Lecture 10

Inner products, Orthogonality.

Recall that for vectors $\bar{x}=(x_1, x_2)$ and $\bar{y}=(y_1, y_2)$ of $\mathbb{R}^2$ the dot product is defined by

$$\bar{x}\cdot\bar{y} = \|\bar{x}\|\|\bar{y}\|\cdot\cos(\widehat{\bar{x}\,\bar{y}}) = x_1y_1 + x_2y_2$$

It is clear that $\bar{x}\perp\bar{y}$ iff $\bar{x}\cdot\bar{y}=0$, and moreover $\|\bar{x}\| = \sqrt{\bar{x}\cdot\bar{x}}$. The dot product is also known as the scalar product and it is defined in any Euclidean space $\mathbb{R}^n$ by

$$\bar{x}\cdot\bar{y} = \sum_{k=1}^{n} x_k y_k .$$

The following definition presents an important generalization of the dot product.

Definition Let V be a vector space. A function $(.|.) : V\times V \to \mathbb{R}$ is called an inner product if

1) $0\neq v\in V \Rightarrow (v|v)>0$
2) $u,v\in V \Rightarrow (u|v)=(v|u)$
3) $u_1,u_2,v\in V \Rightarrow (u_1+u_2|v)=(u_1|v)+(u_2|v)$
4) $u,v\in V, \alpha\in\mathbb{R} \Rightarrow (\alpha u|v)=\alpha\cdot(u|v)$.

It is clear that $(u|0)=0$ and $(v|v)=0$ iff $v=0$.

Besides dot product in $\mathbb{R}^n$ there exist several important examples of inner products.

<u>Ex1</u> Let $a_1, a_2, \dots a_n$ be positive scalars, that is $a_k > 0$ for $k = \overline{1,n}$. The function $F(\bar{x},\bar{y}) = \sum_{k=1}^{n} a_k x_k y_k$ is obviously an inner product in $\mathbb{R}^n$.

<u>Ex2</u> Let $V = \mathbb{R}^{n\times 1}$. The function $G(A,B) = A^T B$ is an inner product in V.

<u>Ex3</u> Let $V = \mathbb{R}^{n\times m}$, $(A|B) = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik} B_{ik}$.

<u>Ex4</u> Let $V = C[a,b]$, $(f|g) = \int_a^b f(t)g(t)\,dt$.

<u>Exercise 1</u> Consider $C[0,6]$. Find $\alpha \in \mathbb{R}$ s.t. $(\alpha + x \,|\, x^2) = 0$.

<u>Solution</u> $0 = (\alpha + x \,|\, x^2) = \int_0^6 (x^3 + \alpha x^2)\,dx = x^3\left(\frac{x}{4} + \frac{\alpha}{2}\right)\Big|_0^6$

Then $\frac{6}{4} + \frac{\alpha}{2} = 0$ and hence $\alpha = -3$.

<u>Exercise 2</u> Find an inner product $(\cdot|\cdot)$ in $\mathbb{R}^3$ s.t. $\bar{v}_1 = (1,2,1)$ and $\bar{v}_2 = (0,-3,0)$ satisfy $(\bar{v}_1|\bar{v}_1) = (\bar{v}_2|\bar{v}_2) = 1$, $(\bar{v}_1|\bar{v}_2) = 0$.

<u>Solution</u> Take a basis $C = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ in $\mathbb{R}^3$ where $\bar{v}_3 = (0,0,7)$. Define $(\bar{x}|\bar{y}) = [x]_C^T \cdot [y]_C$. Then $(\bar{v}_1|\bar{v}_2) = [1,0,0]\cdot\begin{bmatrix}0\\1\\0\end{bmatrix} = 0$, $(\bar{v}_1|\bar{v}_1) = [1\ 0\ 0]\begin{bmatrix}1\\0\\0\end{bmatrix} = 1$, and $(\bar{v}_2|\bar{v}_2) = [0\ 1\ 0]\begin{bmatrix}0\\1\\0\end{bmatrix} = 1$.

The following example is a generalization of Exercise 2 (see, also, Example 2)

<u>Ex 5</u> Let V be a vector space with a basis $B=\{v_1, v_2, \dots v_n\}$ Then the following function $(\cdot|\cdot): V\times V \to \mathbb{R}$:

$$(x|y) = [x]_B^T \cdot [y]_B \qquad (x,y\in V)$$

is an inner product on V.

Indeed, $(v_k|v_p) = \begin{cases} 0 & \text{if } k\neq p \\ 1 & \text{if } k=p \end{cases}$.

