0.1 Vector Spaces

Exercise 0.1.1 Show that the following functions x, $1+x$, $x+\sin^2 x$, x^3-x, and $x+\cos^2 x$ defined on $\mathbb{R}$ are linearly dependent.

Solution:

$x+(1+x)-(x+\sin^2 x)-(x+\cos^2 x)=1-(\sin^2 x+\cos^2 x)=1-1=0.$ □

Exercise 0.1.2 Compute the dimension of the vector subspace

$$V=\operatorname{span}\{(-1,2,3,0),(5,4,3,0),(3,1,0,0)\}$$

of $\mathbb{R}^4$.

Solution: $\begin{bmatrix} -1 & 2 & 3 & 0 \\ 5 & 4 & 3 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[3R_2+R_3]{5R_1+R_2} \begin{bmatrix} -1 & 2 & 3 & 0 \\ 0 & 14 & 18 & 0 \\ 0 & 7 & 9 & 0 \end{bmatrix} \xrightarrow[-2R_3+R_2]{R_3\leftrightarrow R_2} \begin{bmatrix} \underline{-1} & 2 & 3 & 0 \\ 0 & \underline{7} & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

So $\dim(V)=2$. □

Exercise 0.1.3 Find a basis for the row space of A and find the dimension of the row space of A, where $A=\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.

Solution: $A \xrightarrow[R_1+R_4]{-R_1+R_2;-R_1+R_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[-R_3+R_5]{R_3\leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

The basis is $\{(1,0,1,0,0),(0,1,-1,1,0),(0,0,2,0,1),(0,0,0,1,1)\}$. The dimension is 4. □

Exercise 0.1.4 Extend $\{1+x^2, x-x^3\}$ to a basis for the space of polynomials of degree ≤ 3.

Solution: Let $v_1 = 1 + x^2$, $v_2 = x - x^3$, $v_3 = 1$, $v_4 = x$, $v_5 = x^2$, and $v_6 = x^3$. Consider the matrix $[[v_1][v_2][v_3][v_4][v_5][v_6]]$:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[-R_1+R_3]{R_2+R_4} \begin{bmatrix} \underline{1} & 0 & 1 & 0 & 0 & 0 \\ 0 & \underline{1} & 0 & 1 & 0 & 0 \\ 0 & 0 & \underline{-1} & 0 & 1 & 0 \\ 0 & 0 & 0 & \underline{1} & 0 & 1 \end{bmatrix}.$$

We see that the basis is $\{1 + x^2, x - x^3, 1, x\}$. □

Exercise 0.1.5 Find coordinates (the coordinate matrix $[u]_C$) of $u = x - x^2 + x^3$ with respect to the basis $C = \{w_1, w_2, w_3\}$ of the vector space $W = \text{span}\{w_1, w_2, w_3\}$, where $w_1 = x + x^2$, $w_2 = x - x^2$, and $w_3 = x + x^2 + 2x^3$.

Solution:

$$u = x - x^2 + x^3 = w_2 + x^3 = w_2 + \frac{1}{2}(w_3 - w_1) = -\frac{1}{2} \cdot w_1 + 1 \cdot w_2 + \frac{1}{2} \cdot w_3$$

and

$$[u]_C = \begin{bmatrix} -1/2 \\ 1 \\ 1/2 \end{bmatrix}. \qquad \square$$

Exercise 0.1.6 Let $w \in W$ be such that $[w]_C = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, where C is the basis for W defined in 0.1.5. Find the polynomial w.

Solution:

$$w = 1 \cdot w_1 + 2 \cdot w_2 + 0 \cdot w_3 = (x + x^2) + 2(x - x^2) = 3x - x^2. \qquad \square$$

Exercise 0.1.7 Compute the transition matrix $P = P_{B \to C}$ from the basis $B = \{x, x^2, x^3\}$ for W to the basis $C = \{w_1, w_2, w_3\}$ for W defined in 0.1.5.

1-st Solution:

$$2x = w_1 + w_2 \Rightarrow [x]_C = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix};$$

$$2x^2 = w_1 - w_2 \Rightarrow [x^2]_C = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix};$$

$$2x^3 = w_3 - w_1 \Rightarrow [x^3]_C = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}.$$

Hence

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

2-nd Solution: $[w_1\ w_2\ w_3 \mid x\ x^2\ x^3] =$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_2]{\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \xrightarrow{-R_2-R_3+R_1}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] = [I|P]. \quad \square$$

Exercise 0.1.8 Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$.

a) Find a basis for the solution space of $AX = 0$.

b) Find a basis for $\mathbb{R}^3$ that contains the basis constructed in part (a).

Solution: a) $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^{3\times 1}$, $AX = 0$.

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} \underline{1} & 1 & 1 \\ 0 & \underline{-2} & -2 \end{bmatrix}$, so x_3 is free. Find the fundamental solution.

Set $x_3 = 1$ then $x_2 = -1$, $x_1 = 0$. So $X_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

b) Consider the matrix $[X_1\ e_1\ e_2\ e_3]$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} \underline{-1} & 0 & 1 & 0 \\ 0 & \underline{1} & 0 & 0 \\ 0 & 0 & \underline{1} & 1 \end{bmatrix}$$

Hence the basis $\mathbb{R}^{3\times 1}$ is $\{X_1, e_1, e_2\} = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. □

Exercise 0.1.9 Let $\{t, u, v, w\}$ be a basis for a vector space V. Find $\dim(U)$, where $U = \operatorname{span}\{t+2u+v+w, t+3u+v+2w, 3t+4u+2v, 3t+5u+2v+w\}$.

Solution: Let $v_1 = t+2u+v+w$, $v_2 = t+3u+v+2w$, $v_3 = 3t+4u+2v$, $v_4 = 3t+5u+2v+w$. Consider the coordinate matrix $[v_1\ v_2\ v_3\ v_4]$:

$$\begin{bmatrix} 1 & 1 & 3 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \to$$

$$\to \begin{bmatrix} \underline{1} & 1 & 3 & 3 \\ 0 & \underline{1} & -2 & -1 \\ 0 & 0 & \underline{-1} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $\{v_1, v_2, v_3\} = \{t+2u+v+w, t+3u+v+2w, 3t+4u+2v\}$ is a basis for U. It has 3 vectors. Hence $\dim(U) = 3$. □

Exercise 0.1.10

a) Show that $\mathcal{C} = \{(1,1,0), (1,-1,0), (0,0,1)\}$ is a basis for $\mathbb{R}^3$.

b) Let $\mathcal{B} = \{(1,0,0), (0,1,0), (0,0,1)\}$. Find the change of coordinate matrices (that is transition matrices) from $\mathcal{C}$ to $\mathcal{B}$, and from $\mathcal{B}$ to $\mathcal{C}$.

