Lectures 2-3

Row Equivalence. Invertibility

Elementary Matrices.

Def Let A be a matrix. The first non-zero entry (if any) of i-th row of A is called the *leading entry* (of i-th row)

Def If the leading entries of A are of the form $a_{1j_1}, a_{2j_2}, \ldots, a_{rj_r}$ with $j_1 < j_2 < \ldots < j_r$ then A is called an *echelon matrix*.

Examples: 1) $A = \begin{bmatrix} 0 & 2 & -1 & 1 & 3 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is echelon.

2) $[a, b, c, d]$ is echelon for any a, b, c, and d.

3) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ is not echelon.

4) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}$ is echelon.

5) $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$ is not echelon, since $j_2 = 2 = j_3$

Def Let A be an echelon matrix. Then A is called a row-reduced echelon matrix if 1) each leading entry is 1;
2) each leading entry is the unique non-zero entry of its own column.

Example a) $[1, b, c, d]$ is RREM

b) Any identity matrix is RREM

c) Any zero matrix is RREM

d) $\begin{bmatrix} 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ is RREM

e) $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is not RREM

f) $\begin{bmatrix} 1 & 0 & 7 & 3 & 0 \\ 0 & 1 & 8 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ is RREM

g) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is not RREM

Row operations on matrices

There are three types of elementary row operations:

Type I Multiplying i-th row by a nonzero scalar c. (Notation: cR_i)

Type II Interchanging i-th row with j-th row (Notation: $R_i \Leftrightarrow R_j$)

Type III Adding α times i-th row to j-th row (Notation: $\alpha R_i + R_j$)

Examples: a) $\begin{bmatrix} -1 & 2 & 0 & 4 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & -4 & 2 \end{bmatrix} \xrightarrow{(-3)R_2+R_1} \begin{bmatrix} -1 & 5 & -9 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & -4 & 2 \end{bmatrix}$;

b) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 0 & 3 \end{bmatrix} \xrightarrow{R_2 \Leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 3 \\ 2 & 3 & 1 \end{bmatrix}$; c) $\begin{bmatrix} 2 & 4 \\ 0 & -1 \\ 3 & 0 \end{bmatrix} \xrightarrow{5R_3} \begin{bmatrix} 2 & 4 \\ 0 & -1 \\ 15 & 0 \end{bmatrix}$.

Clearly every elementary row operation (ERO for short) has inverse of the same type: cR_i has inverse $\frac{1}{c}R_i$, $R_i \Leftrightarrow R_j$ coincides with its inverse, and the inverse of $\alpha R_i + R_j$ is $(-\alpha)R_i + R_j$.

One may write ERO's consequentially, e.g.:

$$\begin{bmatrix} 2 & -3 & 6 \\ 0 & 1 & -2 \\ -1 & 5 & 1 \\ 2 & -5 & 1 \end{bmatrix} \xrightarrow[\substack{(-1)R_1+R_4 \\ R_1 \leftrightarrow R_2}]{-2R_3} \begin{bmatrix} 0 & 1 & -2 \\ 2 & -3 & 6 \\ 2 & -10 & -2 \\ 0 & -2 & -5 \end{bmatrix}$$

__Def__ A matrix B is called __row-equivalent__ to a matrix C if C can be obtained by applying a sequence of several row operations to the matrix B.

__Example__

$$\begin{bmatrix} 2 & 6 \\ 0 & -2 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{\substack{(-2)R_3 \\ (-1)R_1+R_4}} \begin{bmatrix} 0 & -2 \\ 2 & 6 \\ 2 & -2 \\ 0 & -5 \end{bmatrix} \xrightarrow{\substack{R_2+R_4 \\ (-\frac{1}{2})R_3 \\ R_1 \leftrightarrow R_2}} \begin{bmatrix} 2 & 6 \\ 0 & -2 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}$$

__Thm__ The __row-equivalence__ is an equivalence relation, that is: a) __reflexivity__: $A \sim A$;
b) __symmetricity__: $A \sim B \Rightarrow B \sim A$;
c) __transitivity__: $A \sim B$ and $B \sim D \Rightarrow A \sim D$.
The proof is obvious. ∎

