Lecture 11

Orthogonal and orthonormal bases and orthogonal projections

<u>Def</u> Let V be an inner product space and $S \subseteq V$.

1) S is called <u>orthogonal</u> if $(v|u)=0$ for every $v,u \in S$ such that $v \neq u$.
2) S is called <u>orthonormal</u> if $(v|v)=1$ when $v \in S$ and if S is orthogonal.

When S is a basis it is called orthogonal (resp. <u>orthonormal Basis</u>) if it is a orthogonal (resp. orthonormal) set.

<u>Theorem 1</u> If $0 \notin S$ is orthogonal then S is a basis for $\langle S \rangle$.

<u>Proof</u> It is enough to show that S is linearly independent. Let $c_1 v_1 + \ldots + c_n v_n = 0$. Multiplying by v_k and using orthogonality one gets $c_k (v_k | v_k) = 0$. Since $v_k \neq 0$ then $c_k = 0$. Since k is arbitrary, we obtain that all $c_k = 0$. ∎

(cf. also Exercise 3 in previous lecture)

Ex1. Vectors $v_1=(1\ 0\ 0)$ $v_2=(0\ 1\ -1)$ $v_3=(0\ 1\ 1)$ are orthogonal. Hence the set $B=\{v_1\ v_2\ v_3\}$ is an orth. basis for $\mathbb{R}^3$. To make it orthonormal we take

$$v_1' = \frac{v_1}{\|v_1\|} = (1,0,0)$$

$$v_2' = \frac{v_2}{\|v_2\|} = \left(0\ \frac{1}{\sqrt{2}}\ -\frac{1}{\sqrt{2}}\right)$$

$$v_3' = \frac{v_3}{\|v_3\|} = \left(0\ \frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\right)$$

Ex2. The set of vectors

$1, \cos x, \cos 2x, \dots \cos nx, \dots$ & $\sin x, \sin 2x, \dots$

is orthogonal in $C[-\pi,\pi]$.

Indeed, for example

$n \neq m$

$$(\sin nx \mid \sin mx) = \int_{-\pi}^{\pi} \sin nx \cos mx\, dx =$$

$$= -\frac{1}{2}\int_{-\pi}^{\pi}\left[\cos(n+m)x - \cos(n-m)x\right]dx =$$

$$= -\frac{1}{2}\left[\frac{\sin(n+m)x}{n+m} - \frac{\sin(n-m)x}{n-m}\right]\Bigg|_{-\pi}^{\pi} = 0-0=0$$

To normalize this set we just remark that $\|1\|=\sqrt{2\pi}$ and $\|\sin kx\| = \|\cos kx\| = \sqrt{\pi}$ for $k=1,2,\dots n,\dots$

Ex3 Evaluate $I = \int_{-\pi}^{\pi} (2+3\cos x - 4\sin x + 5\cos 2x - 6\sin 3x)^2 dx$

Solution Using the orthogonality one gets

$I = (2|2) + (3\cos x|3\cos x) + (-4\sin x|-4\sin x) + (5\cos x|5\cos x) + (-6\sin 3x|-6\sin 3x)$

$= 4\cdot\|1\|^2 + 9\|\cos x\|^2 + 16\|\sin x\|^2 + 25\|\cos x\|^2 + 36\|\sin 3x\|^2 =$

$= 4\cdot 2\pi + 9\cdot\pi + 16\pi + 25\pi + 36\pi = 84\pi$

We know how to construct an inner product in a finite dimensional vector space under which a given basis becomes orthonormal. Namely, let $B = \{v_1, \dots v_n\}$ be a basis for V. Define an inner product by

$$(x|y) := [x]_B^T \cdot [y]_B$$

Then $(v_k|v_p) = 0$ when $k \neq p$ and $(v_k|v_k) = 1$.