Solution: a) The determinant $\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -2 \neq 0$. It means that the rows are linearly independent, so $\mathcal{C}$ is linearly independent in $\mathbb{R}^3$, consequently $\mathcal{C}$ is a basis for $\mathbb{R}^3$ since $\dim(\mathbb{R}^3) = 3$ and $\mathbb{R}^3 \supseteq \text{span}(\mathcal{C})$, where $\mathcal{C}$ has 3 vectors and finally $\dim(\langle\mathcal{C}\rangle) = 3$.

b) Set $u_1 = (1,1,0)$, $u_2 = (1,-1,0)$, $u_3 = (0,0,1)$. Then $\mathcal{C} = \{u_1, u_2, u_3\}$.

$$P_{\mathcal{C}\to\mathcal{B}} = [[u_1]_{\mathcal{B}}\ [u_2]_{\mathcal{B}}\ [u_3]_{\mathcal{B}}] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$P_{\mathcal{B}\to\mathcal{C}} = P_{\mathcal{C}\to\mathcal{B}}^{-1}.$$

$$[P_{\mathcal{C}\to\mathcal{B}}|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = [I|P_{\mathcal{B}\to\mathcal{C}}]. \qquad \square$$

Exercise 0.1.11 Let the set $\{u, v, w\}$ be linearly independent. Show that the set $\{u + 2v, v - 3w, u - v + w\}$ is linearly independent.

Solution: Denote $B = \{u, v, w\}$ is a basis for $\text{span}(B)$. Then we can write

$$[u+2v]_B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad [v-3w]_B = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \quad [u-v+w]_B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Since

$$\det \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 0 & -3 & 1 \end{bmatrix} = 1 \cdot \begin{vmatrix} 1 & -1 \\ -3 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 1 \\ 0 & -3 \end{vmatrix} = -2 - 6 \neq 0.$$

So the set $\{u + 2v, v - 3w, u - v + w\}$ is linearly independent. $\square$

Exercise 0.1.12 Find the value(s) of α if $\begin{bmatrix} \alpha & 2 \\ 0 & 6-\alpha \end{bmatrix}$ is contained in the space

$$\operatorname{span}\left\{\begin{bmatrix} -1 & \alpha \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \alpha & -1 \\ 0 & \alpha^2-\alpha-1 \end{bmatrix}, \begin{bmatrix} \alpha+1 & -3 \\ 0 & \alpha^2-4 \end{bmatrix}\right\}.$$

Solution:

$$\begin{bmatrix} \alpha & 2 \\ 0 & 6-\alpha \end{bmatrix} = x \cdot \begin{bmatrix} -1 & \alpha \\ 0 & 1 \end{bmatrix} + y \cdot \begin{bmatrix} \alpha & -1 \\ 0 & \alpha^2-\alpha-1 \end{bmatrix} + z \cdot \begin{bmatrix} \alpha+1 & -3 \\ 0 & \alpha^2-4 \end{bmatrix}.$$

Thus the following system must be consistent:

$$\begin{cases} -x & + & \alpha y & + & (\alpha+1)z & = & \alpha \\ \alpha x & - & y & - & 3z & = & 2 \\ x & + & (\alpha^2-\alpha-1)y & + & (\alpha^2-4)z & = & 6-\alpha \end{cases}$$

$$\left[\begin{array}{ccc|c} -1 & \alpha & \alpha+1 & \alpha \\ \alpha & -1 & -3 & 2 \\ 1 & \alpha^2-\alpha-1 & \alpha^2-4 & 6-\alpha \end{array}\right] \xrightarrow[R_1+R_3]{\alpha R_1+R_2}$$

$$\rightarrow \left[\begin{array}{ccc|c} -1 & \alpha & \alpha+1 & \alpha \\ 0 & \alpha^2-1 & \alpha^2+\alpha-3 & 2+\alpha^2 \\ 0 & \alpha^2-1 & \alpha^2+\alpha-3 & 6 \end{array}\right] \xrightarrow{-R_2+R_3}$$

$$\rightarrow \left[\begin{array}{ccc|c} -1 & \alpha & \alpha+1 & \alpha \\ 0 & \alpha^2-1 & \alpha^2+\alpha-3 & 2+\alpha^2 \\ 0 & 0 & 0 & 4-\alpha^2 \end{array}\right].$$

Hence $4-\alpha^2=0$ or $\alpha=\pm 2$. □

Exercise 0.1.13 Given the matrix $\begin{bmatrix} 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \end{bmatrix}$. Show that the dimension of the column space of this matrix is equal to 3. Justify your answer.

Solution:

$$\begin{bmatrix} 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \end{bmatrix} \xrightarrow[R_2\leftrightarrow R_3]{R_1\leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_1+R_2}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 0 & -1 & -3 & 3 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2+R_3} \begin{bmatrix} \underline{1} & 1 & 1 & -1 & 0 \\ 0 & \underline{-1} & -3 & 3 & 0 \\ 0 & 0 & \underline{-3} & 3 & 1 \end{bmatrix}.$$

Hence $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the column space. Therefore its dimension is equal to 3. □

Exercise 0.1.14 Find the value(s) of $\alpha \in \mathbb{R}$ such that $\dim(\text{span}(A)) = 2$, where $A = \{1+2x^2+x^4, 2+x+4x^2+x^3+5x^4, 1+x+2x^2+x^3+\alpha x^4\}$. Justify your answer.

Solution: Put the coefficients in 3×5 matrix

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 2 & 1 & 4 & 1 & 5 \\ 1 & 1 & 2 & 1 & \alpha \end{bmatrix} \xrightarrow[-R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & \alpha-1 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} \underline{1} & 0 & 2 & 0 & 1 \\ 0 & \underline{1} & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & \underline{\alpha-4} \end{bmatrix}.$$

The dimension $\dim(\text{span}(A))$ is the same with the dimension of column or row space of the matrix. To make it equal to 2 one must have $\alpha - 4 = 0$ or $\alpha = 4$. □

Exercise 0.1.15 Given two bases $B = \{u+v, u-v, w\}$ and $C = \{u+w, v, v-w\}$ for the vector space spanned by $\{u, v, w\}$.

a) Find the transition matrix $P_{B\to C}$ from B to C.

b) Find the transition matrix $P_{C\to B}$ from C to B.