Hence for $x\in V$ with the coordinate matrix

$$[x]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

we obtain that $(x|x) = x_1^2 + x_2^2 + \dots + x_n^2 = 0$ iff $[x]_B$ is zero matrix, that is $x=0$.

$(x|y) = [x]_B^T \cdot [y]_B = ([x]_B^T [y]_B)^T = [y]_B^T [x]_B = (y|x)$. The properties $(x+z|y) = (x|y)+(z|y)$ and $(\alpha x|y) = \alpha(x|y)$ follow immediately from the linearity of the matrix multiplication.

<u>Exercise 3</u> Let $S=\{v_1,v_2,\dots v_n\}\subseteq V$ where $V=(V,(\cdot|\cdot))$ is an inner product space. Show that S is linearly independent iff $|G|\neq 0$, where

$$G=G(v_1,v_2,\dots,v_n)=\begin{bmatrix}(v_1|v_1) & (v_2|v_1) & \dots & (v_n|v_1)\\ (v_1|v_2) & (v_2|v_2) & \dots & (v_n|v_2)\\ \vdots & & & \vdots\\ (v_1|v_n) & (v_2|v_n) & \dots & (v_n|v_n)\end{bmatrix}$$

is the <u>Gram matrix</u> of family S.

<u>Solution</u> Let $c_1v_1+c_2v_2+\dots+c_nv_n=0$

Taking the inner product with $v_1,v_2,\dots v_n$ gives us

$$\begin{cases} c_1(v_1|v_1)+c_2(v_2|v_1)+\dots+c_n(v_n|v_1)=0\\ c_1(v_1|v_2)+c_2(v_2|v_2)+\dots+c_n(v_n|v_2)=0\\ \text{-------------------}\\ c_1(v_1|v_n)+c_2(v_2|v_n)+\dots+c_n(v_n|v_n)=0\end{cases}$$

Then $C=\begin{bmatrix}c_1\\c_2\\ \vdots\\ c_n\end{bmatrix}$ is a solution of the homogenous system $GX=0$.

Now it is enough to appeal to the fact that $GX=0$ has no nontrivial solutions iff $|G|\neq 0$.

NORMS. ORTHOGONALITY.
ORTHOGONAL AND ORTHONORMAL BASES

Df Let $(.|.)$ be an inner product on a vector space X.

a) Given $x \in X$ the value
$$\|x\| := \sqrt{(x|x)}$$
is called the **norm** (or the **length**) of x.

b) If $x, y \in X$, the **angle** θ **between them** is defined by
$$\cos\theta := \frac{(x|y)}{\|x\| \cdot \|y\|}.$$

c) Two vectors $x, y \in X$ are called **orthogonal** if $(x|y) = 0$.

Ex 6. The norm of a vector $(\sqrt{3}, -3, -2)$ in $\mathbb{R}^3$ with respect to the standard inner product (=dot product) is:

$$\|(\sqrt{3}, -3, -2)\| = \sqrt{(\sqrt{3})^2 + (-3)^2 + (-2)^2} = \sqrt{3+9+4} = 4$$

Exercise 4 In $C[-\pi,\pi]$ with the integral inner product, find the norms of $f(x) = \sin x$ and $g(x) = 1$ and show that they are orthogonal.

Solution

$$\|f\|^2 = \|\sin x\|^2 = \int_{-\pi}^{\pi} \sin^2 x \, dx = \frac{1}{2}\int_{-\pi}^{\pi}(1-\cos 2x)\,dx =$$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)\Big|_{-\pi}^{\pi} = \pi - \frac{1}{4}\sin 2x\Big|_{-\pi}^{\pi} = \pi.$$

and $\|f\| = \sqrt{\pi} = \|\sin x\|$.

$$\|1\|^2 = (1|1) = \int_{-\pi}^{\pi} 1\,dx = 2\pi, \quad \text{and} \quad \|1\| = \sqrt{2\pi}$$

$$(\sin x | 1) = \int_{-\pi}^{\pi} \sin x \, dx = (-\cos x)\Big|_{-\pi}^{\pi} = 0$$

Exercise 5 Find the angle between $v = (2,-2,0,-1)$ and $w = (1,-1,-1,1)$ in $\mathbb{R}^4$.

Solution

$$(v|w) = 2\cdot 1 + (-2)(-1) + 0(-1) + (-1)\,1 = 3.$$

$$\|v\| = \sqrt{4+4+0+1} = 3$$

$$\|w\| = \sqrt{1+1+1+1} = 2$$

$$\cos\theta = \frac{(v|w)}{\|v\|\cdot\|w\|} = \frac{3}{3\cdot 2} = \frac{1}{2}$$

So, $\theta = 60^\circ = \frac{\pi}{3}$.

