0.1 Diagonalization and its Applications

Exercise 0.1.1 Let A be a 3×3-matrix whose characteristic polynomial is $\Delta_A(t) = t^3 - 2t^2 + 3t + 2$.

a) Express A^{-1} in term of powers A^0, A^1, and A^2 of A.

b) Express A^5 in term of powers A^0, A^1, and A^2 of A.

Solution: a) By the Cayley — Hamilton theorem, $A^3 - 2A^2 + 3A + 2I = 0$. Hence

$$I = -\frac{3}{2}A + A^2 - \frac{1}{2}A^3 = A \cdot \left(-\frac{3}{2}I + A - \frac{1}{2}A^2\right).$$

Thus

$$A^{-1} = -\frac{3}{2}I + A - \frac{1}{2}A^2.$$

b) Since $A^3 = 2A^2 - 3A - 2I$, we have

$$A^5 = A^2 \cdot A^3 = A^2 \cdot (2A^2 - 3A - 2I) = 2A^1 \cdot A^3 - 3A^3 - 2A^2 =$$

$$= 2A(2A^2 - 3A - 2I) - 3(2A^2 - 3A - 2I) - 2A^2 = 4A^3 - 6A^2 - 4A - 6A^2 + 9A + 6I - 2A^2 =$$

$$= 4A^3 - 14A^2 + 5A + 6I = 4(2A^2 - 3A - 2I) - 14A^2 + 5A + 6I =$$

$$= 8A^2 - 12A - 8I - 14A^2 + 5A + 6I = -2I - 7A - 6A^2. \quad \square$$

Exercise 0.1.2 Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

a) Find an invertible matrix P such that $P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

b) Find the matrix $P^{-1}A^{-1}P$ for the matrix P which is found in a).

c) Find an invertible matrix Q such that $Q^{-1}A^3Q = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 27 \end{bmatrix}$.

Solution: a) We have $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 2$, where λ is from $(A - \lambda I)X = 0$.

$\lambda_1 = -1 \Rightarrow \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$; y is free, hence $P_1 = \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}$.

$\lambda_2 = 3 \Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$; z is free, hence $P_2 = \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}$.

$\lambda_3 = 2 \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$; x is free, hence $P_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$P = \begin{bmatrix} -\frac{1}{3} & \frac{3}{2} & 1 \\ 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

b) $P^{-1}A^{-1}P = (P^{-1}AP)^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$.

c) $Q^{-1}A^3Q = \begin{bmatrix} 2^3 & 0 & 0 \\ 0 & (-1)^3 & 0 \\ 0 & 0 & 3^3 \end{bmatrix} = D^3$ for $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Hence $Q^{-1}AQ = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

$\lambda_1 = 2 \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$; x is free, hence $Q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$\lambda_2 = -1 \Rightarrow \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$; y is free, hence $Q_2 = \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}$.

$\lambda_3 = 3 \Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$; z is free, hence $Q_3 = \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}$.

$$Q = \begin{bmatrix} 1 & -\frac{1}{3} & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}. \quad \square$$

Exercise 0.1.3 Let $B = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 3 & x & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

What must be value of x so that B is diagonalizable?

Solution: $\Delta_B(t) = \det(B - \lambda I) = (\lambda - 2)(\lambda - 3)^2(\lambda - 1)$; $\lambda_2 = \lambda_3 = 3$.

B is diagonalizable iff there are two linearly independent eigenvectors corresponding to $\lambda = 3$.

$\lambda = 3$, $(B - \lambda I)X = 0$, $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & x & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & x & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

In order to have two fundamental solutions, x must be zero. □

Exercise 0.1.4 Find all eigenvalues of the matrix $A = \begin{bmatrix} 5 & 5 \\ 6 & 4 \end{bmatrix}$.