__Def__ __Elementary matrices__ are matrices of the form $\mathcal{E}(I)$, where $\mathcal{E}$ is an ERO.

__Example__ a) $(R_1 \leftrightarrow R_2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$;

b) $(7R_3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

c) $(3R_3+R_1)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&3&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$

<u>Theorem 1.</u> For any matrix A and ERO $\mathcal{E}$, we have

$$\mathcal{E}(I)\cdot A = \mathcal{E}(A).$$

For the proof, consider three types of ERO. ■

<u>Theorem 2.</u> (Thm 1.5.3 and 1.5.6 in the textbook)
Any matrix A can be transformed by applying some ERO $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n$ to a RREM A', which is <u>uniquely determined</u> by A:

$$A' = \mathcal{E}(A) = \mathcal{E}_n\mathcal{E}_{n-1}\dots\mathcal{E}_1(A) = \underbrace{\mathcal{E}_n(I)\cdot\mathcal{E}_{n-1}(I)\cdot\dots\cdot\mathcal{E}_1(I)}_{\mathcal{E}(I)}\cdot A = \mathcal{E}(I)\cdot A.$$

In other words, if $B=\mathcal{E}(A)$ and $P=\mathcal{E}(I)$ then $B=PA$:

$$[A\,|\,I] \xrightarrow{\mathcal{E}} [B\,|\,P] \Rightarrow B = PA$$ ■

<u>Exercise</u> Find P such that PA is a RREM for:

1) $A=\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}$ Answer: $P=\begin{bmatrix}1&-1\\0&1\end{bmatrix}$

2) $A=\begin{bmatrix}2&3\\0&4\end{bmatrix}$ Answer: $P=\begin{bmatrix}1/2&-3/8\\0&1/4\end{bmatrix}$

3) $A=\begin{bmatrix}2&3&4\\6&9&12\end{bmatrix}$ Answer: $P=\begin{bmatrix}1/2&0\\-3&1\end{bmatrix}$

4) $A=\begin{bmatrix}1&6&6\\0&2&3\\1&2&0\end{bmatrix}$ Answer: $P=\begin{bmatrix}0&-1&1\\0&1/2&0\\1&-2&-1\end{bmatrix}$

Solution:

1) $[A \mid I] = \left[\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{(-1)R_2+R_1} \Big[\overbrace{\begin{matrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{matrix}}^{B} \Big| \overbrace{\begin{matrix} 1 & -1 \\ 0 & 1 \end{matrix}}^{P}\Big]$

$B = PA$ is RREM.

2) $\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{array}\right] \xrightarrow{\left(-\frac{3}{4}\right)R_2+R_1} \left[\begin{array}{cc|cc} 2 & 0 & 1 & -\frac{3}{4} \\ 0 & 4 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{4}R_2]{\frac{1}{2}R_1} \Big[\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{B} \Big| \underbrace{\begin{matrix} \frac{1}{2} & -\frac{3}{8} \\ 0 & \frac{1}{4} \end{matrix}}_{P}\Big]$

$B = PA$ is RREM.

3) $\left[\begin{array}{ccc|cc} 2 & 3 & 4 & 1 & 0 \\ 6 & 9 & 12 & 0 & 1 \end{array}\right] \xrightarrow{(-3)R_1+R_2} \left[\begin{array}{ccc|cc} 2 & 3 & 4 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|cc} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -3 & 1 \end{array}\right]$

4) $\left[\begin{array}{ccc|ccc} 1 & 6 & 6 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-1)R_3+(-2)R_2+R_1} \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow$

$\xrightarrow[\frac{1}{2}R_2]{R_1 \Leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right] \xrightarrow{(-2)R_2+R_1} \Big[\underbrace{\begin{matrix} 1 & 0 & -3 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{matrix}}_{B} \Big| \underbrace{\begin{matrix} 0 & -1 & 1 \\ 0 & \frac{1}{2} & 0 \\ 1 & -2 & -1 \end{matrix}}_{P}\Big]$

$B = PA$ is RREM.

Def If $CD = I$ then:

1.) C is called a left inverse of the matrix D.

2.) D is called a right inverse of the matrix C.

Def A square matrix A is called <u>invertible</u> if there exists A' s.t. $A'A = AA' = I$.

