In our course Math 260 we study methods of linear algebra mostly in the case of scalars which are
real numbers. We denote them by \( \mathbb{R} \). We also denote integers by \( \mathbb{Z} \) and naturals
by \( \mathbb{N} \). We shall use abbreviations "iff" for "if and only if", "Def" for "Definition",
"Thm" for "Theorem", etc.

**Def** An \( m \times n \)-matrix \( A \) is a rectangular array of \( m \times n \) scalars

\[
A = \begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
A_{21} & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{m1} & A_{m2} & \cdots & A_{mn}
\end{bmatrix}
\]

Matrix \( A \) has \( m \) rows and \( n \) columns.
For any $1 \leq i \leq m$ the $(1 \times n)$-matrix
\[
\begin{bmatrix}
A_{i1} & A_{i2} & \cdots & A_{in}
\end{bmatrix}
\] is called the $i$-th row of $A$.

For any $1 \leq j \leq n$ the $(m \times 1)$-matrix
\[
\begin{bmatrix}
A_{1j} \\
A_{2j} \\
\vdots \\
A_{mj}
\end{bmatrix}
\] is called the $j$-th column of $A$.

The element $A_{ij}$ is called the $(i,j)$-th entry of $A$.

**Def.** Given an $m \times n$ matrix $A$, the pair $(m,n)$ is called the size of $A$. Two matrices $A$, $B$ are said to be equal if they have the same size and $A_{ij} = B_{ij}$ for every position $ij$ of the entry in both matrices.

Remark that any $m \times n$ matrix may be considered as a function of pairs $(i,j)$ of natural numbers $1 \leq i \leq m$, $1 \leq j \leq n$. 
For example, the function $\delta = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ gives rise to the identity matrix of the size $n \times n$

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

The function $f(i, j) = 0$ gives rise to zero matrices of arbitrary size $m \times n$.

Now, let us consider operations on matrices. First of them is changing rows with columns and vice versa. This operation is called the transpose of a matrix and is denoted by $A^T$, so

$$(A^T)_{kp} := A_{pk}$$

for any entry $A_{pk}$ of the matrix $A$. 
Clearly, the transpose operation makes $(n \times m)$-matrix from an $(m \times n)$-matrix, and
$A^T = A^T$. Our next matrix operation is the scalar multiplication by $\alpha \in \mathbb{R}$:

$$(\alpha \cdot \mathbf{A})_{ij} := \alpha \cdot A_{ij}$$

This operation, as well as the transpose, is applicable to matrices of arbitrary size.

The summation of matrices is applicable to matrices of the same size only:

$$(\mathbf{A} + \mathbf{B})_{ij} := A_{ij} + B_{ij}$$

Above introduced matrix operations possess the following natural properties:
Lemma Assuming that the operations below are defined, then:

a) \( A + B = B + A \) (commutativity of addition)

b) \( (A + B) + C = (A + (B + C)) \) (associativity)

c) \( \alpha \cdot (A + B) = \alpha \cdot A + \alpha \cdot B \)

d) \( 1 \cdot A = A \)

e) \( (\alpha \cdot A)^T = \alpha \cdot A^T \)

f) \( (A + B)^T = A^T + B^T \)

Our next operation is the **matrix multiplication**.

It is applicable iff the number of columns of the first factor is equal to the number of rows of the second factor.

**Def** If \( A \) is an \( m \times n \)-matrix and \( B \) is an \( n \times p \)-matrix then \( A \cdot B \) is an \( m \times p \)-matrix defined by:

\[
(AB)_{ij} = \sum_{k=1}^{n} A_{ik} \cdot B_{kj},
\]

where \( 1 \leq i \leq m \), and \( 1 \leq j \leq p \).
Theorem (elementary properties of the matrix multiplication)

If the operations below are well defined then:

a) \((A \cdot B) \cdot C = A \cdot (B \cdot C)\)

b) \(A \cdot (B + C) = A \cdot B + A \cdot C\)

c) \((B + C) \cdot D = B \cdot D + C \cdot D\)

d) \(\alpha \cdot (A \cdot B) = (\alpha \cdot A) \cdot B = A \cdot (\alpha \cdot B)\)

Proof. We prove a) only:

\[
\left( (A \cdot B) \cdot C \right)_{ij} = \sum_{k=1}^{p} (AB)_{ik} \cdot C_{kj} = \\
= \sum_{k=1}^{p} \left( \sum_{l=1}^{q} A_{il} \cdot B_{lk} \right) \cdot C_{kj} = \\
= \sum_{l=1}^{q} \left( \sum_{k=1}^{p} A_{il} \cdot B_{lk} \right) \cdot C_{kj} = \sum_{l=1}^{q} A_{il} \cdot (BC)_{lj} = \\
= (A \cdot (B \cdot C))_{ij} \]

Remark that the matrix multiplication is not commutative in general.

