0.1 Matrices

Exercise 0.1.1 Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix}.$$

Find A^{-1} (the inverse of A) if it exists.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -4 & -2 & -3 & 0 & 1 \end{array}\right] \xrightarrow[-4R_2+R_3]{R_2+R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 5 & -4 & 1 \end{array}\right] \xrightarrow[-R_2]{\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 5/2 & -2 & 1/2 \end{array}\right] \xrightarrow{-R_3+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 5/2 & -2 & 1/2 \end{array}\right].$$

So $A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1/2 & 1 & -1/2 \\ 5/2 & -2 & 1/2 \end{bmatrix}$. □

Exercise 0.1.2 Let B, C, and D be $n \times n$ matrices such that BC is right invertible and D is a right inverse of BC. Show that B is right invertible and find a right inverse of B.

Solution: $BCD = I$ and hence CD is a right inverse of B. By the theorem, if a square matrix has a right inverse then it is invertible. Thus B is invertible and $B^{-1} = CD$. □

Exercise 0.1.3 Let

$$A = \begin{bmatrix} 3 & -3 & 7 & 2 \\ 1 & -1 & 3 & 0 \\ 1 & -1 & 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & k & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix}.$$

a) Find a row reduced echelon matrix R that is row equivalent to A.

b) Find the value(s) of k (if exist) for which A is row equivalent to B.

Solution: a)

$$\begin{bmatrix} 3 & -3 & 7 & 2 \\ 1 & -1 & 3 & 0 \\ 1 & -1 & 2 & 1 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 3 & -3 & 7 & 2 \\ 1 & -1 & 3 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 & 3 & 0 \\ 3 & -3 & 7 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \rightarrow$$

$$\xrightarrow{-3R_1+R_2} \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{-2R_3+R_2} \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-R_2]{3R_2+R_1}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

b) $B \sim A \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & k & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow k = 1.$ □

Exercise 0.1.4 Let C, D, L, M, and K be 2×4 matrices such that $C \xrightarrow{R_1 \leftrightarrow R_2} L \xrightarrow{2R_2} K$ and $D \xrightarrow{2R_2+R_1} M \xrightarrow{3R_1} K$.

Find an invertible matrix P such that $PC = D$ and write P as a product of four elementary matrices (accordingly to the diagrams above).

Solution:

$$C \xrightarrow[\mathcal{E}_1]{R_1 \leftrightarrow R_2} L \xrightarrow[\mathcal{E}_2]{2R_2} K \xrightarrow[\mathcal{E}_3]{\frac{1}{3}R_1} M \xrightarrow[\mathcal{E}_4]{-2R_2+R_1} D.$$

$$P = \mathcal{E}_4(I) \cdot \mathcal{E}_3(I) \cdot \mathcal{E}_2(I) \cdot \mathcal{E}_1(I) = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1/3 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 1/3 \\ 2 & 0 \end{bmatrix}.$$

$$P = \begin{bmatrix} -4 & 1/3 \\ 2 & 0 \end{bmatrix}. \quad \square$$

Exercise 0.1.5 Let $A = \begin{bmatrix} -2 & 6 & 2 & -2 \\ 1 & -1 & 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 1/4 & 3/2 \\ 1/4 & 1/2 \end{bmatrix}$.

a) Find P^{-1}.

b) Find a row reduced echelon matrix R and the invertible matrix Q such that $A = QR$.

Solution: a) $\left[\begin{array}{cc|cc} 1/4 & 3/2 & 1 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{cc|cc} 1/4 & 3/2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \xrightarrow[-R_2]{4R_1}$

$\rightarrow \left[\begin{array}{cc|cc} 1 & 6 & 4 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right] \xrightarrow{-6R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 6 \\ 0 & 1 & 1 & -1 \end{array}\right].$

$$P^{-1} = \begin{bmatrix} -2 & 6 \\ 1 & -1 \end{bmatrix}.$$

b) $\left[\begin{array}{cccc|cc} -2 & 6 & 2 & -2 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|cc} 1 & -1 & 0 & 1 & 0 & 1 \\ -2 & 6 & 2 & -2 & 1 & 0 \end{array}\right] \xrightarrow{2R_1+R_2}$

$\rightarrow \left[\begin{array}{cccc|cc} 1 & -1 & 0 & 1 & 0 & 1 \\ 0 & 4 & 2 & 1 & 1 & 2 \end{array}\right] \xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{cccc|cc} 1 & -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1/2 & 0 & 1/4 & 1/2 \end{array}\right] \xrightarrow{R_2+R_1}$

$\rightarrow \left[\begin{array}{cccc|cc} 1 & 0 & 1/2 & 1 & 1/4 & 3/2 \\ 0 & 1 & 1/2 & 0 & 1/4 & 1/2 \end{array}\right] = [R|P].$

So we have $PA = R$ consequently $A = P^{-1}R$ and finally

$$Q = P^{-1} = \begin{bmatrix} -2 & 6 \\ 1 & -1 \end{bmatrix}; \qquad R = \begin{bmatrix} 1 & 0 & 1/2 & 1 \\ 0 & 1 & 1/2 & 0 \end{bmatrix}. \quad \square$$

Exercise 0.1.6 Let $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ -1 & -1 & 3 & 4 \\ 2 & 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 3 & 3 \\ r & r & 3 & 1 \\ 1 & 1 & 6 & 5 \end{bmatrix}$.

a) Find the row reduced echelon form of A.

b) Find $r \in \mathbb{R}$ for which the matrices A and B are row equivalent.

Solution: a) $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ -1 & -1 & 3 & 4 \\ 2 & 2 & 3 & 1 \end{bmatrix} \xrightarrow[-2R_1+R_3]{R_1+R_2} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 \end{bmatrix} \xrightarrow[-R_2+R_3]{\frac{1}{3}R_2}$

$$\to \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

b) $B = \begin{bmatrix} 0 & 0 & 3 & 3 \\ r & r & 3 & 1 \\ 1 & 1 & 6 & 5 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 6 & 5 \\ r & r & 3 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix} \xrightarrow{-rR_1+R_2}$

$$\to \begin{bmatrix} 1 & 1 & 6 & 5 \\ 0 & 0 & 3-6r & 1-5r \\ 0 & 0 & 3 & 3 \end{bmatrix} \xrightarrow{-2R_3+R_1} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 3-6r & 1-5r \\ 0 & 0 & 3 & 3 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow R_2]{-\frac{1}{3}R_3}$$

$\to \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3-6r & 1-5r \end{bmatrix} \sim R$ if and only if $3-6r = 1-5r$.