Consider the problem of constructing of orthogonal basis from $B = \{v_1, \dots v_n\}$ in the case when we already have an inner product $(\cdot|\cdot)$ in V. The following Gram-Schmidt orthogonalization process gives a solution to the problem.

<u>Theorem 2</u> Let $B = \{v_1, \dots v_k\}$ be a basis for an inner product space $(V, (\cdot|\cdot))$. Then the following set $C = \{u_1, \dots u_k\}$ is an orthogonal basis for V:

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{(v_2|u_1)}{\|u_1\|^2} u_1$$

$$u_3 = v_3 - \frac{(v_3|u_1)}{\|u_1\|^2} u_1 - \frac{(v_3|u_2)}{\|u_2\|^2} u_2$$

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$$u_k = v_k - \frac{(v_k|u_1)}{\|u_1\|^2} u_1 - \frac{(v_k|u_2)}{\|u_2\|^2} u_2 - \dots - \frac{(v_k|u_{k-1})}{\|u_{k-1}\|^2} u_{k-1}$$

Moreover for any $1 \le m \le k$

$$u_m \in \langle v_1, v_2, \dots v_m \rangle$$

and

$$v_m \in \langle u_1, u_2, \dots u_m \rangle$$

<u>Proof</u> We prove the theorem by induction. For $k=1$ it is trivial. Suppose it is true for $k-1$. We have to show that $u_k \perp u_p$ when $p<k$. Indeed

$$(u_p|u_k) = (u_p|v_k) - \frac{(v_k|u_1)}{\|u_1\|^2}(u_p|u_1) - \dots - \frac{(v_k|u_{k-1})}{\|u_{k-1}\|^2}(u_p|u_{k-1}) =$$

$$= \left[\begin{matrix}\text{using that } p<k \text{ and the} \\ \text{orthogonality of } \{u_1, u_2, \dots u_{k-1}\}\end{matrix}\right] = (u_p|v_k) - \frac{(v_k|u_p)}{\|u_p\|^2} \cdot \|u_p\|^2 = 0$$

∎

Corollary Every finite dimensional inner product space X possesses an orthonormal basis.

Proof Since $\dim(X) = n < \infty$, there exists a linear basis $B_1 = \{x_1, x_2, \dots, x_n\}$ for X. Apply the Gram-Schmidt orthogonalization to B_1 and obtain $B_2 = \{y_1, y_2, \dots, y_n\}$ that is an orthogonal basis. Finally, apply normalization and obtain the orthonormal basis $B_3 = \left\{\frac{y_1}{\|y_1\|}, \frac{y_2}{\|y_2\|}, \dots, \frac{y_n}{\|y_n\|}\right\}$ for X. ∎

Ex 4

Construct an orthonormal basis for $\mathcal{P}_2[0,1]$.

Step I Take any basis for $\mathcal{P}_2[0,1]$, for example, $B_1 = \{1, x, x^2\}$.

Step II Apply the Gram-Schmidt orthogonalization

$$y_1 = x_1 = 1.$$

$$y_2 = x_2 - \frac{(x_2|y_1)}{(y_1|y_1)}y_1 = x - \frac{(x|1)}{(1|1)}\cdot 1 = x - \int_0^1 x\,dx = x - \frac{1}{2}.$$

$$y_3 = x_3 - \frac{(x_3|y_1)}{(y_1|y_1)}y_1 - \frac{(x_3|y_2)}{(y_2|y_2)}y_2;$$

$$x_3 = x^2, \quad y_1 = 1, \quad y_2 = x - \frac{1}{2}.$$

$$(x_3|y_1) = \int_0^1 x^2\,dx = \frac{1}{3},$$

$$(y_1|y_1) = \int_0^1 1\,dx = 1;$$

$$(x_3 | y_2) = \int_0^1 x^2(x - \tfrac{1}{2})\,dx = \int_0^1 (x^3 - \tfrac{x^2}{2})\,dx = \tfrac{1}{4} - \tfrac{1}{6} = \tfrac{1}{12}.$$

