Lecture 14

LINEAR TRANSFORMATIONS

(Df) Let U and V be vector spaces.
A function $T: U \to V$ is called a **linear transformation** from U to V if it preserves the linear operations:

$$T(\alpha \cdot u + \beta \cdot v) = \alpha \cdot T(u) + \beta \cdot T(v)$$

for all $\alpha, \beta \in \mathbb{R}$ and for all $u, v \in U$.

Examples

1) $U = \mathcal{P}(\mathbb{R})$, $V = C(\mathbb{R})$ and T is given by

$$T(p(x)) = \frac{d}{dx}(p(x)).$$

2) $U = \mathbb{R}^2$, $V = \mathbb{R}^3$, and T is given by

$$T((x_1, x_2)) = (x_1 + x_2, \; x_1, \; x_2 - x_1).$$

3) $U = \mathbb{R}^3$, $V = \mathbb{R}^2$, $T: \mathbb{R}^3 \to \mathbb{R}^2$:

$$T((x_1, x_2, x_3)) = (x_1 + x_2, \; x_2 - x_3).$$

4) Let $U = M^{n \times m}$, $V = M^{p \times m}$.
Then for any $p \times n$-matrix A, the mapping $T: U \to V$ given by

$$T(X) = \underset{p \times n}{A} \cdot \underset{n \times m}{X}$$

is a linear transformation.

5) Let $U = M^{n\times n}$, $V = M^{n\times n}$
Then the mapping $T: U \to V$ given by
$$T(A) = A^T \quad (\text{the transpose of } A)$$
is linear transformation.

6) $U = M^{n\times n}$, $V = \mathbb{R}$.
$T: U \to V$ given by $T(A) = \operatorname{Det}(A)$ is <u>not</u> a linear transformation if $n > 1$, since
$$T(\alpha \cdot A) = \alpha^{n^2} T(A) \neq \alpha T(A) \text{ if } n > 1.$$

7) $U = \mathbb{R}$, $V = \mathbb{R}$. The mapping $R: U \to V$ given by $R(x) = x^2$ is not linear since $R(\alpha x) = \alpha^2 R(x) \neq$ $\neq \alpha x$ in general.

8) $U = C[a, b]$, $V = \mathbb{R}$. $L: U \to V$ given by
$$L(f) = \int_a^b f(x)\,dx$$
is a linear transformation;
$$T(f) = \int_a^b f(x) \cdot \sin x \, dx$$
is a linear transformation;
$$Q(f) = \int_a^b \sin(f(x))\,dx$$
is <u>not</u> linear
$$G(f) = \sin\left(\int_a^b f(x)\,dx\right)$$
is <u>not</u> linear.

Theorem Let U, V, and W be vector spaces then

a) for any linear transformations $S: U \to V$ and $T: U \to V$, the mapping $(S+T): U \to V$ given by $(S+T)(u) = Su + Tu$ is a linear transformation.

b) for linear transformations $S: U \to V$ and $R: V \to W$, the mapping $R \circ S$ given by $(R \circ S)(u) = R(S(u))$ is a linear transformation

Thus, the linear transformations are stable under taking the sum and the composit if we shall treat their domains U, V, W carefully.

Theorem Let T be a linear transformation from U to V, and let $X \subseteq U$, $Y \subseteq V$ be subspaces of U and of V Then $T(X)$ is a subspace of V and $T^{-1}(Y)$ is a subspace of U.

Df $T(X)$ is called the **image** os T (and denoted by $im(T)$) if $X = U$.

$T^{-1}(Y)$ is called the **kernel** (= kernel space) of T (denoted by $ker(T)$)

Theorem Let $T: U \to V$ be a linear transformation then

a) $\dim(T(U)) \leq \dim(U)$

b) $\dim(U) = \underbrace{\dim(T(U))}_{= im(T)} + \underbrace{\dim(T^{-1}(\{0\}))}_{= ker(T)}$

Example

Find $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}^{2010}$.

Solution The rotation by angle φ is given by the matrix

$$R_\varphi = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}.$$

In our case we have to find the following matrix: $R^{2010}_{\left(-\frac{\pi}{6}\right)}$.