Thus $3 - r = 1$ and $r = 2$. □

Exercise 0.1.7 Given $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, solve the following matrix equations.

a) Find a matrix X such that $AX = B$.

b) Find a matrix Y such that $YA = B$.

c) Find a matrix Z such that $AZ^TB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution: $\left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & -1 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_2}$

$$\to \left[\begin{array}{cc|cc} 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right] = [I|A^{-1}].$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1/3 \end{array}\right] \xrightarrow{-2R_2+R_1}$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & 1 & -2/3 \\ 0 & 1 & 0 & 1/3 \end{array}\right] = [I|B^{-1}].$$

a) $AX = B \Rightarrow X = A^{-1}B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -9 \\ -1 & 4 \end{bmatrix}.$

b) $YA = B \Rightarrow Y = BA^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -3 & 6 \end{bmatrix}$.

c) $AZ^TB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow Z^T = A^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} B^{-1} = A^{-1}B^{-1}$.

So $Z^T = A^{-1}B^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 3 & -11/3 \\ -1 & 4/3 \end{bmatrix}$.

Finally $Z = (Z^T)^T = \begin{bmatrix} 3 & -1 \\ -11/3 & 4/3 \end{bmatrix}$. □

Exercise 0.1.8 Let A be a square matrix. Show that $A^T - A$ is a skew-symmetric matrix.

Solution: $(A^T - A)^T = A^{TT} - A^T = A - A^T = -(A^T - A)$. □

Exercise 0.1.9 Let B be a square matrix. Show that B^TB is a symmetric matrix.

Solution: $(B^TB)^T = B^TB^{TT} = B^TB$. □

Exercise 0.1.10 Find x, y, z, and t if $-2 \begin{bmatrix} x & -1 \\ 3 & 1 \end{bmatrix} + 3 \begin{bmatrix} 2 & y \\ z & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ z & t \end{bmatrix}$.

Solution:
$$\left\{ \begin{array}{rcl} -2x+6 & = & 2 \\ -6+3z & = & z \\ 2+3y & = & 0 \\ -2+12 & = & t \end{array} \right. \Rightarrow \left\{ \begin{array}{rcl} -2x & = & -4 \\ 2z & = & 6 \\ 3y & = & -2 \\ t & = & 10 \end{array} \right. \Rightarrow \left\{ \begin{array}{rcl} x & = & 2 \\ y & = & -2/3 \\ z & = & 3 \\ t & = & 10 \end{array} \right. . \square$$

Exercise 0.1.11 Find all matrices of the form $X = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$ satisfying $X^2 - I = 0$.

Solution: $X^2 = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

Hence $a^2 = 1$. Thus $a = \pm 1$ and $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $X = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. □

Exercise 0.1.12 Show that

$$\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}^n = \begin{bmatrix} 1 & \frac{3}{2}(1-(-1)^n) \\ 0 & (-1)^n \end{bmatrix}$$

for any positive integer n.

[Hint: 1) Show that it is true for $n = 1$. 2) Show that when it is true for $n = m$ then it is true also for $n = m + 1$.]

1-st solution: If $n = 1$ then

$$\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}^1 = \begin{bmatrix} 1 & \frac{3}{2}(1-(-1)^1) \\ 0 & (-1)^1 \end{bmatrix}.$$

If $n = m + 1$ then

$$\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}^{m+1} = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}^m \cdot \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2}(1-(-1)^m) \\ 0 & (-1)^m \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 3-\frac{3}{2}(1-(-1)^m) \\ 0 & (-1)^{m+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2}(1+(-1)^m) \\ 0 & (-1)^{m+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2}(1-(-1)^{m+1}) \\ 0 & (-1)^{m+1} \end{bmatrix}.$$

2-nd solution: $A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$,

$$A^2 = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Thus $A^3 = A$, $A^4 = I$, ... Hence

$$A^n = \begin{cases} A \text{ if } n \text{ is odd} \\ I \text{ if } n \text{ is even} \end{cases}.$$

Remark that the same is true for $B_n = \begin{bmatrix} 1 & \frac{3}{2}(1-(-1)^n) \\ 0 & (-1)^n \end{bmatrix}$, namely

$$B^n = \begin{cases} A \text{ if } n \text{ is odd} \\ I \text{ if } n \text{ is even} \end{cases}.$$

Hence $A^n = B_n$ for all $n \geq 0$. □

Exercise 0.1.13 Let A, B, C, and D be 3×3 matrices such that $A \xrightarrow{2R_1+R_2} B$ and $D \xrightarrow{R_1\leftrightarrow R_3} C \xrightarrow{-R_2+R_3} B$.

a) Find an invertible matrix P such that $PA = D$.

b) Write P as a product of three elementary matrices (accordingly to the three row operations in the diagrams above).

Solution: a) $A \underset{\mathcal{E}_1}{\xrightarrow{2R_1+R_2}} B \underset{\mathcal{E}_2}{\xrightarrow{R_2+R_3}} C \underset{\mathcal{E}_3}{\xrightarrow{R_1\leftrightarrow R_3}} D.$

$$P = \mathcal{E}_3\mathcal{E}_2\mathcal{E}_1(I) = \mathcal{E}_3\mathcal{E}_2\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \mathcal{E}_3\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

b) $P = \mathcal{E}_3\mathcal{E}_2\mathcal{E}_1(I) = \mathcal{E}_3(I)\cdot\mathcal{E}_2(I)\cdot\mathcal{E}_1(I) =$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad \square$$

Exercise 0.1.14 Find the values of x, y, and z for which the following matrix is skew-symmetric $\begin{bmatrix} x+y-3 & -1 & 3 \\ x & 0 & -2 \\ -3 & x+z & z-x \end{bmatrix}$.