Solution:

$|\lambda I - A| = \begin{vmatrix} \lambda - 5 & -5 \\ -6 & \lambda - 4 \end{vmatrix} = (\lambda-5)(\lambda-4)-30 = \lambda^2-9\lambda+20-30 = \lambda^2-9\lambda-10.$

$$\lambda_{1,2} = \frac{9 \pm \sqrt{121}}{2} = 10, 1. \quad \square$$

Exercise 0.1.5 The eigenvalues of the matrix $B = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ are $\lambda_1 = \lambda_2 = -2$, $\lambda_3 = 5$. Show that B is not diagonalizable.

Solution: 1) $\lambda = \lambda_1 = \lambda_2 = -2$, $(\lambda I - B)X = 0$, $\begin{bmatrix} -8 & -3 & 8 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$

$\begin{bmatrix} -8 & -3 & 8 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 \\ -8 & -3 & 8 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the variable z is free.

So there is only one fundamental solution $F_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

2) $\lambda = \lambda_3 = 5$, $(\lambda I - B)X = 0$, $\begin{bmatrix} -1 & -3 & 8 \\ 0 & 7 & 0 \\ -1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$

$\begin{bmatrix} -1 & -3 & 8 \\ 0 & 7 & 0 \\ -1 & 0 & 8 \end{bmatrix} \to \begin{bmatrix} -1 & -3 & 8 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & -3 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the variable z is free.

So there is only one fundamental solution $F_2 = \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$.

Since we have only two linearly independent eigenvectors correspondent to λ_1, λ_2, and λ_3, the matrix B is not diagonalizable. □

Exercise 0.1.6 The eigenvalues of the real symmetric matrix $C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ are $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 2$. Diagonalize C by means of an orthogonal matrix.

Solution: $\lambda = \lambda_1 = \lambda_2 = -1$, $\lambda I - C = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the variables y and z are free. $P_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$\lambda = \lambda_3 = 2$, $\lambda I - C = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow[\frac{1}{2}R_1 + R_2]{\frac{1}{2}R_1 + R_3} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{bmatrix} \to$ $\begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, the variable z is free. $P_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Use the Gram — Schmidt orthogonalization.

$$Q_1 = P_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}. \quad Q_2 = P_2 - \frac{(P_2|Q_1)}{(Q_1|Q_1)} \cdot Q_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}.$$

$$Q_3 = P_3 - \frac{(P_3|Q_1)}{(Q_1|Q_1)} \cdot Q_1 - \frac{(P_3|Q_2)}{(Q_2|Q_2)} \cdot Q_2 = P_3 - 0 \cdot Q_1 - 0 \cdot Q_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Normalizing $\{Q_1, Q_2, Q_3\}$ one gets

$$\tilde{Q_1} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \tilde{Q_2} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \quad \tilde{Q_3} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

Hence the orthogonal matrix which diagonalizes C is

$$\tilde{Q} = [\tilde{Q_1}, \tilde{Q_2}, \tilde{Q_3}] = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}. \qquad \square$$

Exercise 0.1.7 Determine whether or not the following matrix is diagonalizable, and if it is, find a diagonalizing matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

a) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & -1 \end{bmatrix}$. b) $B = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Solution: a) $|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & -3 & \lambda + 1 \end{vmatrix} = (\lambda - 2)(\lambda^2 - 4) = (\lambda - 2)^2(\lambda + 2) = 0$. The eigenvalues are $\lambda_{1,2} = 2$ and $\lambda_3 = -2$.

$\lambda = 2$, $(2I - A)X = 0$ with $X = [x, y, z]^T$. $2I - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & \underline{1} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the variables x and z are free. $P_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

$\lambda = -2$, $(-2I - A)X = 0$. $-2I - A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & -3 & -1 \end{bmatrix} \to \begin{bmatrix} \underline{1} & 0 & 0 \\ 0 & \underline{3} & 1 \\ 0 & 0 & 0 \end{bmatrix}$, the variable z is free. $P_3 = \begin{bmatrix} 0 \\ -1/3 \\ 1 \end{bmatrix}$.