Since $2010\left(-\frac{\pi}{6}\right) = -335\pi = 334\pi - \pi = -\pi \pmod{2\pi}$, we obtain:

$$R^{2010}_{\left(-\frac{\pi}{6}\right)} = R_{(-\pi)} = \begin{bmatrix} \cos(-\pi) & -\sin(-\pi) \\ \sin(-\pi) & \cos(-\pi) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

MATRIX REPRESENTATION OF LINEAR TRANSFORMATIONS

Let $T: X \to Y$ be a linear operator. Then for any subspace $X_0 \subseteq X$ $T(X_0)$ is a subspace of Y, and for any subspace $Y_0 \subseteq Y$ its inverse image $T^{-1}(Y_0)$ is a subspace of X.

Df $T(X)$ is called the **image** of T and is denoted by $im(T)$.
The subspace $T^{-1}(\{0\})$ is called the **kernel** of T and is denoted by $ker(T)$.

Theorem Let $T: X \to Y$ be a linear operator then $\dim(X) = \dim(im(T)) + \dim(ker(T))$.

Theorem The inverse S^{-1} of a linear operator S is a linear operator (if exists).
If $\dim(X) < \infty$ then, for any $T: X \to X$, T is invertible iff $ker(T) = \{0\}$.

Exercise Find a basis for **ker** $(T^2 + T)$, where $T: \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ is the operation of differentiation.

Solution →

Take $a+bx+cx^2+dx^3 \in \mathcal{P}_3(\mathbb{R})$ is an arbitrary polynomial.

$$(T^2+T)(a+bx+cx^2+dx^3) = (a+bx+cx^2+dx^3)'' +$$
$$+ (a+bx+cx^2+dx^3)' = 2c + 6dx + b + 2cx + 3dx^2 = 0$$

This implies

$$\begin{cases} b+2c = 0 \\ 6d+2c=0 \\ 3d=0. \end{cases}$$

Then $d=c=b=0$ and $\ker(T^2+T) = \langle 1 \rangle$, and $\{1\}$ is a basis for $\ker(T^2+T)$.

Exercise Find a basis for $\operatorname{im}(T^2+T)$

(T is from the previous Ex.)

Solution

$\dim(\mathcal{P}_3(\mathbb{R})) = 4$
$\dim(\ker(T^2+T)) = 1$ $\xRightarrow{\text{By Theorem!}}$ $\dim(\operatorname{im}(T^2+T)) = 4-1=3$.

It is enough to find 3 linearly independent vectors $(p_1(x), p_2(x), p_3(x))$ in $\operatorname{im}(T^2+T)$.

Take vectors x, x^2, x^3, they are linearly independent in $\mathcal{P}_3(\mathbb{R})$.

$$P_1(x) = (T^2+T)x = \frac{d^2}{dx^2}(x) + \frac{d}{dx}(x) = 1,$$

$$P_2(x) = (T^2+T)x^2 = \frac{d^2}{dx^2}(x^2) + \frac{d}{dx}(x^2) = 2+2x;$$

$$P_3(x) = \frac{d^2}{dx^3}(x^3) + \frac{d}{dx}(x3) = 6x + 3x^2$$

Coordinates are columns of echelon matrix with leading entries.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\{1, 2+2x, 6x+3x^2\}$ is a basis for $\operatorname{im}(T^2+T)$.

Theorem Let V be a vector space with a basis $B = \{v_1, \dots, v_n\}$ and let W be a vector space then

a) any linear operator $T : V \to W$ is uniquely determined by vectors $Tv_1, Tv_2, \dots, Tv_n$;

b) If $C = \{w_1, \dots, w_m\}$ is a basis for W then there exists a unique matrix $A_T \in M^{m \times n}$ such that

$$[T(v)]_C = A_T \cdot [v]_B.$$

This matrix is called the matrix of T relative to bases B and C.