Solution: a)

$$[u+v]_C = [(u+w)+(v-w)]_C = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$[u-v]_C = [(u+w)+(v-w)-2v]_C = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

$$[w]_C = [v-(v-w)]_C = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Hence $P_{B\to C} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

b) $P_{C\to B} = (P_{B\to C})^{-1}$. Write $[P_{B\to C}|I] =$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3+R_2}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[-\frac{1}{2}R_2]{-R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \to$$

$$\xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] = \left[I|\,(P_{B\to C})^{-1}\right].$$

Hence

$$P_{C\to B} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 0 & -2 \end{bmatrix}. \qquad \square$$

Exercise 0.1.16 a) Determine whether the following subsets are subspace (giving reasons for your answers).

(i) $U = \{A \in \mathbb{R}^{2\times 2} | A^T = A\}$ in $\mathbb{R}^{2\times 2}$. ($\mathbb{R}^{2\times 2}$ is the vector space of all real 2×2 matrices under usual matrix addition and scalar-matrix multiplication.)

(ii) $W = \{(x, y, z) \in \mathbb{R}^3 | x \geq y \geq z\}$ in $\mathbb{R}^3$.

b) Find a basis for U. What is the dimension of U? (Show all your work by explanations.)

c) What is the dimension of $\mathbb{R}^{2\times 2}$? Extend the basis of U to a basis for $\mathbb{R}^{2\times 2}$.

Solution: a-i)

1) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in U$ since $A = A^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $U \neq \emptyset$.

2) Let $A, B \in U$. Then $A = A^T$ and $B = B^T$. Then $A + B \in U$, since $(A + B)^T = A^T + B^T = A + B$. So U is closed under addition.

3) Let $c \in \mathbb{R}$ and $A \in U$. Then $A = A^T$. Then $c \cdot A \in U$ since $(c \cdot A)^T = c \cdot A^T = c \cdot A$. Thus U is closed under scalar multiplication.

So we proved that U is a subspace in $\mathbb{R}^{2\times 2}$.

a-ii) W is not a subspace in $\mathbb{R}^3$. Since $(2,1,1) \in W$, however $(-1)\cdot(2,1,1) = (-2,-1,-1) \notin W$, that is W is not closed under scalar multiplication..

b) Let $A \in U$. Then $A = A^T$ i.e. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ for all $a, b, c, d \in \mathbb{R}$. Thus a and d are arbitrary real numbers and $c = b$. So any matrix $A \in U$ can be written as

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are linearly independent, and any matrix in U can be written as a linear combination of these matrices, these matrices form a basis for U, namely $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for U. Thus $\dim(U) = 3$.

c) The space $\mathbb{R}^{2\times 2}$ has the standard basis

$$C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

therefore $\dim(\mathbb{R}^{2\times 2}) = 4$. We can extend the basis B for U to a basis for $\mathbb{R}^{2\times 2}$ by

$$\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_4}$$

$$\rightarrow \left[\begin{array}{ccc|cccc} \underline{1} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \underline{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \underline{1} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \underline{-1} & 0 & 1 \end{array}\right].$$

Thus a basis for $\mathbb{R}^{2\times 2}$ containing vectors of B is

$$D = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}. \quad \square$$

Exercise 0.1.17 Let $p \in \mathcal{P}_2$. The coordinate matrix of p relative to the standard ordered basis $B = \{1, x, x^2\}$ is $[p]_B = [2, -1, 5]^T$. Find the change of coordinate matrix from the ordered basis $B = \{1, x, x^2\}$ to the ordered basis $C = \{1, 1-x, 1+x+x^2\}$ and the coordinate matrix of p relative to C, $[p]_C$.

Solution: $[p]_B = [2, -1, 5]^T$ then $p = 2 \cdot 1 + (-1) \cdot x + 5 \cdot x^2$.

$$[P_{C \to B}|I] = \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_3+R_1]{-R_3+R_2} = \left[\begin{array}{rrr|rrr} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow$$

$$\xrightarrow[-R_2]{R_2+R_1} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = [I|P_{B \to C}],$$

where

$$P_{B \to C} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

is the change of coordinate matrix from the basis B to the basis C.

$[p]_C = P_{B \to C} \cdot [p]_B$. Thus

$$[p]_C = \begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 \\ 6 \\ 5 \end{bmatrix}. \quad \square$$

Exercise 0.1.18 Let $B = \{u, v\}$ be a basis of $\mathbb{R}^2$ and let $A = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$. Show that A is invertible iff $C = \{\alpha u + \beta v, \gamma u + \delta v\}$ is a basis of $\mathbb{R}^2$.

Solution: First, assume A is invertible. Let $c_1 \cdot (\alpha u + \beta v) + c_2 \cdot (\gamma u + \delta v) = 0$. Then $(c_1\alpha + c_2\gamma)u + (c_1\beta + c_2\delta)v = 0$ since u and v are linearly independent.

So we have the system

$$\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1}$$

By our assumption, there exists A^{-1}. Hence this homogeneous system has only trivial solution, namely $c_1 = c_2 = 0$. So C consists of two linearly independent vectors and consequently it is a basis for $\mathbb{R}^2$ since $\dim(\mathbb{R}^2) = 2$.

Conversely, assume C is a basis for $\mathbb{R}^2$, then the system (1) has only the trivial solution, so $AX = 0$ consequently $RX = 0$ for some R which is the row echelon reduced matrix. If $RX = 0$ has only trivial solution then $R = I$ which proves that A is invertible. $\Box$

Exercise 0.1.19 Consider the following list of statements. In each case either prove the statement if it is true or give an example showing that it is false.

i) If V is a subspace of $\mathbb{R}^3$ containing two linearly independent vectors, then V is equal to all of $\mathbb{R}^3$.

ii) If vectors v_1 and v_2 are linearly dependent and $u \notin \text{span}(v_1, v_2)$ then the vectors $u + v_1$ and $u + v_2$ are linearly dependent.

iii) If vectors v_1 and v_2 are linearly independent and $u \notin \text{span}(v_1, v_2)$ then the vectors $u + v_1$ and $u + v_2$ are linearly independent.

Solution: i) False.

