Lecture 15 (OPTIONAL)

QUADRATIC FORMS

(Df) A **quadratic form** is an expression $X^T A X$ or, in general, an expression $X^T A X + BX + c$, where $X \in M^{n \times 1}$, A is an $n \times n$ symmetric matrix, $c \in \mathbb{R}$, and $B \in M^{n \times 1}$.

The equation $X^T A X = d$ or, in general, $X^T A X + BX = d$ is called a **quadratic equation** of a conic (if $n = 2$) or a surface ($n = 3$).

We denote $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then

$$X^T A X = [x_1 \ x_2 \ \dots \ x_n] \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} =$$

$$= [x_1 \ x_2 \ \dots \ x_n] \cdot \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{bmatrix} =$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right) x_i = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i a_{ij} x_j$$

This gives us a way how to obtain a matrix A if the quadratic form is given as a polynomial of two vector variables

$F(X) = \mathcal{P}(X, X)$, $X \in M^{n \times 1}$, of degree ≤ 2.

Example

Write the following equation as a matrix form:

(1) $-x^2 + 2xy + 2y^2 = 7$

$a_{11} = -1$

$\begin{cases} a_{12} + a_{21} = 2 \\ a_{12} = a_{21} \end{cases} \Rightarrow a_{12} = a_{21} = 1$

(since A is symmetric)

$a_{22} = 2$

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$X^T A X = 7.$$

(2) $3t^2 - s^2 + 6(ts + su + ut + 2u^2) = 3$

$$X = \begin{bmatrix} t \\ s \\ u \end{bmatrix}$$

$a_{11} = 3$ $\quad a_{12} = a_{21} = \frac{6}{2} = 3$

$a_{22} = -1$ $\quad a_{13} = a_{31} = \frac{6}{2} = 3$

$a_{33} = 12$ $\quad a_{23} = a_{32} = \frac{6}{2} = 3$

$$A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & -1 & 3 \\ 3 & 3 & 12 \end{bmatrix}$$

$$X^T A X = 3$$

(3) $x^2 - 2xy + y^2 = 1$

$a_{11} = 1$ $\quad a_{12} = a_{21} = -1$

$a_{22} = 1$

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$X^T A X = 1.$$

Example Identify the conic
$$2x_1^2 + 2\sqrt{2}\, x_1 x_2 + 3x_2^2 = 16.$$

Solution $A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$, $X^T A X = 16.$

It is enough to find eigenvalues

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & \sqrt{2} \\ \sqrt{2} & \lambda-3 \end{vmatrix} = (\lambda-2)(\lambda-3) - 2 = \lambda^2 - 5\lambda + 4$$

$$\lambda_{1\,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{5 \pm \sqrt{25-16}}{2} = \frac{5 \pm 3}{2}$$

$$\lambda_1 = 4, \ \lambda_2 = 1$$

Then, for some orthogonal matrix Q, we have
$$Q^T A Q = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{D}.$$

Denote by $Y = Q^T X$ a new variable,
then $X = QY$ since $Q^{-1} = Q^T$.
Substitute in $X^T A T = 16$:

$$(QY)^T A (QY) = 16 \iff Y^T (Q^T A Q) Y = 16 \iff$$
$$\iff Y^T \mathcal{D} Y = 16 \iff [y_1 \ y_2] \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 16 \iff$$
$$\iff 4y_1^2 + y_2^2 = 16 \iff$$
$$\boxed{\frac{1}{4} y_1^2 + \frac{1}{16} y_2^2 = 1}$$

It is an ellipse with principal axes $a = 2$ and $b = 4$ since the equation of an ellipse is $\dfrac{y_1^2}{a^2} + \dfrac{y_2^2}{b^2} = 1.$

Example (p. 246 of the textbook).

Find the standard equation of the quadratic surface

$$\underbrace{2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1}_{P(X,\,X)} \underbrace{-\sqrt{2}\,x_1 - \sqrt{2}\,x_3}_{G(X)} = 27$$

Solution $\quad P(X,X) + G(X) = 27$

$P(X,X) = X^T A X \qquad G(X) = BX$

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} \sqrt{2} & 0 & -\sqrt{2} \end{bmatrix}$$

is known matrix focus.

The eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 3$.

The diagonalizing orthogonal matrix Q is, for example,

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

New variable is given by $Y = Q^T X$,

while $X = QY$, $\quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Finally, $P(X,X) + G(X) = 27$ becomes

$$X^T A X + BX = 27$$

$$(QY)^T A (QY) + B(QY) = 27$$

$$Y^T (Q^T A Q) Y + (BQ) Y = 27$$

$$Y^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} Y + (BQ) Y = 27$$

$$3y_2^2 + 3y_3^2 + \underbrace{\begin{bmatrix} \sqrt{2} & 0 & -\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}}_{=\begin{bmatrix} 0 & -2 & 0 \end{bmatrix}} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 27$$

$$3y_2^2 + 3y_3^2 + \begin{bmatrix} 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 27$$

$$3y_2^2 + 3y_3^2 - 2y_2 = 27$$

Completing to perfect squares gives us

$$3\left(y_2^2 - \frac{2}{3}y_2 + \frac{1}{9}\right) + 3y_3^2 - \frac{3}{9} = 27$$

$$3\left(y_2 - \frac{1}{3}\right)^2 + 3y_3^2 = 27 + \frac{1}{3} = \frac{82}{3}$$

$$y_2 - \frac{1}{3} := z_2$$

$$y_3 := z_3$$

$$y_1 := z_1$$

$$\boxed{z_2^2 + z_3^2 = \frac{82}{9}}$$

This is the standard equation of a circular cylinder in coordinates (z_1, z_2, z_3)

Exercise Find the standard equation of the conic $x^2 + xy + y^2 = 18$.

Solution $A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$ $\quad X^T A X = 18$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -\frac{1}{2} \\ -\frac{1}{2} & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - \frac{1}{4} = \lambda^2 - 2\lambda + \frac{3}{4}$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 3}}{2} = \frac{2 \pm 1}{2}$$

$$\lambda_1 = \frac{3}{2}, \quad \lambda_2 = \frac{1}{2}$$

$$Y = Q^T X, \quad X = QY$$

$$Y^T (Q^T A Q) Y = 18$$

$$Y^T \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} Y = 18$$

$$\frac{3}{2} y_1^2 + \frac{1}{2} y_2^2 = 18$$

$$\frac{1}{12} y_1^2 + \frac{1}{36} y_2^2 = 1$$

$$\frac{y_1^2}{(\sqrt{12})^2} + \frac{y_2^2}{6^2} = 1$$

is the standard equation for an elypse.

Exercise Show that a nonzero nilpotent matrix cannot be diagonalizable.

Solution

Suppose $A \neq 0$, $A^k = 0$, and A is diagonalizable. Then there exists an invertible matrix P s.t.

$P^{-1}AP = D$ is diagonal matrix.

Then $AP = PD$ and $A = PDP^{-1} = 0$

Since $D^k = P^{-1}A^kP = 0$, so $D = 0$.

This contradicts to the assumption $A \neq 0$.

Exercise

Let $A \in M^{n \times n}$ such that $P^T A P$ is diagonal for some invertible P. Show that A is symmetric.

Solution

$$P^T A P = D \Rightarrow P^T A = D P^{-1} \Rightarrow$$

$$\Rightarrow A = (P^T)^{-1} D P^{-1} = (P^{-1})^T D P^{-1}.$$

Then $A^T = ((P^{-1})^T D P^{-1})^T = (P^{-1})^T D P^{-1} = A$,

so A is symmetric.