Lecture 13

DIAGONALIZATION OF REAL SYMMETRIC MATRICES

Df A square matrix A is called **symmetric** if $A^T = A$.

We consider the **standard inner product**

$$(X|Y) = X^T \cdot Y \qquad (X, Y \in M^{n \times 1})$$

on the space of all $n \times 1$-matrices.

Theorem Let A be a real symmetric matrix.

Then

a) all eigenvalues are real.
(this means that all roots of $\Delta_A(\lambda) = 0$ are reac numbers).

b) eigenvectors corresponding to distinct eigenvalues are orthogonal.
(This means that if P_1 corresponds to λ_1 and P_2 corresponds to λ_2, and $\lambda_1 \neq \lambda_2$, then $(P_1|P_2) = P_1^T \cdot P_2 = 0$)

Proof (b) only)

Let $\lambda_1 \neq \lambda_2$ be eigenvalues corresponging to P_1 and P_2.

Then $\lambda_1 (P_1|P_2) = (\lambda P_1|P_2) = (AP_1|P_2) = (AP_1)^T \cdot P_2 =$

$= P_1^T A^T P_2 = P_1^T A P_2 = P_1^T \lambda_2 P_2 = \lambda_2 (P_1|P_2)$.

Since $\lambda_1 \neq \lambda_2$, the equality $\lambda_1 (P_1|P_2) = \lambda_2 (P_1|P_2)$ implies $(P_1|P_2) = 0$. $\blacksquare$

Df A square matrix Q is called **orthogonal** if $Q^T Q = I$.
(In other words, i Q is invertible and $Q^{-1} = Q^T$.)

Theorem A square matrix $Q = [Q_1 Q_2 \dots Q_n]$ is orthogonal iff its columns $\{Q_1, Q_2, \dots, Q_n\}$ form the orthonormal basis for $M^{n\times 1}$

Proof Q is orthogonal $\Leftrightarrow$

$$\Leftrightarrow \quad Q^T Q = \begin{bmatrix} Q_1^T \\ Q_2^T \\ \vdots \\ Q_n^T \end{bmatrix} \cdot [Q_1 Q_2 \dots Q_n] = I \quad \Leftrightarrow$$

$$\Leftrightarrow (Q^T Q)_{ij} = Q_i^T Q_j = \delta_{ij} \text{ for every } i, j \Leftrightarrow$$

$$\Leftrightarrow (Q_i \mid Q_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

∎

Theorem **If** A is a real symmetric matrix then A is **diagonalizable** and there exists an orthogonal matrix Q such that $Q^{-1} A Q$ is diagonal.

Comments to the theorem

The only nontrivial part of its proof is to show that A is diagonalizable.
After this we have P s.t. $P^{-1}AP = \mathcal{D} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{bmatrix}$.
Collect all columns $\{P_1, P_2, \dots, P_{m_1}\}$ corresponding to λ_1.
Apply Gram-Shmidt to $\{P_1, \dots, P_{m_1}\}$ to find an <u>orthonormal</u> basis $\{Q_1, \dots, Q_{m_1}\}$ for $\operatorname{span}(P_1, \dots, P_{m_1})$.

Repeat this with each eigenvector λ_j.
Then we obtain an <u>orthonormal</u> family

$$\underbrace{Q_1, \dots, Q_{m_1}}_{\lambda_1}, \underbrace{Q_{m_1+1}, \dots, Q_{m_2}}_{\lambda_2}, \dots\dots, \underbrace{Q_{m_{k-1}+1}, \dots, Q_{m_k}}_{\lambda_k}$$

of eigenvectors which contains n vectors.

Therefore $Q_1, Q_2, \dots, Q_n$ is an orthonormal basis for $M^{n\times 1}$ and

$$Q^{-1} A Q = \mathcal{D},$$

where $Q = [Q_1\ Q_2\ \dots\ Q_n]$.

The matrix Q is orthogonal since $\{Q_1, \dots, Q_n\}$ is an orthonormal basis.
So $Q^T A Q = \mathcal{D}$.

Example

Diagonalize $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

by means of an orthogonal matrix.

Solution

$A = A^T$ is symmetric.

The characteristic polynomial of A is $|\lambda I - A|$.

$$0 = \begin{vmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda-2 & 1 \\ 1 & 1 & \lambda-2 \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda-2 & 1 \\ 1 & \lambda-2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & \lambda-2 \end{vmatrix} + \begin{vmatrix} 1 & \lambda-2 \\ 1 & 1 \end{vmatrix} =$$

$$= (\lambda-2)\left((\lambda-2)^2-1\right) - (\lambda-2-1) + (1-(\lambda-2)) =$$

$$= (\lambda-2)(\lambda^2-4\lambda+3) - 2\lambda+6 = \lambda^3 - 4\lambda^2 + 3\lambda - 2\lambda^2 + 8\lambda \cancel{-6} - 2\lambda \cancel{+6} =$$

$$= \lambda^3 - 6\lambda^2 + 9\lambda = \lambda(\lambda^2-6\lambda+9) = \lambda(\lambda-3)^2 = \lambda(\lambda-3)(\lambda-3).$$

Thus, the eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = 3, \quad \lambda_3 = 3$$

Remark that if all eigenvalues ξ_1, ξ_2, ξ_3 of a symmetric 3×3-matrix are distinct, we need only to find eigenvectors P_1, P_2, P_3 and normalize them to get $\{Q_1, Q_2, Q_3\}$, which we use to construct an orthogonal matrix $Q = [Q_1\ Q_2\ Q_3]$.