$\dim(\mathbb{R}^3) = 3$, so we need at least 3 vectors to span $\mathbb{R}^3$. Consider $V = \{v_1, v_2\} = \{(1, 1, 0), (0, 1, 0)\}$. The vectors v_1 and v_2 are linearly independent since $c_1(1, 1, 0)+ c_2(0, 1, 0) = (c_1, c_1 + c_2, 0) = (0, 0, 0)$ iff $c_1 = c_2 = 0$.

But $V \neq \mathbb{R}^3$ since $(0, 0, 1) \in \mathbb{R}^3$ but $(0, 0, 1) \neq k_1(1, 1, 0) + k_2(0, 1, 0) = (k_1, k_1 + k_2, 0) = (0, 0, 0)$. Thus $(0, 0, 1) \notin V$.

ii) False.

v_1 and v_2 are linearly dependent means that $v_1 = k \cdot v_2$. So $u + v_1 = u + k \cdot v_2$.

$$c_1(u + k \cdot v_2) + c_2(u + v_2) = 0 \Rightarrow (c_1 + c_2)u + (kc_1 + c_2)v_2 = 0.$$

But $u \notin \text{span}(v_1, v_2)$ hence $c_1 + c_2 = 0 = kc_1 + c_2$ i.e. $(k - 1)c_1 = 0$ that is $k = 1$ or $c_1 = 0$.

So when $k \neq 1$ we have $c_1 = c_2 = 0$ i.e. $u+v_1$ and $u+v_2$ are linearly independent.

For example, $u = (1, 1)$, $v_1 = (2, 0)$, $v_2 = (1, 0)$. Then $u + v_1 = (3, 1)$ and $u + v_2 = (2, 1)$ are not linearly dependent.

iii) True.

If $c_1(u+v_1)+c_2(u+v_2) = 0$ then $(c_1+c_2)u+c_1v_1+c_2v_2 = 0$. But v_1 and v_2 are linearly independent and $u \notin \text{span}(v_1, v_2)$, hence $c_1 = c_2 = 0$ and $c_1 + c_2 = 0$, so $u + v_1$ and $u + v_2$ are linearly independent.. $\Box$

Exercise 0.1.20 Given three ordered bases $B = \{v_1, v_2, v_3\}$, $C = \{u_1, u_2, u_3\}$, and $D = \{w_1, w_2, w_3\}$ with the transition matrix $P_{C\to D} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$, satisfying $v_1 = u_1 + u_2 + u_3$, $v_2 = u_2 + u_3$, and $v_3 = u_1 - u_2$.

a) Write down the vector $2u_1 - 3u_2 + 4u_3$ as a linear combination of w_1, w_2, and w_3.

b) Find the transition matrix $P_{D\to C}$.

c) Let $\bar{C} = \{u_2, u_3, u_1\}$ and $\bar{D} = \{w_3, w_2, w_1\}$. Find the transition matrix $P_{\bar{C}\to\bar{D}}$.

d) Find the transition matrix $P_{B\to D}$.

Solution: a) $v = 2u_1 - 3u_2 + 4u_3$ then $[v]_C = [2, -3, 4]^T$. So

$$[v]_D = P_{C\to D} \cdot [v]_C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 26 \end{bmatrix}.$$

And thus $v = 3w_1 + 8w_2 + 26w_3$.

b) $P_{D\to C} = P_{C\to D}^{-1}$.

$$[P_{C\to D}|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_2]{-R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 8 & -1 & 0 & 1 \end{array}\right] \xrightarrow[-3R_2+R_3]{-R_2+R_1}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 2 & -3 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -3/2 & 1/2 \end{array}\right] \xrightarrow[-2R_3+R_2]{R_3+R_1}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -5/2 & 1/2 \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & 1 & -3/2 & 1/2 \end{array}\right] = [I|P_{D\to C}].$$

c) $P_{\bar{C}\to\bar{D}} = [[u_2]_{\bar{D}}[u_3]_{\bar{D}}[u_1]_{\bar{D}}] = \begin{bmatrix} 4 & 9 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

$$u_2 = w_1 + 2w_2 + 4w_3 = 4w_3 + 2w_2 + w_1 \Rightarrow [u_2]_{\bar{D}} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}.$$

$$u_3 = w_1 + 3w_2 + 9w_3 = 9w_3 + 3w_2 + w_1 \Rightarrow [u_3]_{\bar{D}} = \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix}.$$

$$u_1 = w_1 + w_2 + w_3 = w_3 + w_2 + w_1 \Rightarrow [u_1]_{\bar{D}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

d) $P_{B\to D} = P_{C\to D} \cdot P_{B\to C} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 6 & 5 & -1 \\ 14 & 13 & -3 \end{bmatrix}.$

Here we used $v_1 = u_1 + u_2 + u_3$, $v_2 = u_2 + u_3$, and $v_3 = u_1 - u_2$, hence $P_{B\to C} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$. □

0.2 Inner Product Spaces

Exercise 0.2.1 Find a non-zero polynomial of degree ≤ 2 orthogonal to the set $\{1, x\}$ with respect to the integral inner product $(p|q) = \int_0^1 p(x)q(x)\,dx$.

Solution: Let $p(x) = ax^2 + bx + c$. Then we want $0 = (1|p)$ and $0 = (x|p)$. That is $0 = (1|p) = \int_0^1 (ax^2 + bx + c)\,dx$ which gives $2a + 3b + 6c = 0$ and $0 = (x|p) = \int_0^1 (ax^3 + bx^2 + cx)\,dx$ yields $3a + 4b + 6c = 0$. Consider the matrix

$$\begin{bmatrix} 2 & 3 & 6 \\ 3 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 6 \\ 0 & -1/2 & -3 \end{bmatrix}.$$

If we take $c = 1$ then $-\frac{1}{2}b = 3$. i.e. $b = 6$ and $a = 6$. So one of the required polynomial is $6x^2 - 6x + 1$. $\square$

Exercise 0.2.2 Orthogonalize by the Gram – Schmidt process the basis $\{v_1, v_2, v_3\} = \{(1,0,1), (0,1,0), (0,-1,2)\}$ for $\mathbb{R}^3$ with respect to the standard inner product $((x_1, y_1, z_1)|(x_2, y_2, z_2)) = x_1x_2 + y_1y_2 + z_1z_2$.