(A matrix A is called skew-symmetric if $A^T = -A$ and A is called symmetric if $A^T = A$).

Solution: $A^T = -A \Rightarrow$

$$\begin{bmatrix} x+y-3 & x & -3 \\ -1 & 0 & x+z \\ 3 & -2 & z-x \end{bmatrix} = \begin{bmatrix} -x-y+3 & 1 & -3 \\ -x & 0 & 2 \\ 3 & -x-z & -z+x \end{bmatrix}.$$

So $x = 1$ and $x + z = 2$ since $z = 2 - 1 = 1$.

$x + y - 3 = -x - y + 3 \Rightarrow x + y = 3 \Rightarrow y = 3 - 1 = 2.$

Thus $x = 1$, $y = 2$, $z = 1$. $\square$

Exercise 0.1.15 Given a real or complex square matrix A. Find a symmetric matrix S and a skew-symmetric matrix K such that $A = S + K$.

(Hint: First show that for any matrix B the matrix $B + B^T$ is symmetric and $B - B^T$ is skew-symmetric).

Solution: $(B + B^T)^T = B^T + (B^T)^T = B^T + B = B + B^T$ hence $B + B^T$ is symmetric.

$(B - B^T)^T = B^T - (B^T)^T = B^T - B = -(B - B^T)$ hence $B - B^T$ is skew-symmetric.

$A = \frac{1}{2}A + \frac{1}{2}A = \frac{1}{2}A + \frac{1}{2}A^T - \frac{1}{2}A^T + \frac{1}{2}A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = S + K$, where $S = \frac{1}{2}(A + A^T)$ is symmetric and $K = \frac{1}{2}(A - A^T)$ is skew-symmetric. □

Exercise 0.1.16 Apply 0.1.15 to the matrix $A = \begin{bmatrix} 2i & 0 \\ -i & -1 \end{bmatrix}$.

Solution: $A = S + K$.

$S = \frac{1}{2}(A + A^T) = \frac{1}{2}\left(\begin{bmatrix} 2i & 0 \\ -i & -1 \end{bmatrix} + \begin{bmatrix} 2i & -i \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} 2i & -i/2 \\ -i/2 & -1 \end{bmatrix}$.

$K = \frac{1}{2}(A - A^T) = \frac{1}{2}\left(\begin{bmatrix} 2i & 0 \\ -i & -1 \end{bmatrix} - \begin{bmatrix} 2i & -i \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} 0 & i/2 \\ -i/2 & 0 \end{bmatrix}$.

Thus $\begin{bmatrix} 2i & 0 \\ -i & -1 \end{bmatrix} = \begin{bmatrix} 2i & -i/2 \\ -i/2 & -1 \end{bmatrix} + \begin{bmatrix} 0 & i/2 \\ -i/2 & 0 \end{bmatrix}$. □

Exercise 0.1.17 Let $A = \begin{bmatrix} 1 & 2 & -2 & 7 \\ -1 & 1 & 2 & -1 \\ 1 & 5 & -2 & 13 \end{bmatrix}$.

a) Find a row-reduced echelon matrix R which is row equivalent to A.

b) Find an invertible matrix P such that $R = PA$.

c) Find an invertible matrix Q such that $A = QR$.

Solution: a) and b). $[A|I] =$

$$= \left[\begin{array}{rrrr|rrr} 1 & 2 & -2 & 7 & 1 & 0 & 0 \\ -1 & 1 & 2 & -1 & 0 & 1 & 0 \\ 1 & 5 & -2 & 13 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3]{R_1+R_2} \left[\begin{array}{rrrr|rrr} 1 & 2 & -2 & 7 & 1 & 0 & 0 \\ 0 & 3 & 0 & 6 & 1 & 1 & 0 \\ 0 & 3 & 0 & 6 & -1 & 0 & 1 \end{array}\right] \xrightarrow[-R_2+R_3]{\frac{1}{3}R_2}$$

$$\rightarrow \left[\begin{array}{rrrr|rrr} 1 & 2 & -2 & 7 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{rrrr|rrr} 1 & 0 & -2 & 3 & 1/3 & -2/3 & 0 \\ 0 & 1 & 0 & 2 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right] =$$

$[R|P]$, where

$$R = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \qquad P = \begin{bmatrix} 1/3 & -2/3 & 0 \\ 1/3 & 1/3 & 0 \\ -2 & -1 & 1 \end{bmatrix}$$

c) $R = PA \Rightarrow P^{-1}R = P^{-1}PA = A$. Hence $Q = P^{-1}$. Calculate P^{-1}:

$$[P|I] = \left[\begin{array}{rrr|rrr} 1/3 & -2/3 & 0 & 1 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1 & 0 \\ -2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[3R_1]{3R_2} \left[\begin{array}{rrr|rrr} 1 & -2 & 0 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 & 3 & 0 \\ -2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_2]{2R_1+R_3}$$

$$\rightarrow \left[\begin{array}{rrr|rrr} 1 & -2 & 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & -3 & 3 & 0 \\ 0 & -5 & 1 & 6 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{3}R_2]{5R_2+R_3} \left[\begin{array}{rrr|rrr} 1 & -2 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 5 & 1 \end{array}\right] \xrightarrow{2R_2+R_1}$$

$$\rightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 5 & 1 \end{array}\right] = [I|Q = P^{-1}].$$

So $Q = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix}$. □

Exercise 0.1.18 Let $A = \begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ -3 & 3 & -6 & -3 & 3 \\ 2 & -2 & 5 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 1 & -2 & x \\ 0 & 0 & 0 & 0 & 1 \\ 1 & y & 2 & 1 & z \end{bmatrix}$.

a) Find the row reduced echelon matrices R and S which are row equivalent to A and B respectively. At each step write the elementary row operation that you use.

b) Find the values of x, y, z, for which the matrices A and B are row equivalent.

