A'_{nj} \end{bmatrix} + \beta \cdot \begin{bmatrix} A''_{1j} \\ \vdots \\ A''_{nj} \end{bmatrix} \right] \cdots \begin{bmatrix} A_{1n} \\ \vdots \\ A_{nn} \end{bmatrix} \right| =$$

$$= \alpha \cdot \left| \begin{bmatrix} A_{11} \\ \vdots \\ A_{n1} \end{bmatrix} \cdots \begin{bmatrix} A'_{1j} \\ \vdots \\ A'_{nj} \end{bmatrix} \cdots \begin{bmatrix} A_{1n} \\ \vdots \\ A_{nn} \end{bmatrix} \right| + \beta \cdot \begin{vmatrix} A_{11} \dots & A''_{1j} & \dots A_{1n} \\ \dots & \dots & \dots \\ A_{n1} \dots & A''_{nj} & \dots A_{nn} \end{vmatrix}$$

b) It follows from a) that

$$\left| \begin{matrix} A_{11} \\ \vdots \\ A_{n1} \end{matrix} \cdots \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{matrix} A_{1n} \\ \\ A_{nn} \end{matrix} \right| = 0$$

c) The determinant is alternating function of rows (and of columns):

$$D(\dots R_k \dots R_\ell \dots) = (-1)\, D(\dots R_\ell \dots R_k \dots)$$

Therefore $D(\ldots V \ldots V \ldots) = 0$

d) $D(A^T) = D(A)$

e) $D(AB) = D(A) \cdot D(B)$

For a proof see the textbook.

f) $\begin{vmatrix} A & C \\ 0 & B \end{vmatrix} = |A| \cdot |B|$ and $\begin{vmatrix} E & 0 \\ F & G \end{vmatrix} = |E||G|$

where A, B, E, and G are square matrices.

g) $\begin{vmatrix} C & B \\ A & 0 \end{vmatrix} = (-1)^{kn} \cdot |A| \cdot |B|$.

Where A is $k \times k$-matrix and B is $n \times n$-matrix.

h) $|I| = 1$.

We continue with solutions of some exercises.

Ex 1 It follows from f) that for any upper / lower triangular matrix $A_{n \times n}$

$$|A| = A_{11} \cdot A_{22} \cdots A_{nn}.$$

Ex 2 $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix} = (-2) \cdot 2 \cdot 1 \cdot 5 = -20$

Ex 3 Let $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$. Show that the matrix equation $(X+A)^2=\begin{bmatrix}1&2\\3&4\end{bmatrix}-A^2$ has no solutions among real 2×2 matrices.

Solution Suppose such X exists. Notice that $A^2=\begin{bmatrix}2&2\\2&2\end{bmatrix}$. Thus $(X+A)^2=\begin{bmatrix}1&2\\3&4\end{bmatrix}-\begin{bmatrix}2&2\\2&2\end{bmatrix}=\begin{bmatrix}-1&0\\1&2\end{bmatrix}$ and

$$0\le|X+A|\cdot|X+A|=|(X+A)^2|=\begin{vmatrix}-1&0\\1&2\end{vmatrix}=-2$$

The contradiction shows that such an X does not exist.

Ex 4 (Ex. 8, p119) Let $E^2=E\neq I$, det

Show that $|E|=0$.

Solution $|E|=|E^2|=|E|\cdot|E|\Rightarrow|E|=0$ or $|E|=1$. Suppose $|E|=1$. Then by the invertibility of E one gets

$$I=E^{-1}E=E^{-1}E^2=E^{-1}EE=E$$ which contradicts to assumption that $E\neq I$. Thus $|E|=0$.

Ex 5 Show that any invertible A has non-zero determinant.

Solution $A\cdot A^{-1}=I\Rightarrow|A||A^{-1}|=1\Rightarrow|A|\neq0$.

Ex 6 (Ex 9, p 119)

Show that the determinant of any skew-symmetric matrix of odd order is zero.

Solution Let $\underset{n\times n}{A}^T = -A$, $n = 2k-1$

Then $|A^T| = |-A| = (-1)^{2k-1}|A| = -1 \cdot |A^T|$

Hence $|A^T| = 0$ and $|A| = 0$.