Solution: Choose $w_1 = v_1 = (1, 0, 1)$.

$$w_2 = v_2 - \frac{(v_2|w_1)}{(w_1|w_1)} \cdot w_1 = (0,1,0) - 0 \cdot w_1 = (0,1,0).$$

$$w_3 = v_3 - \frac{(v_3|w_1)}{(w_1|w_1)} \cdot w_1 - \frac{(v_3|w_2)}{(w_2|w_2)} \cdot w_2 = (0,-1,2) - \frac{2}{2} \cdot (1,0,1) - \frac{-1}{1} \cdot (0,1,0) =$$

$$= (0,-1,2) - (1,0,1) + (0,1,0) = (-1,0,1).$$

The answer is $\{(1,0,1), (0,1,0), (-1,0,1)\}$. $\square$

Exercise 0.2.3 Let $\mathbb{R}^{2\times 2}$ be the vector space of all real 2×2 matrices with inner product given by

$$(A|B) = tr(B^T \cdot A),$$

where tr is the trace of a matrix (i.e. sum of the diagonal entries of a matrix). Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

a) Find $(A|B)$ and $\|B\|$, where $\|\cdot\|$ denotes the norm (length) induced by the above inner product.

b) Are A and B orthogonal?

c) Determine the scalar c such that $A - cB$ is orthogonal to A.

Solution: a)

$$(A|B) = tr(B^T \cdot A) = tr\left(\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\right) = tr\left(\begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix}\right) = 0+2 = 2.$$

$$\|B\| = \sqrt{(B|B)} = [tr(B^T \cdot B)]^{1/2} = \left[tr\left(\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\right)\right]^{1/2} =$$

$$= \left[tr\left(\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}\right)\right]^{1/2} = \sqrt{4+2} = \sqrt{6} = \|B\|.$$

b) A and B are not orthogonal since, by part a), $(A|B) = 2 \neq 0$.

c) $A - cB$ and A are orthogonal iff $(A - cB|A) = 0$.

$$(A - cB|A) = (A|A) - c(B|A) = tr(A^T \cdot A) - c(A|B) =$$

$$= tr\left(\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\right) - c \cdot 2 = tr\left(\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\right) - 2c = 3 - 2c.$$

We have $A - cB$ and A are orthogonal iff $3 - 2c = 0$ so $c = 3/2$. □

Exercise 0.2.4 Let u_1 and u_2 be two vectors in an inner product space V such that $\|u_1\| = \|u_2\| = 1$, $(u_1|u_2) = 0$.

a) Find the cosine of the angle between the vectors $2u_1 + 3u_2$ and $4u_1 - 2u_2$.

b) Find a vector $v \in \text{span}(u_1, u_2)$ such that $v \perp (2u_1 + 3u_2)$ and $\|v\| = 1$.

Solution: a)

$$\cos\theta = \frac{(2u_1 + 3u_2|4u_1 - 2u_2)}{\sqrt{(2u_1 + 3u_2|2u_1 + 3u_2) \cdot (4u_1 - 2u_2|4u_1 - 2u_2)}}.$$

$(2u_1 + 3u_2|4u_1 - 2u_2) = 8(u_1|u_1) - 6(u_2|u_2) = 8 - 6 = 2.$

$(2u_1 + 3u_2|2u_1 + 3u_2) \cdot (4u_1 - 2u_2|4u_1 - 2u_2) = (4 + 9) \cdot (16 + 4) = 260.$

We used facts that $(u_1|u_2) = 0$ and $\|u_1\| = \sqrt{(u_1|u_1)} = \|u_2\| = \sqrt{(u_2|u_2)} = 1.$ Thus

$$\cos\theta = \frac{2}{\sqrt{260}}.$$

b) Let $v = x \cdot u_1 + y \cdot u_2$. Find x and y.

$$0 = (v|2u_1 + 3u_2) = (xu_1 + yu_2|2u_1 + 3u_2) = 2x + 3y \Rightarrow y = -\frac{2}{3}x.$$

$$1 = (v|v) = \|v\|^2 = (xu_1 + yu_2|xu_1 + yu_2) = x^2 + y^2 = x^2 + \left(-\frac{2}{3}\right)^2 x^2 = \frac{13}{9}x^2.$$

So we have

$$x^2 = \frac{9}{13} \Rightarrow x = \pm\sqrt{\frac{9}{13}} = \pm\frac{3}{\sqrt{13}}.$$

Finally

$$v = \frac{3}{\sqrt{13}}u_1 - \frac{2}{\sqrt{13}}u_2 \qquad \text{and} \qquad v = -\frac{3}{\sqrt{13}}u_1 + \frac{2}{\sqrt{13}}u_2. \quad \square$$

Exercise 0.2.5 Let $v_1 = (1, 1, 1, 1)$, $v_2 = (1, 1, 2, 0)$, and $v_3 = (2, 3, 0, 0)$ be vectors in $\mathbb{R}^4$ equipped with the standard inner product.

a) Find the orthogonal complement for span$\{v_1, v_2\}$ in $\mathbb{R}^4$.

b) Find the orthogonal basis to span$\{v_1, v_2, v_3\}$.

c) Find the orthogonal projection of $(1, 1, -1, -1)$ to span$\{v_1, v_2\}$.

Solution: a) $(v_1|(x, y, z, u)) = 0$ and $(v_2|(x, y, z, u)) = 0$. So we have the system

$$\begin{cases} x + y + z + u &= 0 \\ x + y + 2z &= 0 \end{cases}.$$

Or in matrix notation $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Find the fundamental solutions of this system. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} \underline{1} & 1 & 1 & 1 \\ 0 & 0 & \underline{1} & -1 \end{bmatrix}$.
The variables y and u are free.