Hence $P = [P_1\ P_2\ P_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/3 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

b) $|\lambda I - B| = \begin{vmatrix} \lambda - 4 & 0 & 1 \\ 0 & \lambda - 3 & 0 \\ -1 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 3)((\lambda - 4)(\lambda - 2) + 1) = (\lambda - 3)(\lambda^2 - 6\lambda + 9) = (\lambda - 3)^3 = 0$. So, the only eigenvalue is $\lambda = 3$.

$\lambda = 3$, $(3I - B)X = 0$. $3I - B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} \underline{-1} & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the variables y and z are free.

Thus we have only two fundamental solutions which are not enough for diagonalizing a matrix. Hence B is not diagonalizable.□

Exercise 0.1.8 Given a diagonal matrix $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and an orthogonal matrix $P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$, find a real symmetric matrix A such that $P^{-1}AP = D$.

Solution: Since P is orthogonal, $P^{-1} = P^T$. Then $A = PDP^{-1} = PDP^T$.

$$PD = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 2 & 0 & 1 \end{bmatrix}, \qquad P^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

$$A = PD \cdot P^T = \left(\frac{1}{\sqrt{2}}\right)^2 \cdot \begin{bmatrix} 2 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 3/2 \end{bmatrix}. \quad \square$$

Exercise 0.1.9 The characteristic polynomial of an invertible 3×3-matrix A is given by $x^3 - 3x^2 - 6x + 8$.

a) Write A^{-1} as a polynomial matrix in A.

b) Write A^5 as a linear combination of I, A, A^2, A^3.

Solution: a) By the Cayley-Hamilton theorem, $A^3 - 3A^2 - 6A + 8I = 0$. Hence $A^2 - 3A - 6I + 8A^{-1} = 0$ and

$$A^{-1} = -\frac{1}{8}A^2 + \frac{3}{8}A^1 + \frac{3}{4}A^0 = \frac{3}{4}I + \frac{3}{8}A^1 - \frac{1}{8}A^2.$$

b) From (a), $A^2 - 3A - 6I + 8A^{-1} = 0$ then $A^5 - 3A^4 - 6A^3 + 8A^2 = 0$ and thus $A^5 = 3A^4 + 6A^3 - 8A^2$.

Analogously $A^4 - 3A^3 - 6A^2 + 8A = 0$ and thus $A^4 = 3A^3 + 6A^2 - 8A$. So we have

$$A^5 = 3(3A^3 + 6A^2 - 8A) + 6A^3 - 8A^2 = 9A^3 + 18A^2 - 24A + 6A^3 - 8A^2 =$$

$$= -24A + 10A^2 + 15A^3. \quad \square$$

Exercise 0.1.10 Let $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$.

a) Find eigenvalues of A.

b) Determine whether A is invertible or not.

c) Find an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$.

Solution: a) $0 = |A - \lambda I| = \begin{vmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{vmatrix} = (-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 2 & -\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & -\lambda \\ 2 & 2 \end{vmatrix} = (-\lambda)(\lambda^2 - 4) - 2(-2\lambda - 4) + 2(4 + 2\lambda) = (\lambda + 2)(-\lambda^2 + 2\lambda + 8) = -(\lambda + 2)^2(\lambda - 4)$ then $\lambda_{1,2} = -2$ and $\lambda_3 = 4$.

b) The determinant $|A| = \begin{vmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{vmatrix} = (-2)\begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix} = (-2)(-4) + 2 \cdot 4 = 16$. Since $16 \neq 0$, the matrix A is invertible.

c) $\lambda = 4$, $\lambda I - A = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 0 & 6 & -6 \\ -2 & 4 & -2 \\ 0 & -6 & 6 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 2 \\ 0 & 6 & -6 \end{bmatrix}$,

the variable z is free. $y = z$; $x = 2y - z = z$. $P_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$\lambda = -2$, $\lambda I - A = \begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the variables y and z are free. $x = -y - z$. The fundamental solutions are $P_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Use the Gram — Schmidt orthogonalization.