$$(y_2 | y_2) = \int_0^1 (x - \tfrac{1}{2})^2\,dx = \int_0^1 (x^2 - x + \tfrac{1}{4})\,dx = \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{1}{12}.$$

$$y_3 = x^2 - \tfrac{1}{3} - \frac{1/12}{1/12}\cdot(x - \tfrac{1}{2}) = x^2 - \tfrac{1}{3} + \tfrac{1}{2} - x = x^2 - x + \tfrac{1}{6}.$$

$$B_2 = \{1,\ x - \tfrac{1}{2},\ x^2 - x + \tfrac{1}{6}\}.$$

Step III

$$\|1\| = \sqrt{\int_0^1 1\,dx} = 1$$

$$\|x - \tfrac{1}{2}\| = \sqrt{\int_0^1 (x - \tfrac{1}{2})^2\,dx} = \sqrt{\tfrac{1}{12}} = \frac{1}{2\sqrt{3}}$$

$$\|x^2 - x + \tfrac{1}{6}\| = \sqrt{\int_0^1 (x^2 - x + \tfrac{1}{6})^2\,dx} = \sqrt{\frac{1}{36\cdot 5}} = \frac{1}{6\sqrt{5}}$$

$$\int_0^1 (x^2 - x + \tfrac{1}{6})^2\,dx = \int_0^1 (x^2 - x + \tfrac{1}{6})(x^2 - x + \tfrac{1}{6})\,dx =$$

$$= \int_0^1 \left(x^4 - x^3 + \frac{x^2}{6} - x^3 + x^2 - \frac{x}{6} + \frac{x^2}{6} - \frac{x}{6} + \frac{1}{36}\right)dx =$$

$$= \int_0^1 \left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{x}{3} + \frac{1}{36}\right)dx = \left(\frac{x^5}{5} - \frac{2x^4}{4} + \frac{4x^3}{3\cdot 3} - \frac{x^2}{6} + \frac{x}{36}\right)\Big|_0^1 =$$

$$= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36} = \frac{36 - 90 + 80 - 30 + 5}{36\cdot 5} = \frac{1}{36\cdot 5}.$$

$$B_3 = \{z_1, z_2, z_3\}$$

$$z_1 = \frac{y_1}{\|y_1\|} = y_1 = 1$$

$$z_2 = \frac{y_2}{\|y_2\|} = \frac{x - \frac{1}{2}}{\frac{1}{2\sqrt{3}}} = 2\sqrt{3}\,x - \frac{2\sqrt{3}}{2} = 2\sqrt{3}\,x - \sqrt{3}$$

$$z_3 = \frac{y_3}{\|y_3\|} = (x^2 - x + \tfrac{1}{6}) \Big/ \frac{1}{6\sqrt{5}} = 6\sqrt{5}\,x^2 - 6\sqrt{5}\,x + \sqrt{5}$$

$$B_3 = \{1,\ 2\sqrt{3}\,x - \sqrt{3},\ 6\sqrt{5}\,x^2 - 6\sqrt{5}\,x + \sqrt{5}\}.$$

B_3 is an orthonormal basis for $\mathcal{P}_2[0,1]$.

Ex 5 Find an orthogonal basis for the L.7

span ((1, 1, 1, 0, 0, 0), (1, 1, 1, 1, 1, 1), (0, 0, 1, 1, 0, 0)) in $\mathbb{R}^6$.

Solution The given set of three vectors is obviously linearly independent.

$x_1 = (1, 1, 1, 0, 0, 0)$

$x_2 = (1, 1, 1, 1, 1, 1)$

$x_3 = (0, 0, 1, 1, 0, 0)$.