So we have $F_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $F_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Finally

$$\operatorname{span}\{v_1, v_2\}^{\perp} = \langle(-1, 1, 0, 0), (-2, 0, 1, 1)\rangle.$$

b) By the Gram –Schmidt, $w_1 = v_1 = (1, 1, 1, 1)$.

$$w_2 = v_2 - \frac{(v_2|w_1)}{(w_1|w_1)} \cdot w_1 = (1, 1, 2, 0) - \frac{4}{4} \cdot (1, 1, 1, 1) = (0, 0, 1, -1).$$

$$w_3 = v_3 - \frac{(v_3|w_1)}{(w_1|w_1)} \cdot w_1 - \frac{(v_3|w_2)}{(w_2|w_2)} \cdot w_2 =$$

$$= (2, 3, 0, 0) - \frac{5}{4} \cdot (1, 1, 1, 1) - \frac{0}{2} \cdot (0, 0, 1, -1) = \left(\frac{3}{4}, \frac{7}{4}, -\frac{5}{4}, -\frac{5}{4}\right).$$

The orthogonal basis is $\left\{(1, 1, 1, 1), (0, 0, 1, -1), \left(\frac{3}{4}, \frac{7}{4}, -\frac{5}{4}, -\frac{5}{4}\right)\right\}$.
c) Since $((1, 1, -1, -1)|v_1) = 0$ and $((1, 1, -1, -1)|v_2) = 0$ then

$$(1, 1, -1, -1) \in \operatorname{span}\{v_1, v_2\}^{\perp}$$

and hence

$$\operatorname{pr}_{\operatorname{span}\{v_1, v_2\}}((1, 1, -1, -1)) = (0, 0, 0, 0). \qquad \square$$

Exercise 0.2.6 Let $\mathbb{R}^4$ be the inner product space relative to the standard inner product. Let $B = \{(1, 1, 0, 0), (0, 1, 1, 0), (1, -1, 1, 1)\}$ be a basis for $L = \operatorname{span}(B)$.

a) Orthogonalize the basis B by means of the Gram –Schmidt orthogonalization process.

b) Find the closest vector to $g = (1, 1, 1, 0)$ in L.

Solution: a) $w_1 = v_1 = (1,1,0,0)$.

$$w_2 = v_2 - \frac{(v_2|w_1)}{(w_1|w_1)} \cdot w_1 = (0,1,1,0) - \frac{1}{2} \cdot (1,1,0,0) = \left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right).$$

$$w_3 = v_3 - \frac{(v_3|w_1)}{(w_1|w_1)} \cdot w_1 - \frac{(v_3|w_2)}{(w_2|w_2)} \cdot w_2 = (1,-1,1,1) - 0 \cdot w_1 - 0 \cdot w_2 = (1,-1,1,1).$$

The obtained orthogonal basis for L is

$$\{w_1, w_2, w_3\} = \left\{(1,1,0,0), \left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right), (1,-1,1,1)\right\}.$$

b) The vector closest to g is the orthogonal projection of g in L, that is

$$\mathrm{pr}_L(g) = \frac{(g|w_1)}{(w_1|w_1)} \cdot w_1 + \frac{(g|w_2)}{(w_2|w_2)} \cdot w_2 + \frac{(g|w_3)}{(w_3|w_3)} \cdot w_3 = \frac{2}{2} \cdot w_1 + \frac{2}{3} \cdot w_2 + \frac{1}{4} \cdot w_3 =$$

$$= (1,1,0,0) + \left(-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 0\right) + \left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = \left(1 - \frac{1}{12}, 1 + \frac{1}{12}, \frac{11}{12}, \frac{1}{4}\right) =$$

$$= \frac{1}{12} \cdot (11, 13, 11, 3). \quad \square$$

Exercise 0.2.7 Consider the vector space $\mathbb{R}^3$ with the standard inner product and let $S = \{(2,-1,1), (1,2,3), (3,1,4)\}$.

a) Find a basis for the orthogonal complement $S^\perp$ of S.

b) Find the orthogonal projection of $(1,1,1)$ on the subspace spanned by S.

Solution: a) All vectors $v \in S^\perp$ satisfy $(v|u) = 0$, where $u \in S$. So to find a basis of $S^\perp$ we need to solve the system $\begin{cases} 2x - y + z &= 0 \\ x + 2y + 3z &= 0 \\ 3x + y + 4z &= 0 \end{cases}$.

Or in matrix notation $A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Find the fundamental solutions of this system.

$$A \xrightarrow[-R_2+R_3]{-R_1+R_3} \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{-2R_2+R_1} \begin{bmatrix} \underline{1} & 2 & 3 \\ 0 & \underline{-5} & -5 \\ 0 & 0 & 0 \end{bmatrix}.$$

The variable z is free.

So we have $P_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. Hence $\{u = (-1,-1,1)\}$ is a basis for $S^\perp$.

b) Note $v = (1,1,1)$. Since

$$v = \mathrm{pr}_{\langle S \rangle}(v) + \mathrm{pr}_{\langle S^\perp \rangle}(v)$$

then

$$\mathrm{pr}_{\langle S \rangle}(v) = v - \mathrm{pr}_{\langle S^\perp \rangle}(v).$$

$$\mathrm{pr}_{\langle S^\perp \rangle}(v) = \frac{(v|u)}{\|u\|^2} \cdot u = \frac{((1,1,1)|(-1,-1,1))}{((-1,-1,1)|(-1,-1,1))} \cdot (-1,-1,1) =$$

$$= \left(-\frac{1}{3}\right) \cdot (-1,-1,1) = \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right).$$

Hence

$$\mathrm{pr}_{\langle S \rangle}(v) = (1,1,1) - \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right) = \left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}\right). \qquad \square$$

Exercise 0.2.8 If v and w are two vectors of an inner product space, prove that

$$\|\mathrm{v}+\mathrm{w}\|^2 + \|\mathrm{v}-\mathrm{w}\|^2 = 2(\|\mathrm{v}\|^2 + \|\mathrm{w}\|^2).$$

Solution:

$$\|\mathrm{v}+\mathrm{w}\|^2 + \|\mathrm{v}-\mathrm{w}\|^2 = (\mathrm{v}+\mathrm{w}|\mathrm{v}+\mathrm{w}) + (\mathrm{v}-\mathrm{w}|\mathrm{v}-\mathrm{w}) =$$

$$= [(\mathrm{v}|\mathrm{v}) + 2(\mathrm{v}|\mathrm{w}) + (\mathrm{w}|\mathrm{w})] + [(\mathrm{v}|\mathrm{v}) - 2(\mathrm{v}|\mathrm{w}) + (\mathrm{w}|\mathrm{w})] =$$

$$= 2[(\mathrm{v}|\mathrm{v}) + (\mathrm{w}|\mathrm{w})] = 2(\|\mathrm{v}\|^2 + \|\mathrm{w}\|^2). \quad \square$$

Exercise 0.2.9 Let $\mathbb{R}^4$ be the inner product space with the standard inner product $(\cdot|\cdot)$. Let $S = \text{span}\{(1,1,0,1),(1,0,1,0),(0,1,-1,1)\} \subseteq \mathbb{R}^4$.

a) Find a basis B for the orthogonal complement to S in $\mathbb{R}^4$.

b) Applying the Gram –Schmidt orthogonalization to the basis B constructed in a), find an orthonormal basis for the orthogonal complement $S^\perp$ of S.

c) Find the orthogonal projection of $\text{v} = (0,0,0,1)$ on S.