Exercise 6 Normalize vectors:

a) $x = (2, -1, 1, -2) \in \mathbb{R}^4$

b) $y = (1, 1, 2, 2) \in \mathbb{R}^4$

c) $z = \begin{bmatrix} 0 & -1 \\ 1 & \sqrt{3} \end{bmatrix} \in M^{2\times 2}$

d) $v = x^2 \in C[0,1]$

Solutions

a) $\|x\| = \sqrt{2^2 + (-1)^2 + 1^2 + (-2)^2} = \sqrt{10}$

The **normalized vector** (i.e. $\|\bar{x}\| = 1$):

$$\bar{x} = \frac{x}{\|x\|} = \frac{x}{\sqrt{10}} = \left(\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right).$$

b) $\|y\| = \sqrt{1^2 + 1^2 + 2^2 + 2^2} = \sqrt{10}$

$$\bar{y} = \frac{y}{\|y\|} = \frac{y}{\sqrt{10}} = \left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right).$$

c) $\|z\| = \sqrt{0^2 + 1^2 + (-1)^2 + (\sqrt{3})^2} = \sqrt{5}$

$$\bar{z} = \frac{z}{\|z\|} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \sqrt{\frac{3}{5}} \end{bmatrix}$$

d) $\|v\| = \|x^2\| = \sqrt{\int_0^1 x^2 dx} = \sqrt{\frac{1}{3}x^3\Big|_0^1} = \frac{1}{\sqrt{3}}$

$$\bar{v} = \frac{v}{\|v\|} = \frac{x^2}{\frac{1}{\sqrt{3}}} = \sqrt{3}\,x^2.$$

Exercise 7

Let $X = \mathcal{P}_1[0,1]$ be a vector space with the integral norm. Find normalized elements orthogonal to x

Solution

$X = \mathrm{span}(1, x)$, therefore the general form of elements of X is $ax+b$.

By definition, $y \perp x$ if $\int_0^1 (ax+b)x\,dx = 0$.

$$\int_0^1 (ax^2+bx)\,dx = \left(\frac{ax^3}{3} + \frac{bx^2}{2}\right)\Big|_0^1 = \frac{a}{3} + \frac{b}{2} = 0$$

$$\Rightarrow a = -\frac{3}{2}b.$$

Hence $\{x\}^\perp = \left\{-\frac{3}{2}bx + b : b \in \mathbb{R}\right\}$.

$\|y_0\| = 1 = \|y_0\|^2$

$$\int_0^1 \left(-\frac{3}{2}bx + b\right)^2 dx = b^2\int_0^1 \left(-\frac{3}{2}x+1\right)^2 dx = b^2\int_0^1\left(\frac{9}{4}x^2 - 3x + 1\right)dx =$$

$$= b^2\left(\frac{9}{4}\frac{x^3}{3} - \frac{3x^2}{2} + x\right)\Big|_0^1 = b^2\left(\frac{3}{4} - \frac{3}{2} + 1\right) = \frac{b^2}{4} = 1.$$

$$b_0 = \pm 2$$

So, $y_0 \in \{-3x+2,\ 3x-2\}$.

Ex 7 Find the angle between the vectors:

a) $A = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ in $M^{2\times 2}$.

$\|A\| = \sqrt{4+1+4} = 3$

$\|B\| = \sqrt{1+1} = \sqrt{2}$

$(A|B) = 2(-1) + 1\cdot 0 + 0\cdot 0 + (-2)\cdot 1 = -4$

$$\cos\theta = \frac{(A|B)}{\|A\|\cdot\|B\|} = \frac{-4}{3\sqrt{2}} = \frac{-2\sqrt{2}}{3}$$

$$\theta = \arccos\left(\frac{-2\sqrt{2}}{3}\right).$$

b) $x = t+1$, $y = t$ in $C[-1,1]$.

