Basic Linear Algebra

Lecture 1 Matrices

In our course Math 260 we study methods of linear algebra mostly in the case of scalars which are real numbers. We denote them by $\mathbb{R}$. We also denote integers by $\mathbb{Z}$ and naturals by $\mathbb{N}$. We shall use abbreviations „iff" for „if and only if", „Def" for „Definition", „Thm" for „Theorem", etc.

Def An $m \times n$-matrix A is a rectangular array of $m \cdot n$ scalars

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

Matrix A has m rows and n columns

For any $1 \le i \le m$ the $(1 \times n)$-matrix

$[A_{i1} \; A_{i2} \ldots A_{in}]$ is called the <u>i-th row</u> of A.

For any $1 \le j \le n$ the $(m \times 1)$-matrix

$$\begin{bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{mj} \end{bmatrix}$$ is called the <u>j-th column</u> of A.

The element A_{ij} is called the <u>ij-th entry</u> of A.

<u>Def</u> Given $m \times n$ matrix A, the pair (m, n) is called the <u>size</u> of A. Two matrices A, B are said to be <u>equal</u> if they have the same size and $A_{ij} = B_{ij}$ for every position ij of the entry in both matrices.

Remark that any $m \times n$ matrix may be considered as a function of pairs (i, j) of natural numbers $1 \le i \le m$, $1 \le j \le n$

For example, the function δ (= Kronecker delta)

$$\delta(ij) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$ gives rise to

the identity matrix of the size $n \times n$

$$I = \left.\underbrace{\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ - & - & - & - & - \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}}_{n}\right\} n$$

The function $f(ij) \equiv 0$ gives rise to zero matrices of arbitrary size $m \times n$.

Now, let us consider operations on matrices. First of them is changing rows with columns and vice versa. This operation is called the traspose of a matrix and is denoted by A^T, so

$$(A^T)_{kp} := A_{pk}$$

for any entry A_{pk} of the matrix A.

Clearly, the transpose operation makes (n×m)-matrix from an (m×n)-matrix, and $A^{TT} = A^{T}$.

Our next matrix operation is the scalar multiplication by $\alpha \in \mathbb{R}$:

$$(\alpha A)_{ij} := \alpha \cdot A_{ij}$$

This operation, as well as the transpose, is applicable to matrices of arbitrary size

The <u>summation of matrices is applicable to matrices of the same size only</u>:

$$(A+B)_{ij} := A_{ij} + B_{ij}$$

Above introdused matrix operations possess the following natural properties:

Lemma Assuming that the operations below are defined, then:

a) $A+B = B+A$ (commutativity of addition)

b) $(A+B)+C = A+(B+C)$ (associativity)

c) $\alpha\cdot(A+B) = \alpha\cdot A + \alpha\cdot B$

d) $1\cdot A = A$

e) $(\alpha\cdot A)^T = \alpha\cdot A^T$

f) $(A+B)^T = A^T + B^T$ ▨

Our next operation is the matrix multiplication. It is applicable iff the number of columns of the first factor is equal to the number of rows of the second factor.

Def If A is an $m\times n$-matrix and B is an $n\times p$-matrix then $A\cdot B$ is $m\times p$-matrix defined by:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}\cdot B_{kj},$$

where $1\le i\le m$, and $1\le j\le p$.

Theorem (elementary properties of the matrix multiplication)

If the operations below are well defined then:

a) $(A\cdot B)\cdot C = A\cdot(B\cdot C)$

b) $A\cdot(B+C) = A\cdot B + A\cdot C$

c) $(B+C)\cdot D = B\cdot D + C\cdot D$

d) $\alpha\cdot(A\cdot B) = (\alpha\cdot A)\cdot B = A\cdot(\alpha\cdot B)$

Proof We prove a) only:

$$\left((A\cdot B)\cdot C\right)_{ij} = \sum_{k=1}^{p} (AB)_{ik}\cdot C_{kj} =$$

$$= \sum_{k=1}^{p} \left(\sum_{l=1}^{r} A_{il}\cdot B_{lk}\right)\cdot C_{kj} =$$

$$= \sum_{l=1}^{r} A_{il} \sum_{k=1}^{p} B_{lk}\cdot C_{kj} = \sum_{l=1}^{r} A_{il}\cdot (BC)_{lj} =$$

$$= \left(A\cdot(B\cdot C)\right)_{ij} \quad \blacksquare$$

Remark that the matrix multiplication is <u>not</u> commutative in general.

<u>Example</u>

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

So $AB \neq BA$.