Ex 7 Calculate

$$\begin{vmatrix} 1 & 2 & 3 & 5 & 6 \\ 2 & 3 & 0 & 9 & 10 \\ 4 & 5 & 0 & 13 & 15 \\ 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 8 & 1 \end{vmatrix} \overset{\text{solution}}{=} \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 4 & 5 & 0 \end{vmatrix} \cdot \begin{vmatrix} 7 & 1 \\ 8 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} \cdot (-1) = 3(-2)(-1) = 6$$

Ex 8

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \end{vmatrix} \overset{\text{Solution}}{=} (-1)^{3\cdot 2}\begin{vmatrix} 4 & 5 \\ 9 & 10 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 3 & 1 \end{vmatrix} = (-5)\cdot(3+6) = -45$$

Ex 9

$$\begin{vmatrix} 7 & 5 & 9 & 1 & 2 & 3 \\ 11 & 8 & 6 & 2 & 3 & 0 \\ 12 & 5 & 1 & 4 & 5 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 & 0 \end{vmatrix} \overset{\text{solution}}{=} (-1)^{3\cdot 3}\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 4 & 5 & 0 \end{vmatrix}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 2 & 5 & 0 \end{vmatrix} = -1\cdot 3\cdot\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}\begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} = -3\cdot(-2)\cdot 4 = 24$$

Ex 10 (Ex 3, p 117)

given that $\begin{vmatrix} 1 & a & 2 \\ -1 & 1 & b \\ a & 2 & 3b \end{vmatrix} = 4$ compute the following determinants:

a) $$\begin{vmatrix} x^2 & ax & 2x \\ -x & 1 & b \\ ax & 2 & 3b \end{vmatrix} = \begin{vmatrix} x\begin{bmatrix} x \\ -1 \\ a \end{bmatrix} & \begin{matrix} ax \\ 1 \\ 2 \end{matrix} & \begin{matrix} 2x \\ b \\ 3b \end{matrix} \end{vmatrix} = x \begin{vmatrix} [x & ax & 2x] \\ -1 & 1 & b \\ a & 2 & 3b \end{vmatrix} =$$

$$= x^2 \begin{vmatrix} 1 & a & 2 \\ -1 & 1 & b \\ a & 2 & 3b \end{vmatrix} = 4x^2$$

b) $$\begin{vmatrix} x & a+bx & 2 \\ -x & 1-bx & b \\ ax & 2+abx & 3b \end{vmatrix} = x \begin{vmatrix} \begin{matrix} 1 \\ -1 \\ a \end{matrix} & \begin{bmatrix} a \\ 1 \\ 2 \end{bmatrix} + bx \begin{bmatrix} 1 \\ -1 \\ a \end{bmatrix} & \begin{matrix} 2 \\ b \\ 3b \end{matrix} \end{vmatrix} = x \begin{vmatrix} 1 & a & 2 \\ -1 & 1 & b \\ a & 2 & 3b \end{vmatrix} = 4x$$

c) $$\begin{vmatrix} -1 & -2a & -2 \\ -2 & 4 & 2b \\ a & 4 & 3b \end{vmatrix} = 2 \begin{vmatrix} -1 & -a & -2 \\ -2 & 2 & 2b \\ a & 2 & 3b \end{vmatrix} = 4 \cdot \begin{vmatrix} -1 & -a & -2 \\ -1 & 1 & b \\ a & 2 & 3b \end{vmatrix} = 4 \cdot 4 = 16$$

d) $$\begin{vmatrix} a+1 & a+2 & 2+3b \\ -1 & 1 & b \\ a & 2 & 3b \end{vmatrix} = \begin{vmatrix} 1 & a & 2 \\ -1 & 1 & b \\ a & 2 & 3b \end{vmatrix} = 4$$

Ex 11 (Ex 5c, p 119)

$$\begin{vmatrix} 1 & 4 & 1 & -4x \\ 3 & -1 & 3 & x \\ 2 & -1 & 7 & x \\ 5 & 1 & 1 & -x \end{vmatrix} = \begin{vmatrix} \begin{matrix} 1 & 4 & 1 \\ 3 & -1 & 3 \\ 2 & -1 & 7 \\ 5 & 1 & 1 \end{matrix} & -x\begin{bmatrix} 4 \\ -1 \\ -1 \\ 1 \end{bmatrix} \end{vmatrix} = 0$$