$$\|x\| = \sqrt{\int_{-1}^{1}(t+1)^2\,dt} = \sqrt{\int_{-1}^{1}(t^2+2t+1)\,dt} =$$

$$= \sqrt{\left(\frac{t^3}{3}+t^2+t\right)\Big|_{-1}^{1}} = \sqrt{\left(\frac{1}{3}+2\right)-\left(-\frac{1}{3}+1-1\right)} = \sqrt{\frac{8}{3}}$$

$$\|y\| = \sqrt{\int_{-1}^{1}t^2\,dt} = \sqrt{\frac{t^3}{3}\Big|_{-1}^{1}} = \sqrt{\frac{2}{3}}$$

$$(x|y) = \int_{-1}^{1}(t^2+t)\,dt = \left(\frac{t^3}{3}+\frac{t^2}{2}\right)\Big|_{-1}^{1} =$$

$$= \left(\frac{1}{3}+\frac{1}{2}\right) - \left(-\frac{1}{3}+\frac{1}{2}\right) = \frac{2}{3}$$

$$\cos\theta = \frac{(x|y)}{\|x\|\cdot\|y\|} = \frac{\frac{2}{3}}{\sqrt{\frac{8}{3}}\cdot\sqrt{\frac{2}{3}}} = \frac{\frac{2}{3}}{\sqrt{\frac{16}{9}}} = \frac{\frac{2}{3}}{\frac{4}{3}} = \frac{1}{2}$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

Df Let S be a subset of an inner product space X. Then the set

$$\{x \in X : (x|y) = 0 \text{ for all } y \in S\}$$

is called the **orthogonal complement** of S. And it is denoted by $S^{\perp}$ (read "S perp").

Theorem 1 Let $(\cdot|\cdot)$ be an inner product on X. Then

a) $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ for all $x, y \in X$ such that $(x|y) = 0$ (**Pythagoras theorem**);

b) $|(x|y)| \leq \|x\| \cdot \|y\|$ for all $x, y \in X$ (**Cauchy - Schwarz inequality**);

c) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (**Triangle inequality**);

d) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in X$ (**Parallelogram low**).

We prove only a) and b).

a) $\|x+y\|^2 = (x+y\,|\,x+y) = (x|x) + 2(x|y) + (y|y) =$

$= \|x\|^2 + \|y\|^2$ since $(x|y) = 0$.

b)

$\longrightarrow$ next page

b) If $x=0$ or $y=0$ the inequality is obvious.
So we may assume $y \neq 0$.
Setting $\lambda = \frac{(x|y)}{\|y\|^2}$, we obtain

$$(x-\lambda y \mid \lambda y) = \lambda(x|y) - \lambda^2 (y|y) =$$
$$= \frac{(x|y)^2}{\|y\|^2} - \frac{(x|y)^2}{\|y\|^4}\cdot\|y\|^2 = 0.$$

Then, by using the Pythagoras theorem,

$$\|x-\lambda y\|^2 + \|\lambda y\|^2 \overset{?}{=} \|(x-\lambda y)+(\lambda y)\|^2 \overset{?}{=} \|x\|^2$$

then

$$0 \leq \|x+\lambda y\|^2 = \|x\|^2 - \|\lambda y\|^2 = \|x\|^2 - |\lambda|^2\cdot\|y\|^2 =$$
$$= \|x\|^2 - \frac{(x|y)^2}{\|y\|^4}\cdot\|y\|^2 \quad \text{and hence}$$
$$\|x\|^2 \geq \frac{(x|y)^2}{\|y\|^2} \quad \text{or} \quad \|x\|\cdot\|y\| \geq |(x|y)|. \quad \blacksquare$$

Theorem 2 Let X be an inner product space and let $Y \subseteq X$ be a subspace.
Then $Y^\perp$ is a subspace of X,
$(Y^\perp)^\perp = Y$, and
$\dim(Y) + \dim(Y^\perp) = \dim(X)$

Here by $Y^\perp$ we denote the <u>orthogonal complement</u> of the subset Y of X:

Exercise 8 Find the orthogonal complement $S^\perp$ of the set $S = \{(1,-1,1), (1,1,0), (2,0,1)\} \subseteq \mathbb{R}^3$.

$$\underset{\vec{u}}{(1,-1,1)}, \quad \underset{\vec{v}}{(1,1,0)}, \quad \underset{\vec{w}}{(2,0,1)}$$

Solution

A vector $x = (x_1, x_2, x_3)$ is orthogonal to S if $(x|u) = (x|v) = (x|w) = 0$ or

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 + x_2 = 0 \\ 2x_1 + x_3 = 0 \end{cases}$$

Then taking $x_3 = t$ as a free parameter and using the back substitution, we obtain

$$x_3 = t$$
$$x_1 = -\frac{1}{2}t$$
$$x_2 = -x_1 = \frac{1}{2}t$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} t$$

Hence

$S^\perp = \{t(-\frac{1}{2}, \frac{1}{2}, 1) ; t \in \mathbb{R}\}$ is a subspace spanned by $(-\frac{1}{2}, \frac{1}{2}, 1)$ in $\mathbb{R}^3$.

Exercise 9 Find a basis for $S^\perp$ if

$S = \{(4, -3, 2, -1), (8, -7, 5, -30)\} \subseteq \mathbb{R}^4$.