But unfortunately, in our case, $\lambda_2 = \lambda_3$

(1) $\lambda = \lambda_1 = 0$

$$\lambda I - A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow[\frac{1}{2}R_1 + R_3]{\frac{1}{2}R_1 + R_2} \begin{bmatrix} -2 & 1 & 1 \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & -\frac{3}{2} \end{bmatrix} \xrightarrow{R_2 + R_3}$$

$$\longrightarrow \begin{bmatrix} -2 & 1 & 1 \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{2}{3}R_2} \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$(\lambda I - A)X = 0$, $\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\begin{cases} -2x + y + z = 0 \\ -y + z = 0 \\ 0 \cdot z = 0 \end{cases}$$

Take z as a free variable

$-2x + 2z = 0$

$y = z$

$$X = \begin{bmatrix} z \\ z \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} z \Rightarrow P_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(2) $\lambda = \lambda_2 = \lambda_3 = 3$

$$\lambda I - A = 3I - A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$X = \begin{bmatrix} u \\ t \\ s \end{bmatrix}$, $\quad u + t + s = 0$

We may take any pair, for example, $\{u, s\}$, as free variables

$$X = \begin{bmatrix} u \\ -u-s \\ s \end{bmatrix} = u\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Remark that we may take, for example, $\tilde{P}_2 = P_2 + P_3$ and $\tilde{P}_3 = P_2 - P_3$, as a basis for span(P_1, P_2) instead of $\{P_1, P_2\}$.

③ Now, what we will do with the family

$$\underbrace{P_1}_{\lambda_1}, \underbrace{P_2, P_3}_{\lambda_2} ?$$

We normalize the 1-st vector and orthonormalize $\{P_2, P_3\}$.

ⓐ $\|P_1\| = \sqrt{3}$, thus $Q_1 = \frac{P_1}{\|P_1\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

ⓑ Apply Gram-Shmidt to $\{P_2, P_3\}$.

$\tilde{P}_2 = P_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$\boxed{\begin{aligned} (P_3 \mid \tilde{P}_2) &= 1 \\ (\tilde{P}_2 \mid \tilde{P}_2) &= 2 \end{aligned}}$$

$\tilde{P}_3 = P_3 - \frac{(P_3 \mid \tilde{P}_2)}{(\tilde{P}_2 \mid \tilde{P}_2)} \tilde{P}_2 = P_3 - \frac{1}{2} P_2$.

$\tilde{P}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

$\|\tilde{P}_2\| = \sqrt{2} \Rightarrow Q_2 = \frac{\tilde{P}_2}{\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$

$\|\tilde{P}_3\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \frac{\sqrt{6}}{2}$

$Q_3 = \frac{\tilde{P}_3}{\|\tilde{P}_3\|} = \frac{\tilde{P}_3}{\left(\frac{\sqrt{6}}{2}\right)} = \frac{2}{\sqrt{6}} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$

$$Q = [Q_1\, Q_2\, Q_3] = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

Remark that the diagonalizing orthogonal matrix Q is uniquely determind iff A possesses n distinct eigenvalues!!! For example, (see p. 243 of the text book)

$Q_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ is another solution of our problem.

Exercise

Diagonalize $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ by means of an orthogonal matrix Q.

Solution

$$|\lambda I - A| = \begin{vmatrix} \lambda-4 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 1 & \lambda \end{vmatrix} = (\lambda-4)(\lambda^2-1) = (\lambda-4)(\lambda-1)(\lambda+1).$$

The eigenvalues are $\lambda_1 = 4$, $\lambda_2 = 1$, $\lambda_3 = -1$.

All eigenvalues are distinct, therefore Q is unique, and we only have to normalize basis of eigenvectors $\{P_1, P_2, P_3\}$.

① $\lambda = 4$.

$$\lambda I - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{(-\frac{1}{4})R_2 + R_3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & \frac{15}{4} \end{bmatrix}$$

$$X = \begin{bmatrix} u \\ t \\ v \end{bmatrix} \qquad \begin{cases} 4t + v = 0 \\ \frac{15}{4} v = 0 \end{cases}$$

$u = u$ – is a free variable.

$$X = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} = u \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow P_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = Q_1 \quad \text{since } \|P_1\| = 1.$$

② $\lambda = 1$

$$\lambda I - A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-R_2 + R_3} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} u \\ t \\ v \end{bmatrix} \qquad \begin{cases} -3u = 0 \\ t + v = 0 \end{cases} \Rightarrow \begin{matrix} u = 0 \\ t = -v \end{matrix}$$

Choose t as a free variable

$$X_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} t \qquad P_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \|P_2\| = \sqrt{2} \Rightarrow Q_2 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

(3) $\lambda = -1$

$$\lambda I - A = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[(-\frac{1}{5})\cdot R_1]{R_2 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} u \\ t \\ v \end{bmatrix} \qquad \begin{cases} u = 0 \\ -t + v = 0 \end{cases}$$

Choose t as a free variable

$$X = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t$$

$$P_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \qquad \|P_3\| = \sqrt{2} \Rightarrow Q_3 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$Q = [Q_1\, Q_2\, Q_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$