Remark The <u>inverse</u> A' of A (if exists) is uniquely determined.

Proof: If $A'A = AA' = I$ and
$A''A = AA'' = I$ then

$$A'' = I \cdot A'' = A'AA'' = A'(A \cdot A'') = A'I = A' \quad \blacksquare$$

Remark If A, B are invertible then
$(AB)^{-1} = B^{-1}A^{-1}$

Proof: $B^{-1}A^{-1} \cdot AB = I = AB \cdot B^{-1}A^{-1} \quad \blacksquare$

Remark Every elementary matrix is invertible

Proof: $E = \mathcal{E}(I)$
$$\mathcal{E}^{-1}(I) \cdot E = \mathcal{E}^{-1}(I) \cdot \mathcal{E}(I) = \mathcal{E}^{-1}\mathcal{E}(I) = I \quad \blacksquare$$

Theorem (Thm. 1.5.5 in the textbook)

For a square matrix A FCAE:

1) $A = \mathcal{E}(I) = \mathcal{E}_n(I) \cdot \mathcal{E}_{n-1}(I) \cdot \ldots \cdot \mathcal{E}_1(I)$, that is:
A is a product of elementary matrices;

2) $A \sim I$, that is: A is equivalent to identity matrix;

3) A is invertible;

4) A is not row-equivalent to a matrix withazero row;

5) A^T is invertible.

Proof:

1) $\Leftrightarrow$ 2): It follows from Theorem 2 and from the definition of the row-equivalence.

2) $\Rightarrow$ 3): If $A \sim I$ then $A = \mathcal{E}(I)$ and hence $\mathcal{E}^{-1}(I)$ is the inverse of A, that is:
$\mathcal{E}^{-1}(I) \cdot A = \mathcal{E}^{-1} \cdot \mathcal{E}(I) = I = \mathcal{E}(I) \cdot \mathcal{E}^{-1}(I) = A \cdot \mathcal{E}^{-1}(I)$.

3) $\Rightarrow$ 4): Let A be invertible. Suppose that a matrix $C = \mathcal{E}(A)$ has k-th row zero.

Then the matrix $C \cdot A^{-1} \cdot \mathcal{E}^{-1}(I)$ has the same k-th row zero. But this is impossible, since

$$C \cdot A^{-1}\mathcal{E}^{-1}(I) = \mathcal{E}(A) A^{-1}\mathcal{E}^{-1}(I) = \mathcal{E}(I)A A^{-1}\mathcal{E}^{-1}(I) = \mathcal{E}(I) I \cdot \mathcal{E}^{-1}(I) = I.$$

The contradiction shows that $\mathcal{E}(A) \sim A$ has no zero rows for any sequence $\mathcal{E} = \mathcal{E}_n \circ \mathcal{E}_{n-1} \ldots \mathcal{E}_1$ of ERO's. So, there is no matrices equivalent to A which may have zero rows.

4) $\Rightarrow$ 2) Since the only RREM matrix which is square matrix is the identity matrix, assuming that it has no zero rows.

3) $\Leftrightarrow$ 5) If A^{-1} exists then the matrix $(A^{-1})^T$ is the inverse to A^T since

$$(A^{-1})^T A^T = (A \cdot A^{-1})^T = I = (A^{-1}A)^T = A^T \cdot (A^{-1})^T.$$

Remark also that if A^T is invertible then

$$((A^T)^{-1})^T A = ((A^T)^{-1})^T A^{TT} = (A^T (A^T)^{-1})^T =$$

$$= I^T = I = ((A^T)^{-1} A^T)^T = A^{TT}((A^T)^{-1})^T = A((A^T)^{-1})^T.$$

$\blacksquare$

Hence $A^{-1} = ((A^T)^{-1})^T$.

Notice that in the theorem we may not replace rows by columns in 4). Indeed, the matrix $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$ is not row equivalent to any matrix with zero columns, yet it is not invertible.