c) By using the row operation in a) properly, write $B = \mathcal{E}_k \ldots \mathcal{E}_2\mathcal{E}_1 A$ with $k \leq 10$, where $\mathcal{E}_1$, $\mathcal{E}_2$, ..., $\mathcal{E}_k$ are elementary matrices.

d) Show that the system $A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$ is consistent for each p, q, r.

e) Solve $AX = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution: a) $A = \begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ -3 & 3 & -6 & -3 & 3 \\ 2 & -2 & 5 & 0 & 0 \end{bmatrix} \xrightarrow[-2R_1+R_3]{3R_1+R_2} \begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix} \to$

$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow[-2R_2+R_1]{\frac{1}{3}R_3} \begin{bmatrix} 1 & -1 & 0 & 5 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = R.$

$B = \begin{bmatrix} 0 & 0 & 1 & -2 & x \\ 0 & 0 & 0 & 0 & 1 \\ 1 & y & 2 & 1 & z \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & y & 2 & 1 & z \\ 0 & 0 & 1 & -2 & x \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to$

$\xrightarrow[-xR_3+R_2]{-zR_3+R_1} \begin{bmatrix} 1 & y & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & y & 0 & 5 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = S.$

b) For two matrices to be row equivalent, they should have the same row reduced echelon matrix. Thus $R = S$ and $y = -1$; x and z can be any numbers.

c) $B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot$

$\cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot$

$\cdot \begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ -3 & 3 & -6 & -3 & 3 \\ 2 & -2 & 5 & 0 & 0 \end{bmatrix}.$

d) $\left[\begin{array}{rrrrr|c} 1 & -1 & 2 & 1 & 0 & p \\ -3 & 3 & -6 & -3 & 3 & q \\ 2 & -2 & 5 & 0 & 0 & r \end{array}\right] \xrightarrow[3R_1+R_2]{-2R_1+R_3} \left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 1 & 0 & p \\ 0 & 0 & 0 & 0 & 3 & q+3p \\ 0 & 0 & 1 & -2 & 0 & r-2p \end{array}\right] \rightarrow$

$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 1 & 0 & p \\ 0 & 0 & 1 & -2 & 0 & r-2p \\ 0 & 0 & 0 & 0 & 3 & q+3p \end{array}\right].$

So the system is consistent for each p, q, r.

e) $\left[\begin{array}{rrrrr|c} 1 & -1 & 2 & 1 & 0 & 1 \\ -3 & 3 & -6 & -3 & 3 & 1 \\ 2 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[3R_1+R_2]{-2R_1+R_3} \left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 1 & -2 & 0 & -1 \end{array}\right] \rightarrow$

$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 3 & 4 \end{array}\right].$

$$\left\{\begin{array}{rcr} x_1 - x_2 + 2x_3 + x_4 & = & 1 \\ 3x_5 & = & 4 \\ x_3 - 2x_4 & = & -1 \end{array}\right. \Leftrightarrow$$

$x_1 = 1 + x_2 - 2x_3 - x_4 = 1 + x_2 - 4x_4 + 2 - x_4 = 3 + x_2 - 5x_4$; $x_5 = 4/3$; $x_3 = 2x_4 - 1$.

$$X = \begin{bmatrix} 3 + x_2 - 5x_4 \\ x_2 \\ -1 + 2x_4 \\ x_4 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 4/3 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}. \ \square$$

Exercise 0.1.19 Consider the following list of statements. In each case, either prove the statement is true or give an example showing that it is false.

a) For a square matrix A, $A + A^T$ is symmetric.

b) For a square matrix A, $A - A^T$ is skew-symmetric.

c) For square matrices A and B, $(A + B)(A - B) = A^2 - B^2$.

d) If $A^2 = I$ then $A = I$ or $A = -I$.

e) For square matrices A and B, if $AB = 0$ then $BA = 0$.

f) A square matrix P is called idempotent if $P^2 = P$. If P is idempotent so is $Q = P + AP - PAP$ for any square matrix A.

g) If A^2 is invertible then A is invertible.

Solution: a) True. $(A + A^T)^T = A^T + (A^T)^T = A^T + A$.

b) True. $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$.

c) False. Consider as example $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$. We have

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\right) = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \neq$$

$$\neq \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}^2 - \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.$$

d) False. An example A or B from c).

e) False. As example take $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$. We have

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

but

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

f) True. We have $P \cdot Q = P^2 + PAP - P^2AP = P + PAP - PAP = P$ and $Q \cdot P = P^2 + AP^2 - PAP^2 = P + AP = PAP = Q$. Thus $Q^2 = (QP)Q = Q(PQ) = QP = Q$, so Q is idempotent.

g) True. If A^2 is invertible then there is B such that $A^2B = I$ (and $BA^2 = I$) then $A(AB) = I$ then A has a left inverse. By Theorem 2.2.1 in the book, A is invertible. □

Exercise 0.1.20 Let $C = [A|B]$ be the augmented matrix of a system $AX = B$ of linear equations with a square coefficient matrix A. Assume that C is row equivalent to a matrix D with a zero row. Show that the matrix A is not invertible.

Solution: Let $C \sim D$ then $PC = D$ for some invertible matrix P. That is $P[A|B] = [PA|PB] = C$. Hence the square matrix PA has a zero row. Since $A \sim PA$, then A is not invertible. □

Exercise 0.1.21 Let A, B be $n \times n$ matrices such that AB is invertible. Show that A is invertible.

Solution: Let C be the inverse of AB, that is $CAB = ABC = I$. Hence $A(BC) = I$. Thus a square matrix A has a right inverse, namely BC. Then BC is also a left inverse and hence $A^{-1} = BC$ □

Exercise 0.1.22 Let $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 6 & 4 & 2 \\ -1 & 2 & -2 & 0 \end{bmatrix}$. Find a row-reduced echelon matrix B which is row equivalent to A. Find an invertible matrix P such that $B = PA$.