Solution

$(x, y, z, t) \in S^\perp$ iff

$$\begin{cases} 4x - 3y + 2z - t = 0 \\ 8x - 7y + 5z - 30t = 0 \end{cases} \qquad (*)$$

Since $\dim(\langle S \rangle) = 2$ then $\dim(S^\perp) = \dim(\mathbb{R}^4) - \dim(\langle S \rangle) = 4 - 2 = 2$. Hence, de dimension of the solution space of $(*)$ is two, and we may chooze two free parameters. Let us chooze t and z. The back substitution gives

$$\begin{cases} 4x - 3y = t - 2z \\ 8x - 7y = 30t - 5z \end{cases}$$

$$\begin{cases} 0x - y = 28t - z \\ 4x - 3y = t - 2z \end{cases} \Rightarrow y = z - 28t$$

$$4x = t - 2z + 3y = t - 2z + 3(z - 28t) = t - 2z + 3z - 84t = $$
$$= z - 83t$$

$$x_0 = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{4}z - \frac{83}{4}t \\ z - 28t \\ z \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 1 \\ 1 \\ 0 \end{bmatrix} z + \begin{bmatrix} -\frac{83}{4} \\ -28 \\ 0 \\ 1 \end{bmatrix} t.$$

Hence a basis for $S^\perp$ may be choosen

$\left\{ \left(\frac{1}{4}, 1, 1, 0\right), \left(-\frac{83}{4}, -28, 0, 1\right) \right\}$.

(Df) Let X be an inner product space with the inner product $(\cdot|\cdot)$. A Basis $B \subseteq X$ for X is called an **orthogonal basis** if $(v, w) = 0$ for any $v, w \in B$, $v \neq w$.

A basis B is called **orthonormal** if for any $v, w \in B$

$$(v|w) = \begin{cases} 1, & \text{if } v = w \\ 0, & \text{if } v \neq w \end{cases}$$

Exercise 10 Show that the set

$B = \{1, \cos x, \cos 2x, \dots\} \cup \{\sin x, \sin 2x, \dots, \sin nx, \dots\}$

is an orthogonal basis for $\operatorname{span}(B) \subseteq C[-\pi, \pi]$ with respect to the integral inner product.

Solution

1. B is linearly independent which follows from the next step. Hence B is a basis for $\langle B \rangle$.

2a). If $n \neq m$

$$(\cos nx | \cos mx) = \int_{-\pi}^{\pi} (\cos nx)(\cos mx)\, dx =$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} (\cos (n+m)x + \cos(n-m)x)\, dx = \frac{1}{2}\left(\frac{\sin(n+m)x}{n+m} + \frac{\sin(n-m)x}{n+m} \right)\Big|_{-\pi}^{\pi} = 0.$$

2b) If $n \neq m$

$$(\sin nx | \sin mx) = \int_{-\pi}^{\pi} (\sin nx)(\sin mx)\, dx = 0$$

2c) $(\cos nx | \sin mx) = \int_{-\pi}^{\pi} (\cos nx) \cdot (\sin mx)\, dx = 0$

since the integrand is an odd function.

2d) $(1 | \sin mx) = \int_{-\pi}^{\pi} \sin mx\, dx = 0$

since the integrand is an odd function.

2e) $(1 | \cos mx) = \int_{-\pi}^{\pi} \cos mx\, dx = \frac{\sin mx}{m}\Big|_{-\pi}^{\pi} = 0 - 0 = 0.$

2a) – 2e) Show that B is orthogonal.

If B is linearly dependent then

$$c_1 y_1 + \dots + c_n y_n = 0 \quad \text{for some } y_1, \dots, y_n \in B$$

and $c_1, \dots, c_n \in \mathbb{R}$ which are not all equal to zero.

Let $c_1 \neq 0$.

Take an inner product

$$(c_1 y_1 + \dots + c_n y_n \mid y_1) = (0 \mid y_1) = 0$$

$$\parallel$$

$$c_1(y_1 \mid y_1) + c_2 \underbrace{(y_2 \mid y_1)}_{0} + \dots + c_n \underbrace{(y_n \mid y_1)}_{0} = c_1(y_1 \mid y_1).$$

This contradicts to the fact that all elements of B are nonzero vectors.

The last assertion possesses the following generalization:

Theorem 3 Any pairwise orthogonal set is linearly independent. ▨

Proof is obvious. Let $\alpha w_1 + \dots \alpha_n w_n = 0$. Take the inner product with w_k to see that $\alpha_k = 0$.

(See also the Exercise 3)