Example Let $B = \{v_1, \dots, v_n\}$ and $C = \{w_1, \dots, w_n\}$ be bases for V. Then the coordinate matrix of $Id : V \to V$ is the transition matrix $P_{B \to C}$. $P_{B \to C}$ is determined as

$$P_{B \to C} = \left[[v_1]_C \ [v_2]_C \ \dots \ [v_n]_C \right] = \left[[Id(v_1)]_C \ \dots \ [Id(v_n)]_C \right].$$

This example leads to the proof of the theorem above and to the clear construction of coordinate matrix A_T of the operator T.

$$A_T = \left[[Tv_1]_C \ [Tv_2]_C \ \dots \ [Tv_n]_C \right]$$

$$Tv_1 = a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m$$
$$Tv_2 = a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m$$
$$\dots$$
$$Tv_n = a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_m$$

$$[Tv_1]_C = \begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{bmatrix} \quad \dots \quad [Tv_n]_C = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \dots \\ a_{mn} \end{bmatrix}$$

$$A_T = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Example Let $L : \mathbb{R}^4 \to \mathbb{R}^2$ is given by

$$L(x_1, x_2, x_3, x_4) = (x_1 + x_3, \ -x_2 - x_4).$$

Find the coordinate matrix relative to

$$\mathcal{E} = \{ \underset{v_1}{(1,0,0,0)}, \ \underset{v_2}{(0,1,0,0)}, \ \underset{v_3}{(0,0,1,0)}, \ \underset{v_4}{(0,0,0,1)} \}$$

$$C = \{ \underset{w_1}{(1,1)}, \ \underset{w_2}{(1,-1)} \}$$

$$Lv_1 = L((1,0,0,0)) = (1,0) = \tfrac{1}{2}(1,1) + \tfrac{1}{2}(1,-1)$$
$$Lv_2 = L((0,1,0,0)) = (0,-1) = (-\tfrac{1}{2})(1,1) + \tfrac{1}{2}(1,-1).$$
$$Lv_3 = L((0,0,1,0)) = (1,0) = \tfrac{1}{2}(1,1) + \tfrac{1}{2}(1,-1)$$
$$Lv_4 = L((0,0,0,1)) = (0,-1) = (-\tfrac{1}{2})(1,1) + \tfrac{1}{2}(1,-1).$$

$$A = \left[\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} \right] = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

If $v = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ then

$$[L(v)]_C = A \cdot [v]_{\mathcal{E}} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x_1 - x_2 + x_3 - x_4 \\ x_1 + x_2 + x_3 + x_4 \end{bmatrix}$$

Example Let $L : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$L(x_1, x_2, x_3, x_4) = (2x_1 + x_2 - x_4,\ x_3 + x_4,\ x_1 - x_3)$

Find the matrix of L relative to the standard basis.

Solution $A = \Big[\underbrace{[\ ]\ [\ ]\ [\ ]\ [\ ]}\Big]$

$L((1,0,0,0)) = (2,0,1)$
$L((0,1,0,0)) = (2,0,0)$
$L((0,0,1,0)) = (0,1,-1)$
$L((0,0,0,1)) = (-1,1,0)$

$$A = \begin{bmatrix} 2 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

Exercise 1 Let $\mathcal{P}_3(\mathbb{R}) \xrightarrow{T} \mathcal{P}_3(\mathbb{R})$ be a linear operator such that

$T(1) = 1 + t, \quad T(t) = t + t^2, \quad T(t^2) = t^2 + t^3, \quad T(t^3) = 1.$

a) Find $T(2 - t + t^2 - t^3) =$

$= 2 \cdot L(1) + (-1)\,T(t) + T(t^2) + (-1)\,T(t^3) = \ldots$

b) Compute the matrix A_T relative to the basis $B = \{1, t, t^2, t^3\}$

c) Compute the matrix A_T relative to the basis $C = \{1 + t, 1 + t^2, t^2 + t^3, 1\}$

d) Compute A_T with respect to (B, C).

Exercise 2

Compute the matrix $A_{\mathcal{D}}$ relative to $B = \{1, x, x^2, \cos x, \sin x\}$,

where $\mathcal{D} = \dfrac{d}{dx}$, $\quad \mathcal{D}: V \to V$, $\quad V = \mathrm{span}(B)$.