Solution: Remark that matrix A is not square, thus there are infinitely many such matrix P. We find one of them.

$$[A|I] = \left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 6 & 4 & 2 & 0 & 1 & 0 \\ -1 & 2 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_3+R_2} \left[\begin{array}{cccc|ccc} 0 & 4 & 1 & 1 & 1 & 0 & 1 \\ 0 & 8 & 2 & 2 & 0 & 1 & 1 \\ -1 & 2 & -2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R_3]{-2R_1+R_2} \left[\begin{array}{cccc|ccc} 0 & 4 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & 1 & -1 \\ 1 & -2 & 2 & 0 & 0 & 0 & -1 \end{array}\right] \xrightarrow[R_1\leftrightarrow R_3]{\frac{1}{4}R_1;\ R_3\leftrightarrow R_2}$$

$$\left[\begin{array}{cccc|ccc} 1 & -2 & 2 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & -2 & 1 & -1 \end{array}\right] \xrightarrow{2R_2+R_1}$$

$$\left[\begin{array}{cccc|ccc} 1 & 0 & 5/2 & 1/2 & 1/2 & 0 & -1/2 \\ 0 & 1 & 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & -2 & 1 & -1 \end{array}\right] = [B|P].$$

Hence $P = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 1/4 & 0 & 1/4 \\ -2 & 1 & -1 \end{bmatrix}$. □

Exercise 0.1.23 Let $C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 6 & 4 \\ -1 & 2 & -2 \end{bmatrix}$. Is C invertible? If no, explain why C is not invertible.

Solution: Since $C \xrightarrow[R_3+R_2]{R_3+R_1 \;\; -2R_1+R_2} \begin{bmatrix} 0 & 4 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & -2 \end{bmatrix}$, then C is row equivalent to a matrix with zero row. Hence C is not invertible. □

Exercise 0.1.24 Let $D = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}$. Is D invertible? If yes, find D^{-1}. If no, explain why C is not invertible.

Solution: $[D|I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & -2 & 1 & 0 \\ 0 & 3 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[-R_2+R_3]{\frac{1}{3}R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right] \xrightarrow[2R_3+R_1]{R_2+R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/3 & -5/3 & 2 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/3 & -5/3 & 2 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right] = [I|D^{-1}].$$

Hence $D^{-1} = \begin{bmatrix} 7/3 & -5/3 & 2 \\ -2/3 & 1/3 & 0 \\ -1 & 1 & -1 \end{bmatrix}$. □

0.2 Systems of linear equations

Exercise 0.2.1 Find the general solution $[x_1\ x_2\ x_3\ x_4\ x_5]^T$ of the following system

$$\begin{array}{ccccccccccc} x_1 & + & 2x_2 & & & - & 3x_4 & + & x_5 & = & 2 \\ & & & & x_3 & + & 4x_4 & - & 2x_5 & = & -1 \\ x_1 & + & 2x_2 & + & x_3 & + & x_4 & - & x_5 & = & 1 \end{array}$$

Solution:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -3 & 1 & 2 \\ 0 & 0 & 1 & 4 & -2 & -1 \\ 1 & 2 & 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{-R_1+R_3} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -3 & 1 & 2 \\ 0 & 0 & 1 & 4 & -2 & -1 \\ 0 & 0 & 1 & 4 & -2 & -1 \end{array}\right] \xrightarrow{-R_2+R_3}$$

$$\rightarrow \left[\begin{array}{ccccc|c} \underline{1} & 2 & 0 & -3 & 1 & 2 \\ 0 & 0 & \underline{1} & 4 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The system is consistent; the variables x_2, x_4, x_5 are free. Find the fundamental solutions of the system.

1) $x_2 = 1$, $x_4 = 0$, $x_5 = 0$. Then $\left\{ \begin{array}{lcl} x_1 + 2 & = & 0 \\ x_3 & = & 0 \end{array} \right.$ and $X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

2) $x_2 = 0$, $x_4 = 1$, $x_5 = 0$. Then $\left\{ \begin{array}{lcl} x_1 - 3 & = & 0 \\ x_3 + 4 & = & 0 \end{array} \right.$ and $X_2 = \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$.

3) $x_2 = x_4 = 0$, $x_5 = 1$. Then $\left\{ \begin{array}{lcl} x_1 + 1 & = & 0 \\ x_3 - 2 & = & 0 \end{array} \right.$ and $X_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$.

Find a partial solution of the system.

$x_2 = x_4 = x_5 = 0$. Then $\begin{cases} x_1 &= 2 \\ x_3 &= -1 \end{cases}$ and $V = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$.

Thus the general solution is $X = V + x_2 \cdot X_1 + x_4 \cdot X_2 + x_5 \cdot X_3$ □

Exercise 0.2.2 Find the value(s) of t for which $[t\ 0\ -1\ 0\ 0]^T$ is a solution of the system in 0.2.1.

Solution:

$$\begin{bmatrix} t \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_5 \cdot \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{cases} -2x_2 + 3x_4 - x_5 = t-2 \\ x_2 = 0 \\ -4x_4 + 2x_5 = 0 \\ x_4 = 0 \\ x_5 = 0 \end{cases} \Rightarrow \quad t-2=0 \quad \Rightarrow \quad t=2. \quad \square$$

Exercise 0.2.3 Find x, y, and z (if exist) for which

$$x \cdot \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + y \cdot \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + z \cdot \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & 0 \end{bmatrix}.$$

Solution: The correspondent system of linear equations is the following:

$$\begin{cases} x + y + z = 2 \\ 2x + 2y + 3z = 5 \\ x + z = -3 \end{cases}$$

Solve it:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 5 \\ 1 & 0 & 1 & -3 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -5 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

The system is consistent. It is equivalent to $\begin{cases} x + y + z = 2 \\ -y = -5 \\ z = 1 \end{cases}$.