$y_1 = x_1 = (1, 1, 1, 0, 0, 0)$

$$y_2 = x_2 - \frac{(x_2 | y_1)}{(y_1 | y_1)} y_1 = (1, 1, 1, 1, 1, 1) - \frac{3}{3}(1, 1, 1, 0, 0, 0) =$$

$= (1, 1, 1, 1, 1, 1) - (1, 1, 1, 0, 0, 0) = (0, 0, 0, 1, 1, 1)$.

$$y_3 = x_3 - \frac{(x_3 | y_1)}{(y_1 | y_1)} y_1 - \frac{(x_3 | y_2)}{(y_2 | y_2)} y_2 =$$

$= x_3 - \frac{1}{3} y_1 - \frac{1}{3} y_2 =$

$= (0, 0, 1, 1, 0, 0) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0\right) - \left(0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) =$

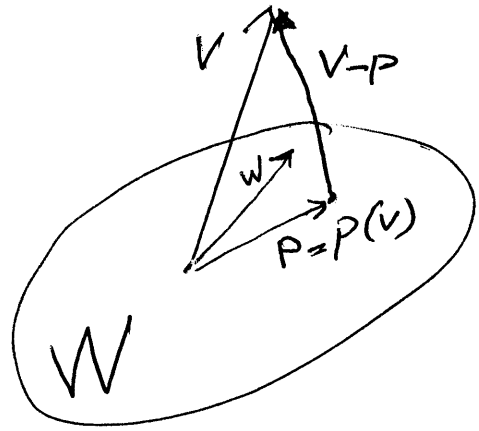
$= \left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$.

$$B = \left\{(1, 1, 1, 0, 0, 0), (0, 0, 0, 1, 1, 1), \left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)\right\}.$$

<u>Def</u> Let W be a subspace of an inner product space V. A vector $p \in W$ is called an <u>orthogonal projection</u> of $v \in V$ if

$$v - p \perp w$$

for all $w \in W$.



The vector $p = p(v) \in W$ is the <u>best approximation</u> of v in W in the sense that

$$\|v - p\| \leq \|v - w\|$$

for all $w \in W$.

<u>Theorem 3</u> Let V be an inner product space and let $W \subseteq V$, $\dim(W) < \infty$ be a subspace with a basis $B = \{w_1, w_2, \ldots w_n\}$. Each $v \in V$ has a uniquely determined orth. projection $p = p(v) = \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n$

$$(w_1 | w_1)\alpha_1 + \ldots + (w_n | w_1)\alpha_n = (v | w_1)$$

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$$(w_1 | w_n)\alpha_1 + \ldots + (w_n | w_n)\alpha_n = (v | w_n)$$

$\|v\| \geq \|p\|$ and $\|v\| = \|p\|$ iff $v = p$.

Moreover

$$p = \frac{(v|w_1)}{\|w_1\|^2} w_1 + \frac{(v|w_2)}{\|w_2\|^2} w_2 + \dots + \frac{(v|w_n)}{\|w_n\|^2} w_n$$

when B is an orthogonal basis. ∎

For a proof consult the textbook on p 225.

__Corollary__ (Bessel's inequality) If $\{w_1, \dots w_n\}$ is an orthonormal set then for each $v \in V$ we have

$$\|v\|^2 \geq \sum_{k=1}^{n} |(v|w_k)|^2$$

__Proof__ Take $W = \langle w_1, \dots w_n \rangle$ and the projection $p = p(v)$ onto W. $p = (v|w_1)w_1 + (v|w_2)w_2 + \dots + (v|w_n)w_n$

Use the Pyhagorean theorem to get $\|p(v)\|^2 = \sum_{k=1}^{n} |(v|w_k)|^2$. By the approximation property $\|v\| \geq \|p(v)\|$. This completes the proof. ∎

__Ex6.__ Find the closed vector to $v = (1, 0, 2\ 3)$ in the span of orthogonal vects $(1\ 0\ 1\ 1), (1\ 1\ {-1}\ 0), (1, -1\ 0\ {-1})$