The matrix exponentiation is defined only for square matrices:

$$\underset{n \times n}{A^p} := \underbrace{A \cdot A \cdot \ldots \cdot A}_{p \text{ times}}$$

It satisfies common rules:

$$A^p \cdot A^q = A^{p+q}$$

$$(A^p)^q = A^{p \cdot q}$$

$$A^0 = I, \text{ where } I \text{ is } n \times n\text{-identity matrix}$$

<u>Thm</u> If $A \cdot B$ is defined then $(AB)^T = B^T \cdot A^T$

<u>Proof</u>: $((A \cdot B)^T)_{ij} = (AB)_{ji} = \sum_{k=1}^{p} A_{jk} B_{ki} =$

$$= \sum_{k=1}^{p} (A^T)_{kj} \cdot (B^T)_{ik} = \sum_{k=1}^{p} (B^T)_{ik} (A^T)_{kj} = (B^T A^T)_{ij}.$$

∎

Now, let us consider several types of square matrices.

<u>Def</u> A <u>diagonal matrix</u> is a square matrix $A = [A_{ij}]_{i,j}$ such that $A_{ij} = 0$ if $i \neq j$

Clearly, the set of all $n \times n$ diagonal matrices is closed under scaler multiplication, matrix addition, and matrix multiplication. Moreover, any diagonal matrix is stable under taking the transpose.

Def (Triangular matrices)

A square matrix A is called

1) upper triangular if

$A_{ij} = 0$ when $i > j$

2) lower triangular if

$A_{ij} = 0$ when $i < j$

It is easy to see that the set of all upper-triangular matrices of the same size is closed under scalar multiplication, matrix addition, and matrix multiplication. The same is true (by taking the transpose) for lower-triangular matrices.

Proof: Let A, B be upper-triangular, then for $i > j$ $A_{ij} = 0 = B_{ij}$ and, hence,

$$(AB)_{ij} = \sum_{k=1}^{p} A_{ik} B_{kj} = \sum_{k=1}^{i-1} \underbrace{A_{ik}}_{0} B_{kj} + \sum_{k=i}^{p} A_{ik} \underbrace{B_{kj}}_{0} = 0$$

∎

Def Let A be a square matrix.

1) A is called symmetric if $A^T = A$

2) A is called skew-symmetric if $A^T = -A$

Theorem Any square matrix can be written as a sum of a symmetric matrix and a skew-symmetric matrix.

Proof Let A be square matrix.

a) $(A+A^T)^T = A^T + A^{TT} = A^T + A = A + A^T$

b) $(A-A^T)^T = A^T - A^{TT} = A^T - A = -(A-A^T)$

Thus $A+A^T$ is symmetric, as well as $\frac{1}{2}(A+A^T)$.
$A-A^T$ is skew-symmetric, as well as $\frac{1}{2}(A-A^T)$.

Finally we get:

$$A = \underbrace{\frac{1}{2}(A+A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A-A^T)}_{\text{skew-symmetric}}$$

∎

Ex Find $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{2013}$

Solution: Use the mathematical induction:

Assume $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{n} = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$. Then

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2+2n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2(n+1) \\ 0 & 1 \end{bmatrix}.$$

Hence the formula $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{n} = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$ is true for all $n \in \mathbb{N}$ by induction.

Substitute $n = 2013$ and get $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{2013} = \begin{bmatrix} 1 & 4026 \\ 0 & 1 \end{bmatrix}$

Ex Find all roots of the matrix equation:

$X^{333} = X = \begin{bmatrix} 0 & t \\ 2 & 0 \end{bmatrix}$, $t \in \mathbb{R}$.

Solution: Notice that $t \neq 0$, otherwise $X^2 = 0$ and $X^{333} = 0$.

$$X^2 = \begin{bmatrix} 0 & t \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & t \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2t & 0 \\ 0 & 2t \end{bmatrix} = (2t) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X^{333} = X^{2 \cdot 166} \cdot X = \left((2t) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{166} \cdot X = (2t)^{166} \cdot X = X$$

Hence $\left((2t)^{166} - 1\right) X = 0$. Since $X \neq 0$ we obtain $2t = \pm 1$ or $t = \pm \frac{1}{2}$. Thus the roots are

$$\begin{bmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix}.$$

Ex

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Calculate $(A+2B)(C-2B)$.

Solution:

$$(A+2B)(C-2B) = AC + 2BC - 2AB - 4B^2 =$$

$$\left\{ \begin{array}{ll} AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & BC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \right\}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$$