So $x = -4$, $y = 5$, $z = 1$ □

Exercise 0.2.4 Find the value(s) of t for which the following matrix equation has no solution.

$$x \cdot \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} + y \cdot \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + z \cdot \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & t \end{bmatrix}.$$

Solution: The correspondent system of linear equations is the following:

$$\begin{cases} x + y + z = 2 \\ 2x + 2y + 3z = 5 \\ x + z = -3 \\ 4x + 3y + 5z = t \end{cases}$$

Solve it:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 5 \\ 1 & 0 & 1 & -3 \\ 4 & 3 & 5 & t \end{array}\right] \xrightarrow{\text{see } 0.2.3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & -5 \\ 0 & 0 & 1 & 1 \\ 4 & 3 & 5 & t \end{array}\right] \xrightarrow[-R_2]{-4R_1+R_4} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & t-8 \end{array}\right]$$

$$\xrightarrow{R_2+R_4} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & t-3 \end{array}\right] \xrightarrow{-R_3+R_4} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & t-4 \end{array}\right].$$

Consequently, the system is inconsistent if $t \neq 4$, thus our equation has no solution if $t \neq 4$. □

Exercise 0.2.5 a) Find the value(s) of r such that the following system of linear equations

$$\begin{cases} 2x + 3y + 7z + 11t = 1 \\ x + 2y + 4z + 7t = 2r \\ 5x + 10z + 5t = r - 1 \end{cases}$$

is consistent.

b) Find fundamental solutions of the following homogeneous system and write down the general solution in terms of them.

$$\begin{cases} 2x + 3y + 7z + 11t = 0 \\ x + 2y + 4z + 7t = 0 \\ 5x + 10z + 5t = 0 \end{cases}.$$

Solution: a) $\left[\begin{array}{cccc|c} 2 & 3 & 7 & 11 & 1 \\ 1 & 2 & 4 & 7 & 2r \\ 5 & 0 & 10 & 5 & r-1 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_2]{-5R_2+R_3;-2R_2+R_1}$

$\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 4 & 7 & 2r \\ 0 & -1 & -1 & -3 & 1-4r \\ 0 & -10 & -10 & -30 & -1-9r \end{array}\right] \xrightarrow[-R_2]{-10R_2+R_3} \left[\begin{array}{cccc|c} 1 & 2 & 4 & 7 & 2r \\ 0 & 1 & 1 & 3 & 4r-1 \\ 0 & 0 & 0 & 0 & 31r-11 \end{array}\right].$

Hence it is consistent iff $31r - 11 = 0$ or $r = \frac{11}{31}$.

b) $\begin{bmatrix} 2 & 3 & 7 & 11 \\ 1 & 2 & 4 & 7 \\ 5 & 0 & 10 & 5 \end{bmatrix} \rightarrow \ldots \rightarrow \begin{bmatrix} \underline{1} & 2 & 4 & 7 \\ 0 & \underline{1} & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

So, the variables z and t are free. Find the fundamental solutions of the system.

$$\begin{cases} x + 2y + 4z + 7t = 0 \\ y + z + 3t = 0 \end{cases}.$$

1) $z = 1$, $t = 0$. Then $X_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

2) $z = 0$, $t = 1$. Then $X_2 = \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

The general solution is $X = z \cdot X_1 + t \cdot X_2 = \begin{bmatrix} -2z - t \\ -z - 3t \\ z \\ t \end{bmatrix}$. □

Exercise 0.2.6 Let $A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & -2 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Consider the homogeneous system $AX = 0$. Find for the system:

a) Free variable(s) and basic variable(s).

b) Fundamental solution(s).

c) The general solution.

d) Is the system $AX = B$ consistent for $B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$?

Solution: a) $\begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & -2 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} \underline{1} & 0 & -2 & 3 \\ 0 & \underline{1} & -1 & 1 \\ 0 & 0 & \underline{1} & 0 \end{bmatrix}$.

So, the variables x_1, x_2, and x_3 are basic. The variable x_4 is free.

b) Put $x_4 = 1$. We have $\begin{cases} x_1 - 2x_3 + 3x_4 &= 0 \\ x_2 - x_3 + x_4 &= 0 \\ x_3 &= 0 \end{cases} \Rightarrow \begin{cases} x_1 + 3 &= 0 \\ x_2 + 1 &= 0 \\ x_3 &= 0 \end{cases}$.

The fundamental solution is $X_1 = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

c) $X = x_4 \cdot X_1 = \begin{bmatrix} -3x_4 \\ -x_4 \\ 0 \\ x_4 \end{bmatrix}$.

d) $\left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 2 \\ 1 & 0 & -2 & 3 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{cccc|c} \underline{1} & 0 & -2 & 3 & 1 \\ 0 & \underline{1} & -1 & 1 & 2 \\ 0 & 0 & \underline{1} & 0 & -2 \end{array}\right]$.

So the system is consistent. □

Exercise 0.2.7 Find fundamental solutions of the following homogeneous system

$$\begin{cases} x_1 & - & x_2 & - & x_3 & - & x_4 & + & x_5 & = & 0 \\ 4x_1 & - & 4x_2 & - & x_3 & - & 9x_4 & + & 6x_5 & = & 0 \\ & & & & 3x_3 & - & 5x_4 & + & 2x_5 & = & 0 \\ & & x_2 & + & x_3 & & & + & x_5 & = & 0 \end{cases}.$$

Solution:

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 & 1 \\ 4 & -4 & -1 & -9 & 6 \\ 0 & 0 & 3 & -5 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{-4R_1+R_2} \begin{bmatrix} 1 & -1 & -1 & -1 & 1 \\ 0 & 0 & 3 & -5 & 2 \\ 0 & 0 & 3 & -5 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & -5 & 2 \\ 0 & 0 & 3 & -5 & 2 \end{bmatrix} \xrightarrow{-R_3+R_4} \begin{bmatrix} \underline{1} & -1 & -1 & -1 & 1 \\ 0 & \underline{1} & 1 & 0 & 1 \\ 0 & 0 & \underline{3} & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \tilde{A}.$$

So, the variables x_4 and x_5 are free. Fundamental solutions are:

1) Solutions of $\tilde{A}X = 0$ corresponding to $x_4 = 1$, $x_5 = 0$.

$$\begin{cases} x_1 - x_2 - x_3 & = & 1 \\ x_2 + x_3 & = & 0 \\ 3x_3 & = & 5 \end{cases} \text{ and } F_1 = \begin{bmatrix} 1 \\ -5/3 \\ 5/3 \\ 1 \\ 0 \end{bmatrix}.$$

2) Solutions of $\tilde{A}X = 0$ corresponding to $x_4 = 0$, $x_5 = 1$.

$$\begin{cases} x_1 - x_2 - x_3 & = & -1 \\ x_2 + x_3 & = & -1 \\ 3x_3 & = & -2 \end{cases} \text{ and } F_2 = \begin{bmatrix} -2 \\ -1/3 \\ -2/3 \\ 0 \\ 1 \end{bmatrix}. \ \square$$

Exercise 0.2.8 Find the relations satisfied by a and b if the system $AX = B$ is consistent, where A is the coefficient matrix from 0.2.7 and $B = \begin{bmatrix} a \\ b \\ 2 \\ 3 \end{bmatrix}$.