$Q_1 = P_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $Q_2 = P_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ since $P_2 \perp P_1$.

$$Q_3 = P_3 - \frac{(P_3|Q_1)}{(Q_1|Q_1)} \cdot Q_1 - \frac{(P_3|Q_2)}{(Q_2|Q_2)} \cdot Q_2 = P_3 - \frac{1}{2}P_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}.$$

Normalizing $\{Q_1, Q_2, Q_3\}$ one gets

$$\tilde{Q_1} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad \tilde{Q_2} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \tilde{Q_3} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}.$$

$$P = [\tilde{Q_1}, \tilde{Q_2}, \tilde{Q_3}] = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}. \qquad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \qquad \square$$

Exercise 0.1.11 Let $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

a) Determine the characteristic polynomial and all eigenvalues of the matrix A.

b) Find the eigenvectors of the matrix A.

c) Diagonalize the matrix A by means of an orthogonal matrix Q such that $Q^T AD$ is diagonal.

Solution: a) $|\lambda I - A| = \begin{vmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & -1 & \lambda-1 \end{vmatrix} = (\lambda+1)(\lambda-1)^2 - (\lambda+1) =$ $(\lambda+1)\cdot\lambda\cdot(\lambda-2) = \lambda^3 - \lambda^2 - 2\lambda$ is a characteristic polynomial of A and $\lambda_1 = -1$, $\lambda_2 = 0$, and $\lambda_3 = 2$ are the eigenvalues of A.

b) $\lambda = -1$, $\lambda I - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, the variable x is free. $y = z = 0$. $P_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$\lambda = 0$, $\lambda I - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, the variable z is free. $x = 0$, $y = -z$. $P_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

$\lambda = 2$, $\lambda I - A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, the variable z is free. $x = 0$, $y = z$. $P_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

c) The vectors P_1,P_2, and P_3 are orthogonal so we need only to norming them.

$$\tilde{P_1} = \frac{P_1}{\|P_1\|} = P_1 \cdot 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\tilde{P_2} = \frac{P_2}{\|P_2\|} = P_2 \cdot 1/\sqrt{2} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

$$\tilde{P_3} = \frac{P_3}{\|P_3\|} = P_3 \cdot 1/\sqrt{2} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

So the required orthogonal matrix is

$$Q = [\tilde{P}_1 \ \tilde{P}_2 \ \tilde{P}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}; \quad Q^{-1} = Q^T = Q.$$

$$D = Q^T A Q = QAQ = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \qquad \square$$

0.2 Linear Transformations

Exercise 0.2.1 Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3)$.

a) Is T one-to-one? Explain.

b) Find a basis for the kernel (= null space) $N = T^{-1}(0)$ of T.

c) Extend the basis which is found in b) to a basis for $\mathbb{R}^3$.

d) Find the dimension of $N = T^{-1}(0)$ and the dimension of $T(\mathbb{R}^3)$.

e) Find the matrix representation of T with respect to the standard bases in $\mathbb{R}^3$ and $\mathbb{R}^2$.

Solution: a) T is not one-to-one, since, for example, $T(1, -1, 1) = (0, 0) = T(0, 0, 0)$.

b) $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3) = (0, 0)$. Thus $\left\{ \begin{array}{rcl} x_1 + x_2 & = & 0 \\ x_2 + x_3 & = & 0 \end{array} \right.$.

$\left[\begin{array}{ccc} \underline{1} & 1 & 0 \\ 0 & \underline{1} & 1 \end{array} \right]$; x_3 is free. $P_1 = \left[\begin{array}{r} 1 \\ -1 \\ 1 \end{array} \right]$ is a single fundamental solution. Hence $B = \{P_1^T\} = \{(1, -1, 1)\}$ is a basis for $T^{-1}(0)$.

c) $[P_1\ e_1\ e_2\ e_3] =$

$$= \left[\begin{array}{rccc} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow[-R_1+R_3]{R_1+R_2} \left[\begin{array}{crcc} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array} \right] \xrightarrow{R_2+R_3} \left[\begin{array}{cccc} \underline{1} & 1 & 0 & 0 \\ 0 & \underline{1} & 1 & 0 \\ 0 & 0 & \underline{1} & 1 \end{array} \right].$$