__Solution__ The orth. projection $p = p(v)$ is

$$p = \frac{(v|w_1)}{\|w_1\|^2} w_1 + \frac{(v|w_2)}{\|w_2\|^2} w_2 + \frac{(v|w_3)}{\|w_3\|^2} w_3 =$$

$$\left[\begin{aligned} &\|w_1\|^2 = \|w_2\|^2 = \|w_3\|^2 = 3 \\ &(v|w_1) = 1 + 3 = 4 \\ &(v|w_2) = 1 - 2 = -1 \\ &(v|w_3) = 1 - 3 = -2 \end{aligned}\right.$$

$$= \frac{4}{3}(1011) - \frac{1}{3}(11{-1}0) - \frac{2}{3}(1{-1}0{-1}) = \left(\frac{1}{3}, \frac{1}{3}, \frac{5}{3}, 2\right).$$

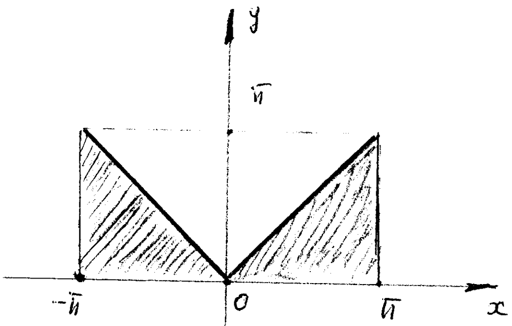
Ex 7 Find the orthogonal projection of $f(x) = |x| = v$ on the space spanned by $\{1, \sin x, \cos x\}$ in $C[-\pi, \pi]$.

$$\underset{v_1}{1}, \quad \underset{v_2}{\sin x}, \quad \underset{v_3}{\cos x}$$

Solution

$$\begin{cases} (v_1 | v_1)\, x_1 + (v_2 | v_1)\, x_2 + (v_3 | v_1)\, x_3 = (v | v_1) \\ (v_1 | v_2)\, x_1 + (v_2 | v_2)\, x_2 + (v_3 | v_2)\, x_3 = (v | v_2) \\ (v_1 | v_3)\, x_1 + (v_2 | v_3)\, x_2 + (v_3 | v_3)\, x_3 = (v | v_3) \end{cases}$$

$$(*) \begin{cases} 2\pi\, x_1 + 0\, x_2 + 0\, x_3 = \pi^2 \\ 0\, x_1 + \pi\, x_2 + 0\, x_3 = 0 \\ 0\, x_1 + 0\, x_2 + \pi\, x_3 = -4 \end{cases}$$



$$(|x| \,|\, 1) = \int_{-\pi}^{\pi} |x|\, dx = \pi^2$$

$$(|x| \,|\, \sin x) = \int_{-\pi}^{\pi} |x| \cdot \sin x \, dx = 0 \quad \text{since } |x| \cdot \sin x \text{ is odd.}$$

$$(|x| \,|\, \cos x) = 2 \int_0^{\pi} x \cos x \, dx = 2\left[x \sin x \Big|_0^{\pi} - \int_0^{\pi} \sin x \, dx \right] =$$

$$= 2\left[0 + \cos x \Big|_0^{\pi} \right] = -4.$$

$$(*) \iff \begin{cases} 2\pi\, x_1 = \pi^2 \\ \pi\, x_2 = 0 \\ \pi\, x_3 = -4 \end{cases} \Rightarrow \quad \begin{aligned} x_1 &= \frac{\pi}{2} \\ x_2 &= 0 \\ x_3 &= -\frac{4}{\pi} \end{aligned}$$

$$P(|x|) = x_1 \cdot v_1 + x_2 \cdot v_2 + x_3 \cdot v_3 = \frac{\pi}{2} \cdot 1 + 0 \cdot \sin x - \frac{4}{\pi} \cdot \cos x \Rightarrow$$

$$\boxed{p(|x|) = \frac{\pi}{2} - \frac{4}{\pi} \cos x}$$