Solution: Augmented matrix is

$$[A|B] = \left[\begin{array}{ccccc|c} 1 & -1 & -1 & -1 & 1 & a \\ 4 & -4 & -1 & -9 & 6 & b \\ 0 & 0 & 3 & -5 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{see sol. 0.2.7}}$$

$$\to \left[\begin{array}{ccccc|c} 1 & -1 & -1 & -1 & 1 & a \\ 0 & 1 & 1 & 0 & 1 & 3 \\ 0 & 0 & 3 & -5 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & b-4a-2 \end{array}\right] = [\tilde{A}|\tilde{B}].$$

So $AX = B$ is equivalent to $\tilde{A}X = \tilde{B}$. Hence $AX = B$ is consistent iff $b-4a-2 = 0$. □

Exercise 0.2.9 Write the general solution of the system in 0.2.8 for $a = 1$ (in terms of fundamental solutions).

Solution: If $a = 1$ then $b - 4 \cdot 1 - 2 = 0$ and hence $b = 6$. The general solution of $AX = B$ is the same as the general solution of $\tilde{A}X = \tilde{B}$ that is $X = x_4 \cdot F_1 + x_5 \cdot F_2 + X'$, where X' is any partial solution of $\tilde{A}X = \tilde{B}$. To find X', one may take $x_4 = 0$ and $x_5 = 0$ in $\tilde{A}X = \tilde{B}$:

$$\begin{cases} x_1 - x_2 - x_3 &= 1 \\ x_2 + x_3 &= 3 \\ 3x_3 &= 2 \end{cases}. \text{ Hence } X' = \begin{bmatrix} 4 \\ 7/3 \\ 2/3 \\ 0 \\ 0 \end{bmatrix} \text{ and}$$

$$X = x_4 \cdot \begin{bmatrix} 1 \\ -5/3 \\ 5/3 \\ 1 \\ 0 \end{bmatrix} + x_5 \cdot \begin{bmatrix} -2 \\ -1/3 \\ -2/3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 7/3 \\ 2/3 \\ 0 \\ 0 \end{bmatrix}. \quad \square$$

Exercise 0.2.10 Given a system

$$\begin{cases} x + 3y - 2z = 1 \\ -x - 5y + 3z = -1 \\ 2x - 8y + 3z = \alpha \end{cases}.$$

a) Determine the value(s) (if exist) of α which makes the system consistent.

b) Find fundamental solutions of the corresponding homogeneous system.

c) Write down a general solution for those α when the system is consistent to the given non-homogeneous system in terms of fundamental solutions that you found in b).

Solution: a) $\left[\begin{array}{rrr|r} 1 & 3 & -2 & 1 \\ -1 & -5 & 3 & -1 \\ 2 & -8 & 3 & \alpha \end{array}\right] \xrightarrow[-2R_1+R_3]{R_1+R_2} \left[\begin{array}{rrr|r} 1 & 3 & -2 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & -14 & 7 & \alpha-2 \end{array}\right] \rightarrow$

$\xrightarrow{-7R_2+R_3} \left[\begin{array}{rrr|r} 1 & 3 & -2 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & \alpha-2 \end{array}\right].$

So only $\alpha = 2$ makes the system consistent.

b) The free variable is z. Put $z = 1$. We have $\left\{\begin{array}{rcl} x + 3y - 2z & = & 0 \\ -2y + z & = & 0 \end{array}\right.$.

$-2y + 1 = 0 \Rightarrow y = 1/2$; $x + 3 \cdot 1/2 - 2 = 0 \Rightarrow x = 1/2$.

Hence there is only one fundamental solution for $z = 1$: $F = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$.

c) A general solution is $X = z \cdot F + X_p$, where X_p is a partial solution of the system ($\alpha = 2$).

$\left\{\begin{array}{rcl} x + 3y - 2z & = & 1 \\ -2y + z & = & 0 \end{array}\right.$.

For X_p take $z = 0$ then $X_p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Thus a general solution is $X = z \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. □

Exercise 0.2.11 Find the conditions on a, b, c, and d for which the matrix system

$$x_1 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 7 & 7 \\ -3 & -3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has

a) no solution;

b) infinitely many solutions.

c) Find the general solution of the equation for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution: The augmented matrix is

$$\left[\begin{array}{rrrr|r} 1 & 1 & 3 & 7 & a \\ 1 & 1 & 3 & 7 & b \\ 1 & -1 & 1 & -3 & c \\ 1 & -1 & 1 & -3 & d \end{array}\right] \xrightarrow[-R_3+R_4]{-R_1+R_2} \left[\begin{array}{rrrr|r} 1 & 1 & 3 & 7 & a \\ 0 & 0 & 0 & 0 & b-a \\ 1 & -1 & 1 & -3 & c \\ 0 & 0 & 0 & 0 & d-c \end{array}\right] \xrightarrow{-R_1+R_3}$$

$$\to \left[\begin{array}{rrrr|r} 1 & 1 & 3 & 7 & a \\ 0 & 0 & 0 & 0 & b-a \\ 0 & -2 & -2 & -10 & c-a \\ 0 & 0 & 0 & 0 & d-c \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{-\frac{1}{2}R_3} \left[\begin{array}{rrrr|r} 1 & 1 & 3 & 7 & a \\ 0 & 1 & 1 & 5 & a/2-c/2 \\ 0 & 0 & 0 & 0 & b-a \\ 0 & 0 & 0 & 0 & d-c \end{array}\right] \xrightarrow{-R_2+R_1}$$

$$\to \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 2 & a/2+c/2 \\ 0 & 1 & 1 & 5 & a/2-c/2 \\ 0 & 0 & 0 & 0 & b-a \\ 0 & 0 & 0 & 0 & d-c \end{array}\right].$$

a) If $b-a \neq 0$ or $d-c \neq 0$ the system has no solution.