Hence $\{P_1, e_1, e_2\}$ is a basis for $\mathbb{R}^{3\times 1}$.

d) $\dim(N) = 1$ and $\dim(T(\mathbb{R}^3)) = 3 - 1 = 2$.

e) $T(1, 0, 0) = (1, 0) = 1 \cdot (1, 0) + 0 \cdot (0, 1)$;

$T(0, 1, 0) = (1, 1) = 1 \cdot (1, 0) + 1 \cdot (0, 1)$;

$T(0, 0, 1) = (0, 1) = 0 \cdot (1, 0) + 1 \cdot (0, 1)$.

$$A_T = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \end{array} \right]. \qquad \square$$

Exercise 0.2.2 The linear transformation T of $\mathbb{R}^3$ id given by $T(x, y, z) = (x + y, 2y + 2z, -x + z)$.

a) Find the matrix representation of T relative to the standard basis for $\mathbb{R}^3$.

b) Find a vector of norm one which is orthogonal to the vector space $T(\mathbb{R}^3)$ relative to the standard inner product in $\mathbb{R}^3$.

Solution: a) $T(e_1) = T(1, 0, 0) = (1, 0, -1)$; $T(e_2) = T(0, 1, 0) = (1, 2, 0)$; $T(e_3) = T(0, 0, 1) = (0, 2, 1)$.

The required matrix is

$$A_T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ -1 & 0 & 1 \end{bmatrix}.$$

b) $(T(e_1)|(x, y, z)) = ((1, 0, -1))|(x, y, z)) = 0 \quad \Rightarrow \quad x - z = 0;$

$(T(e_2)|(x, y, z)) = ((1, 2, 0))|(x, y, z)) = 0 \quad \Rightarrow \quad x + 2y = 0;$

$(T(e_3)|(x, y, z)) = ((0, 2, 1))|(x, y, z)) = 0 \quad \Rightarrow \quad 2y + z = 0.$

We have $\begin{cases} x - z &= 0 \\ x + 2y &= 0 \\ 2y + z &= 0 \end{cases}$.

$$[[x][y][z]] = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow[-R_2+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since z is free, we find the fundamental solution that is $F = \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}$, $\|F\| = \sqrt{(F|F)} = 3/2$. The required vector is

$$P = \pm \frac{F}{\|F\|} = \pm \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}. \qquad \square$$

Exercise 0.2.3 Let $L : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$ be the linear transformation given by $L(A) = \frac{1}{2}(A - A^T)$.

a) Write a basis and find the dimension of the $\text{Ker}(L) = \{A \in \mathbb{R}^{2\times 2} | \ L(A) = 0_{\mathbb{R}^{2\times 2}}\}$.

b) Write a basis and find the dimension of the $\text{Im}(L) = \{L(A) | \ A \in \mathbb{R}^{2\times 2}\}$.

c) Find the matrix representation of L with respect to the standard ordered basis for $\mathbb{R}^{2\times 2}$ which is $\{E_{11}, E_{12}, E_{21}, E_{22}\}$.

Solution: a) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We have $L(A) = \frac{1}{2}\begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix} = 0$, hence $b = c$. Thus

$$\text{Ker}(L) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | \ b = c \right\} = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} | \ a, b, d \in \mathbb{R} \right\}.$$

And the standard basis for $\text{Ker}(L)$ is

$a = 1$, $b = d = 0$: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $a = 0$, $b = 1$, $d = 0$: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $a = b = 0$, $d = 1$: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Hence $\dim(\text{Ker}(L)) = 3$.

b) $L(A) = L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$.

$\begin{cases} 2x = 0 \\ 2y = b - c \\ 2z = c - b \\ 2t = 0 \end{cases}$, so $y = -z$, and z is free. We have $P = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ (is the fundamental solution).