<u>Example</u> Consider 1×2 matrix $A = [1\ 0]$

It is right invertible since $\underset{1\times 2}{[1\ 0]}\cdot \underset{2\times 1}{\begin{bmatrix}1\\0\end{bmatrix}} = [1] = \underset{1\times 1}{I}$ but

A is not left invertible since

$$BA = \underset{2\times 1}{\begin{bmatrix}a\\ b\end{bmatrix}}\underset{1\times 2}{[1\ 0]} = \begin{bmatrix}a & 0\\ b & 0\end{bmatrix} \neq \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \underset{2\times 2}{I} \text{ for any } a, b \in \mathbb{R}.$$

<u>Corollary</u> If a square matrix A has a left inverse or right inverse, then A is invertible.

<u>Proof</u> Let $AR = I$ but A is not invertible. Then, by Theorem 3, $\mathcal{E}(A)$ has some k-th row zero for some $\mathcal{E}(A) \sim A$. Then $\mathcal{E}(A)\cdot R$ has the same k-th row zero, which is impossible since

$$\mathcal{E}(A)\cdot R = \mathcal{E}(I)\cdot A\cdot R = \mathcal{E}(I)\cdot I = \mathcal{E}(I)$$

and since $\mathcal{E}(I)$ is invertible.

Let $LA = I$. Taking the transpose, one gets $A^T L^T = I$. By the first part of proof, A^T is invertible. Then A is invertible too. ∎

The algorithm to determine whether A is invertible and to find A^{-1} when A is invertible.

1) Construct partitioned matrix $[A|I]$;
2) Transform by row-operations $[A|I] \xrightarrow{\mathcal{E}} [R|P]$ where R is row-reduced echelon matrix;
3) if $R \neq I$ then A is not invertible;
4) if $R = I$ then $R = PA$ and hence

$$[A|I] \xrightarrow{\mathcal{E}} [I|A^{-1}].$$

Exercise. Find A^{-1} if $A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$.

Solution:

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 1 & -2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 1 & -2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & -2 & 3 & -\frac{1}{2} & 0 & 1 \end{array}\right] \rightarrow$$

$$\xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -2 & 1 \end{array}\right] \xrightarrow[-R_3+R_2]{-R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & -2 & 1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -2 & 1 \end{array}\right]$$

Thus $A^{-1} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -3 & 1 \\ 1/2 & -2 & 1 \end{bmatrix}$.

Exercise. Find A^{-1} if exists for $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -3 & -4 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 1 & -3 & -4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & -5 & -7 & 0 & -1 & 1 \end{array}\right] \rightarrow$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & 0 & 0 & 3 & -2 & 1 \end{array}\right]$$

matrix in the left has zero row hence A is not invertible by the theorem 3.

Theorem 4 Two $m\times n$-matrices A and B are row equivalent iff $B=PA$ for some invertible $m\times m$-matrix P.

Proof: "$\Rightarrow$" $A\sim B$ implies $B=\mathcal{E}(A)$ and

$$[A\,|\,I] \xrightarrow{\mathcal{E}} [B\,|\,P]$$ and then $B=PA$ where

$P=\mathcal{E}(I)$ is invertible.

"$\Leftarrow$" $B=PA$, P is invertible square matrix

Thus $P=\mathcal{E}(I)$, Then $B=\mathcal{E}(I)\cdot A=\mathcal{E}(A)$. ∎

Exercise Let A, B, C, and D be 3×4 matrices such that $A \xrightarrow{-R_1+R_3} B \xrightarrow{R_1\leftrightarrow R_2} D$ and $C \xrightarrow{3R_2+R_3} D$.

Find an invertible matrix P such that $PA=C$ and write P as a product of three elementary matrices (accordingly to the diagrams above)

Solution $$A \underset{\mathcal{E}_1}{\xrightarrow{-R_1+R_3}} B \underset{\mathcal{E}_2}{\xrightarrow{R_1\leftrightarrow R_2}} D \underset{\mathcal{E}_3}{\xrightarrow{-3R_2+R_3}} C$$

$$P=\mathcal{E}_3(I)\cdot\mathcal{E}_2(I)\cdot\mathcal{E}_1(I)=\begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}=$$

$$=\mathcal{E}_3\mathcal{E}_2\mathcal{E}_1(I)=\mathcal{E}_3\mathcal{E}_2\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}=\mathcal{E}_3\begin{bmatrix}0&1&0\\1&0&0\\-1&0&1\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\-4&0&1\end{bmatrix}$$