Example

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]

\[ AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \]

So \( AB \neq BA \).

The matrix exponentiation is defined only for square matrices:

\[ A^p = A \cdot A \cdot \ldots \cdot A \]

\( n \times n \) \( p \) times

It satisfies common rules:

\[ A^p \cdot A^q = A^{p+q} \]

\[ (A^p)^q = A^{pq} \]

\[ A^0 = I \]

where \( I \) is \( n \times n \) identity matrix.
Then, if $AB$ is defined, then $(AB)^T = B^TA^T$.

Proof: $(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{p} A_{jk} B_{ki} = \\
= \sum_{k=1}^{p} (A^T)_{kj} (B^T)_{ik} = \sum_{k=1}^{p} (B^T)_{ik} (A^T)_{kj} = (BA^T)_{ij}$. 

Now, let us consider several types of square matrices.

**Def** A **diagonal matrix** is a square matrix $A = [A_{ij}]_{ij}$ such that $A_{ij} = 0$ if $i \neq j$.

Clearly, the set of all $n \times n$ diagonal matrices is closed under scalar multiplication, matrix addition, and matrix multiplication. Moreover, any diagonal matrix is stable under taking the transpose.
Def (Triangular matrices)
A square matrix $A$ is called
1) upper triangular if
\[ A_{ij} = 0 \text{ when } i > j \]
2) lower triangular if
\[ A_{ij} = 0 \text{ when } i < j \]

It is easy to see that the set of all upper-triangular matrices of the same size is closed under scalar multiplication, matrix addition, and matrix multiplication. The same is true (by taking the transpose) for lower-triangular matrices.

Proof: Let $A, B$ be upper-triangular, then for $i > j$
\[ A_{ij} = 0 = B_{ij} \text{ and, hence,} \]
\[
(AB)_{ij} = \sum_{k=1}^{p} A_{ik} B_{kj} = \sum_{k=1}^{i-1} A_{ik} B_{kj} + \sum_{k=i}^{p} A_{ik} B_{kj} = 0
\]
Def: Let \( A \) be a square matrix.

1) \( A \) is called **symmetric** if \( A^T = A \)
2) \( A \) is called **skew-symmetric** if \( A^T = -A \)

**Theorem** Any square matrix can be written as a sum of a symmetric matrix and a skew-symmetric matrix.

**Proof** Let \( A \) be a square matrix.

a) \((A + A^T)^T = A^T + A = A + A^T\)

b) \((A - A^T)^T = A^T - A = A - A^T = -(A - A^T)\)

Thus \( A + A^T \) is symmetric, as well as \( \frac{1}{2} (A + A^T) \). \( A - A^T \) is skew-symmetric, as well as \( \frac{1}{2} (A - A^T) \).

Finally we get:

\[
A = \frac{1}{2} (A + A^T) + \frac{1}{2} (A - A^T) \]

**symmetric** **skew-symmetric**
Ex. Find \([1,2]^{2013}\)

Solution: Use the mathematical induction:
Assume \([1,2]^n = [1,2^n]\). Then
\[
[1,2]^{n+1} = [1,2^n] \cdot [1,2] = [1,2^{n+1}] = [1,2^{n+1}].
\]
Hence the formula \([1,2]^n = [0,1]\) is true for all \(n \in \mathbb{N}\) by induction.
Substitute \(n = 2013\) and get \([1,2]^{2013} = [0,1]\).

Ex. Find all roots of the matrix equation:
\[X^{333} = X = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \quad t \in \mathbb{R},\]

Solution: Notice that \(t \neq 0\), otherwise \(X = 0\) and \(X = 0\).

\[
X^{333} = \begin{bmatrix} 2+t & 2 \\ 2 & 2+t \end{bmatrix} = \begin{bmatrix} 2t & 0 \\ 0 & 2t \end{bmatrix} = (2t) \cdot \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.
\]

\[X^{333} = X^{2 \cdot 166} \cdot X = \left((2t) \cdot \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right)^{166} \cdot X = (2t)^{166} \cdot X.
\]

Hence \((2t)^{166} - 1 \cdot X = 0\). Since \(X \neq 0\) we obtain
\[2t = \pm 1 \quad \text{or} \quad t = \pm \frac{1}{2}.
\]
Thus the roots are
\[
\begin{bmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix}.
\]
\[ \begin{align*} 
A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & B &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & C &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\
\text{Calculate } & (A + 2B)(C - 2B). \\
\text{Solution:} & \\\n(A + 2B)(C - 2B) &= AC + 2BC - 2AB - 4B^2 = \\
\begin{cases} 
AB &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & BC &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
AC &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, & B^2 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} 
\end{cases} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - 4\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} 
\end{align*} \]