Solution: a)

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{-R_2+R_1} \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{-R_1+R_3} \begin{bmatrix} \underline{1} & 0 & 1 & 0 \\ 0 & \underline{1} & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The third and forth variables are free. We have $x + z = 0$ and $y - z + t = 0$. Then $y = z - t$ and $x = -z$.

Find the fundamental vectors of the system.

$$z = 0, t = 1. \qquad P_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$z = 1, t = 0. \qquad P_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

So $B = \{P_1, P_2\}$ is a basis for $S^\perp$.

b) $w_1 = P_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$

$$w_2 = P_2 - \frac{(P_2|w_1)}{(w_1|w_1)} \cdot w_1 = P_2 + \frac{1}{2} \cdot w_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1/2 \\ 1 \\ 1/2 \end{bmatrix}.$$

$$\overline{w}_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$

$$\overline{w}_2 = \frac{w_2}{\|w_2\|} = w_2 \cdot \left(\sqrt{1+1/4+1+1/4}\right)^{-1} = \begin{bmatrix} -\sqrt{2}/\sqrt{5} \\ 1/\sqrt{10} \\ \sqrt{2}/\sqrt{5} \\ 1/\sqrt{10} \end{bmatrix}$$

So $B_{ort} = \{\overline{w}_1, \overline{w}_2\}$.

c) $\mathrm{pr}_{\langle S\rangle}(\mathrm{v}) = \mathrm{v} - \mathrm{pr}_{\langle S^\perp\rangle}(\mathrm{v})$.

$$\mathrm{pr}_{\langle S^\perp\rangle}(\mathrm{v}) = (\mathrm{v}|\overline{w}_1) \cdot \overline{w}_1 + (\mathrm{v}|\overline{w}_2) \cdot \overline{w}_2 = \frac{1}{\sqrt{2}}\overline{w}_1 + \frac{1}{\sqrt{10}}\overline{w}_2 =$$

$$= \begin{bmatrix} 0 \\ -1/2 \\ 0 \\ 1/2 \end{bmatrix} + \begin{bmatrix} -1/5 \\ 1/10 \\ 1/5 \\ 1/10 \end{bmatrix} = \begin{bmatrix} -1/5 \\ -2/5 \\ 1/5 \\ 3/5 \end{bmatrix}.$$

$$\mathrm{pr}_{\langle S\rangle}(\mathrm{v}) = \mathrm{v} - \begin{bmatrix} -1/5 \\ -2/5 \\ 1/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \\ -1/5 \\ 2/5 \end{bmatrix}. \quad \square$$

Exercise 0.2.10 Given a basis $B = \{1, t+t^2, t-t^2\}$ for $V = \mathcal{P}_2(\mathbb{R})$. The inner product $(.|.)$ in the vector space V is defined by $(u|v) = [u]_B^T[v]_B$, where $[u]_B^T$ is the transpose of the coordinate matrix $[u]_B$ of a vector u with respect to the basis B.

a) Show that B is an orthonormal basis for V with respect to the inner product $(.|.)$.

b) Find the norm of $v = 1+t+t^2$ with respect to the given inner product $(.|.)$.

c) Find the cosine of the angle between $v = 1+t+t^2$ and $u = 2t$ with respect to the given inner product $(.|.)$.

d) Find the orthogonal projection of $w = 1-t+2t^2$ onto $S = \mathrm{Span}\{1, 1+t^2\}$ with respect to the given inner product $(.|.)$.

Solution: a) Denote $v_1 = 1$, $v_2 = t + t^2$, $v_3 = t - t^2$.

Then $[v_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[v_2]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $[v_3]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$(v_1|v_1) = [1\ 0\ 0] \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1$, $(v_2|v_2) = 1$, $(v_3|v_3) = 1$.

Consequently $\|v_1\| = \sqrt{(v_1|v_1)} = 1$, $\|v_2\| = 1$, $\|v_3\| = 1$.

$(v_1|v_2) = [1\ 0\ 0] \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$, $(v_2|v_3) = 0$, $(v_1|v_3) = 0$.

Hence $B = \{v_1, v_2, v_3\}$ is orthonormal basis.

b) $[v]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ since $v = 1 + t + t^2 = 1 \cdot 1 + 1 \cdot (t + t^2) + 0 \cdot (t - t^2)$.

$(v|v) = [v]_B^T [v]_B = [1\ 1\ 0] \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 2$, $\|v\| = \sqrt{(v|v)} = \sqrt{2}$.

c) $[u]_B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ since $u = 2t = 0 \cdot 1 + 1 \cdot (t + t^2) + 1 \cdot (t - t^2)$, $[v]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

$\|u\| = \sqrt{(u|u)} = \sqrt{2}$, $\|v\| = \sqrt{2}$. $(u|v) = [u]_B^T [v]_B = [0\ 1\ 1] \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1$.

$$\cos \hat{vu} = \cos \hat{uv} = \frac{(u|v)}{\|u\| \cdot \|v\|} = \frac{2}{\sqrt{2} \cdot \sqrt{2}} = \frac{1}{2}.$$

d) $w = 1 - t + 2t^2 = \alpha \cdot 1 + \beta \cdot (t + t^2) + \gamma \cdot (t - t^2)$ then

$$\begin{cases} \alpha = 1 \\ \beta + \gamma = -1 \\ \beta - \gamma = 2 \end{cases} \sim \begin{cases} \alpha = 1 \\ \beta + \gamma = -1 \\ 2\beta = 1 \end{cases} \sim \begin{cases} \alpha = 1 \\ \beta = 1/2 \\ \gamma = -1 + 1/2 = -3/2 \end{cases}.$$

Hence $[w]_B = \begin{bmatrix} 1 \\ 1/2 \\ -3/2 \end{bmatrix}$.