Thus the basis for $\text{Im}(L)$ is $\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$, and hence $\dim(\text{Im}(L)) = 1$.

c) $B = \{E_{11}, E_{12}, E_{21}, E_{22}\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

$$A_L = [[L(E_{11})]_B \ [L(E_{12})]_B \ [L(E_{21})]_B \ [L(E_{22})]_B] =$$

$$= \left[\left[\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right]_B \left[\begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix} \right]_B \left[\begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix} \right]_B \left[\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right]_B \right] =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad \square$$

Exercise 0.2.4 Let $L : V \longrightarrow V$ be a linear transformation such that $L(\mathbf{v}_1) = 2\mathbf{v}_1$ and $L(\mathbf{v}_2) = -\mathbf{v}_2$, where $\mathbf{v}_1 \neq 0$, $\mathbf{v}_2 \neq 0$.

a) Show that $\mathbf{v}_1$, $\mathbf{v}_2$ are linearly independent.

b) Show that there is no $\lambda \in \mathbb{R}$ such that $L(\mathbf{v}_1 + \mathbf{v}_2) = \lambda(\mathbf{v}_1 + \mathbf{v}_2)$.

c) Let $\mathcal{B} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ be a basis for V such that $\mathcal{B}_1 = \{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for $\text{Ker}(L) = \{\mathbf{v} \in V | \ L(\mathbf{v}) = 0\}$. Show that the vectors $L(\mathbf{w}_3)$, $L(\mathbf{w}_4)$ are linearly independent.

Solution: a) Find $\alpha, \beta \in \mathbb{R}$ such that $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = 0$. It means that $0 = L(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = \alpha L(\mathbf{v}_1) + \beta L(\mathbf{v}_2) = 2\alpha\mathbf{v}_1 - \beta\mathbf{v}_2 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = 0$. Then $3\alpha\mathbf{v}_1 = 0$ that is $\alpha = 0$ (since $\mathbf{v}_1 \neq 0$ by the condition) and so $\beta = \alpha = 0$.

Thus $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

b) If such a λ exists then $L(\mathbf{v}_1 + \mathbf{v}_2) = 2\mathbf{v}_1 - \mathbf{v}_2 = \lambda(\mathbf{v}_1 + \mathbf{v}_2)$ that is equivalent to $(2 - \lambda)\mathbf{v}_1 = \mathbf{v}_2 + \lambda\mathbf{v}_2 = (\lambda + 1)\mathbf{v}_2$.

If $\lambda \neq 2$ then $\mathbf{v}_1 = \frac{1+\lambda}{2-\lambda}\mathbf{v}_2$ that contradicts to a) ($\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent).

If $\lambda = 2$ then $(1 + \lambda)\mathbf{v}_2 = 0$ and $\mathbf{v}_2 = 0$ that contradicts to the condition that $\mathbf{v}_1 \neq 0$, $\mathbf{v}_2 \neq 0$.

So there is no such a λ.

c) $\{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for $\text{Ker}(L)$ then $L(\alpha\mathbf{w}_1 + \beta\mathbf{w}_2) = 0 = \alpha L(\mathbf{w}_1) + \beta L(\mathbf{w}_2)$ for all $\alpha, \beta \in \mathbb{R}$.

If $L(\mathbf{w}_3)$ and $L(\mathbf{w}_4)$ are linearly dependent then there are γ and δ ($\gamma \neq 0$, $\delta \neq 0$) such that $\gamma L(\mathbf{w}_3) + \delta L(\mathbf{w}_4) = 0$ that is equivalent to $L(\gamma\mathbf{w}_3 + \delta\mathbf{w}_4) = 0$ i.e. $\gamma\mathbf{w}_3 + \delta\mathbf{w}_4 \in \text{Ker}(L)$.

This means that there are $a, b \in \mathbb{R}$ ($a \neq 0$, $b \neq 0$) such that $\gamma\mathbf{w}_3 + \delta\mathbf{w}_4 = a\mathbf{w}_1 + b\mathbf{w}_2$ that contradicts to the condition that $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$ are linearly independent as basis vectors in $\mathcal{B}$.$\square$