Solutions of Ex 1 and Ex 2

1a) $T(2-t+t^2-t^3) = 2(1+t)-(t+t^2)+(t^2+t^3)-1 =$

$= 2+2t-t-t^2+t^2+t^3-1 = 1+t+t^3$

1b) $Tv_1 = 1+t = 1\cdot v_1 + 1\cdot v_2 + 0\cdot v_3 + 0\cdot v_4$

$Tv_2 = t+t^2 = 0\cdot v_1 + 1\cdot v_2 + 1\cdot v_3 + 0\cdot v_4$

$Tv_3 = t^2+t^3 = 0\cdot v_1 + 0\cdot v_2 + 1\cdot v_3 + 1\cdot v_4$

$Tv_4 = 1 = 1\cdot v_1 + 0\cdot v_2 + 0\cdot v_3 + 0\cdot v_4$

$[Tv_1]_B = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, [Tv_2]_B = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, [Tv_3]_B = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, [Tv_4]_B = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$

$$A_T = \begin{bmatrix}1&0&0&1\\1&1&0&0\\0&1&1&0\\0&0&1&0\end{bmatrix}$$

1d) $Tv_1 = 1+t = 1\cdot w_1$

$Tv_2 = t+t^2 = w_1 + w_2 - 2w_4$

$Tv_3 = t^2+t^3 = w_3$

$Tv_4 = 1 = w_4$

$[Tv_1]_C = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, [Tv_2]_C = \begin{bmatrix}1\\1\\0\\-2\end{bmatrix}, [Tv_3]_C = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, [Tv_4]_C = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

$$A_T = \begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&-2&0&1\end{bmatrix}$$

1c) $Tw_1 = T(1+t) = 1+t+t+t^2 = 1+2t+t^2 = 2w_1 + w_2 - 2w_4$

$Tw_2 = T(1+t^2) = 1+t+t^2+t^3 = w_1 + w_3$

$Tw_3 = T(t^2+t^3) = t^2+t^3+1 = w_3 + w_4$

$Tw_4 = T(1) = 1+t = w_1$

$[Tw_1]_C = \begin{bmatrix}2\\1\\0\\-2\end{bmatrix}, [Tw_2]_C = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}; [Tw_3]_C = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, [Tw_4]_C = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $A_T = \begin{bmatrix}2&1&0&1\\1&0&0&0\\0&1&1&0\\-2&0&1&0\end{bmatrix}$

Solution of Ex 2

$\mathcal{D}(v_1) = 1' = 0$

$\mathcal{D}(v_2) = x' = 1 = 1 \cdot v_1$

$\mathcal{D}(v_3) = (x^2)' = 2x = 2 \cdot v_2$

$\mathcal{D}(v_4) = (\cos x)' = -\sin x = (-1) \cdot v_5$

$\mathcal{D}(v_5) = (\sin x)' = \cos x = 1 \cdot v_4$

$$[\mathcal{D}v_1]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\mathcal{D}v_2]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\mathcal{D}v_3]_B = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\mathcal{D}v_4]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, [\mathcal{D}v_5]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$A_{\mathcal{D}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Example Let A ~~be~~ a 3×3-matrix satisfying $\|A\bar{x}\|_2 \le \|\bar{x}\|_2$ for all $\bar{x} = (x_1, x_2\, x_3) \in \mathbb{R}$, where $\|\bar{x}\|_2 = \sqrt[2]{x_1^2 + x_2^2 + x_3^2}$. Show that $|A| \le 1$.

Solution $A(B(0,1)) \subseteq B(0,1)$, where $B(0,1)$ is the unit ball $\{\bar{x} \in \mathbb{R}^3 : \|\bar{x}\|_2 \le 1\}$. Since $Vol(A(\Omega)) = |A| \cdot Vol(\Omega)$ for every $\Omega \subseteq \mathbb{R}^3$ for which the volume $Vol(\Omega)$ is defined, one gets:

$$|A| \cdot Vol(B(0,1)) = Vol(A(B(0,1))) \le Vol(B(0,1)).$$

Hence $|A| \le 1$.