$\mathrm{pr}_S(w) = (w|w_1)w_1 + (w|w_2)w_2$, where $\{w_1.w_2\}$ is an orthonormal basis for $S = \mathrm{Span}\{1, 1 + t^2\}$. □

Exercise 0.2.11 a) Find a basis for the orthogonal complement of $S = \{(1,2,-1,3),(2,2,1,-3),(1,0,2,-6)\}$ in $\mathbb{R}^4$ with respect to the standard inner product in $\mathbb{R}^4$.

b) Let $w_1 = (1,1,-1,-1)$, $w_2 = (1,2,1,2)$, and $w_3 = (1,1,2,1)$. Find an orthonormal basis for $W = \text{Span}\{w_1, w_2, w_3\}$ with respect to the standard inner product in $\mathbb{R}^4$.

Solution: a) Denote $v_1 = (1,2,-1,-1)$, $v_2 = (2,2,1,-3)$, and $v_3 = (1,0,2,-6)$. Consider the system $\begin{cases} (v|v_1) = x + 2y - z + 3t = 0 \\ (v|v_2) = 2x + 2y + z - 3t = 0 \\ (v|v_3) = x + 0 \cdot y + 2z - 6t = 0 \end{cases}$. Find the fundamental solution of this system. It is a basis for $S^{\perp}$.

$$\begin{bmatrix} 1 & 0 & 2 & -6 \\ 1 & 2 & -1 & 3 \\ 2 & 1 & 1 & -3 \end{bmatrix} \xrightarrow[-R_1+R_2]{-2R_1+R_3} \begin{bmatrix} 1 & 0 & 2 & -6 \\ 0 & 2 & -3 & 9 \\ 0 & 2 & -3 & 9 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} \underline{1} & 0 & 2 & -6 \\ 0 & \underline{2} & -3 & 9 \\ 0 & 0 & 0 & 01 \end{bmatrix}.$$

$x = 6t - 2z$, $y = \frac{3}{2}z - \frac{9}{2}t$. The variables z and t are free. The fundamental solution is

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -2z + 6t \\ \frac{3}{2}z - \frac{9}{2}t \\ z \\ t \end{bmatrix} = z \begin{bmatrix} -2 \\ 3/2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 6 \\ -9/2 \\ 0 \\ 1 \end{bmatrix}.$$

The basis of $S^{\perp}$ is $\{(-2, 3/2, 1, 0), (6, -9/2, 0, 1)\}$.

b) $x_1 = w_1 = (1,1,-1,-1)$.

$x_2 = w_2 - \frac{(w_2|x_1)}{(x_1|x_1)} \cdot x_1 = (1,2,1,2) - 0 \cdot x_1 = (1,2,1,2)$.

$x_3 = w_3 - \frac{(w_3|x_1)}{(x_1|x_1)} \cdot x_1 - -\frac{(w_3|x_2)}{(x_2|x_2)} \cdot x_2 = (1,1,2,1) - \frac{-1}{4}(1,1,-1,-1) - \frac{7}{10}(1,2,1,2) = \left(\frac{5}{4}, \frac{5}{4}, \frac{7}{4}, \frac{3}{4}\right) - \left(\frac{7}{10}, \frac{14}{10}, \frac{7}{10}, \frac{14}{10}\right) = \left(\frac{11}{20}, -\frac{3}{20}, \frac{21}{20}, -\frac{13}{20}\right)$.

$\|x_1\| = \sqrt{(x_1|x_1)} = \sqrt{4} = 2$, $\|x_2\| = \sqrt{(x_2|x_2)} = \sqrt{10}$, $\|x_3\| = \sqrt{(x_3|x_3)} = \sqrt{\frac{11^2+3^2+21^2+13^2}{20^2}} = \frac{\sqrt{640}}{20} = \frac{8\sqrt{10}}{20}$.

The required orthonormal basis is

$$\left\{ \frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \frac{x_3}{\|x_3\|} \right\} = \left\{ \frac{1}{2}(1,1,-1,-1), \frac{1}{\sqrt{10}}(1,2,1,2), \frac{1}{8\sqrt{10}}(11,-3,21,-13) \right\}. \quad \square$$

Exercise 0.2.12 Let $u_1 = t - t^2$, $u_2 = t + t^2$, $u_3 = 2$, $w_1 = 1$, $w_2 = t$, and $w_3 = t^2$.

a) Show that $B = \{u_1, u_2, u_3\}$ is a basis for the vector space $\mathcal{P}_2(\mathbb{R})$ of polynomials of degree ≤ 2.

b) Find the transition matrix $P_{C \to B}$, where $B = \{u_1, u_2, u_3\}$ and $C = \{w_1, w_2, w_3\}$.

c) Calculate the coordinate matrix $[3 - 2t + t^2]_B$, where $B = \{u_1, u_2, u_3\}$.

d) Given $[v]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find the polynomial $v \in \mathcal{P}_2(\mathbb{R})$.

Solution: a) Consider the coordinate matrix $[u_1\ u_2\ u_3]$ in the standard basis of $\mathcal{P}_2(\mathbb{R})$:

$\begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_2+R_3} \begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$. So the vectors u_1, u_2 and u_3 are linearly independent hence they form a basis for $\mathcal{P}_2(\mathbb{R})$.

b) $[v]_B = P_{C \to B}[v]_C = [[w_1]_B\ [w_2]_B\ [w_3]_B][v]_C$

$$\begin{cases} w_1 = 1 = 0 \cdot u_1 + 0 \cdot u_2 + 1/2 \cdot u_3 \\ w_2 = t = 1/2 \cdot u_1 + 1/2 \cdot u_2 + 0 \cdot u_3 \\ w_3 = t^2 = -1/2 \cdot u_1 + 1/2 \cdot u_2 + 0 \cdot u_3 \end{cases}.$$

$$P = \begin{bmatrix} 0 & 1/2 & -1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 \end{bmatrix}.$$

c) $3 - 2t + t^2 = x_1 u_1 + x_2 u_2 + x_3 u_3 = x_1(t - t^2) + x_2(t + t^2) + x_3 \cdot 2$ hence $(x_2 - x_1)t^2 + (x_2 + x_1)t + 2x_3 = t^2 - 2t + 3$ then $x_2 - x_1 = 1$, $x_2 + x_1 = -2$, and $2x_3 = 3$. Finally $x_1 = -3/2$, $x_2 = -1/2$, $x_3 = 3/2$ and

$$[3 - 2t + t^2]_B = \begin{bmatrix} -3/2 \\ -1/2 \\ 3/2 \end{bmatrix}.$$

d)

$$[t - t^2\ \ t + t^2\ \ 2] \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = t - t^2 + 2(t + t^2) + 6 = t^2 + 3t + 6.$$

The required polynomial is $t^2 + 3t + 6$. □