b) If $b-a=0$ and $d-c=0$ the system has infinitely many solutions.

c) We have $b=a=c=d=1$. Then the augmented matrix of the system is equivalent to $\left[\begin{array}{rrrr|r} \underline{1} & 0 & 2 & 2 & a/2+c/2=1 \\ 0 & \underline{1} & 1 & 5 & a/2-c/2=0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

The fundamental solutions are solutions of the correspondent homogeneous system. $\begin{cases} x_1+2x_3+2x_4 &= 0 \\ x_2+x_3+5x_4 &= 0 \end{cases}$. So $x_1=-2x_3-2x_4$ and $x_2=-x_3-5x_4$.

For $x_3=0$ and $x_4=1$: $F_1 = \begin{bmatrix} -2 \\ -5 \\ 0 \\ 1 \end{bmatrix}$.

For $x_3 = 1$ and $x_4 = 0$: $F_2 = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

A partial solution find from $\begin{cases} x_1 + 2x_3 + 2x_4 & = & 1 \\ x_2 + x_3 + 5x_4 & = & 0 \end{cases}$ with $x_3 = x_4 = 0$: $V = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

And finally the general solution is $X = x_3 \cdot F_1 + x_4 \cdot F_2 + V$. □

Exercise 0.2.12 Find the values of a and b for which the system

$$\begin{cases} x + 2y - z + at = 1 \\ ay + (b+1)z = b \\ z + at = b \\ (a-1)t = b \end{cases}$$

has

i) No solution.

ii) A unique solution.

iii) Infinitely many solutions.

Solution: i) If $a = 0$, $b \neq 0$ then the system has **no solution** since

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & b+1 & 0 & b \\ 0 & 0 & 1 & 0 & b \\ 0 & 0 & 0 & -1 & b \end{array}\right] \xrightarrow{(b+1)R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & b+1 & 0 & b \\ 0 & 0 & b+1 & 0 & b(b+1) \\ 0 & 0 & 0 & -1 & b \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & b+1 & 0 & b \\ 0 & 0 & 0 & 0 & b^2 \neq 0 \\ 0 & 0 & 0 & -1 & b \end{array}\right].$$

If $a = 1$, $b \neq 0$ then the system has **no solution** since

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & b+1 & 0 & b \\ 0 & 0 & 1 & 1 & b \\ 0 & 0 & 0 & 0 & b \neq 0 \end{array}\right].$$

ii) If $a \neq 0$, $a \neq 1$, and b is arbitrary then the coefficient matrix $\begin{bmatrix} 1 & 2 & -1 & a \\ 0 & a & b+1 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & a-1 \end{bmatrix}$ is invertible. In this case our system has a **unique solution**.

iii) If $a = 0$, $b = 0$ then the system has **infinitely many solutions** since y is a free variable.

If $a = 1$, $b = 0$ then the system has **infinitely many solutions** since t is a free variable. □

Exercise 0.2.13 Find two different pairs (a, b) of values of a and b for which the homogeneous system

$$\begin{cases} x + 2y - z + (a+1)t + bu = 0 \\ ay + (b+1)z + 2au = 0 \\ z + bt + au = 0 \end{cases}$$

has three fundamental solutions. For one of such a pair (a, b) of numbers a and b find all three correspondent fundamental solutions.

Solution: To get more than 2 free variables, we must have $a = 0$. Moreover, then our system becomes:

$$\begin{cases} x + 2y - z + t + bu = 0 \\ (b+1)z = 0 \\ z + bt = 0 \end{cases}$$

Thus, in order to have 3 free variables, one must also take $b = 0$ or $b = -1$. So the required pairs are $(a, b) = (0, 0)$ and $(a, b) = (0, -1)$.

1) $a = 0$, $b = 0$.

$$[x\ y\ z\ t\ u] : \begin{bmatrix} \underline{1} & 2 & -1 & 1 & 0 \\ 0 & 0 & \underline{1} & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x + 2y - z + t = 0 \\ z = 0 \end{cases};$$

y, t, and u are free.

$$y=1,\ t=0,\ u=0,\ F_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

$$y=0,\ t=1,\ u=0,\ F_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix};$$

$$y=0,\ t=0,\ u=1,\ F_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

2) $a = 0$, $b = -1$.

$$[x\ y\ z\ t\ u] : \begin{bmatrix} \underline{1} & 2 & -1 & 1 & -1 \\ 0 & 0 & \underline{1} & -1 & 0 \end{bmatrix}$$

$$\begin{cases} x + 2y - z + t - u = 0 \\ z - t = 0 \end{cases};$$

y, t, and u are free.

$$y=1,\ t=0,\ u=0,\ G_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

$$y=0,\ t=1,\ u=0,\ G_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix};$$

$$y = 0,\ t = 0,\ u = 1,\ G_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \ \square$$

Exercise 0.2.14 a) Determine whether the system $Ax = B$ is consistent,

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + 5x_5 = 0 \\ -5x_1 - 10x_2 + 3x_3 + 3x_4 + 55x_5 = -8 \\ x_1 + 2x_2 + 2x_3 - 3x_4 - 5x_5 = 14 \\ -x_1 - 2x_2 + x_3 + x_4 + 15x_5 = -2 \end{cases}$$

b) If it is consistent, find the general solution of the form $x_h + x_p$, where x_h is the solution of $Ax = 0$ and x_p is the solution of $Ax = B$. What is the dimension of the solution space of the system $Ax = 0$, please, explain.

Solution: a) Take the coefficient matrix.

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & 5 & 0 \\ -5 & -10 & 3 & 3 & 55 & -8 \\ 1 & 2 & 2 & -3 & -5 & 14 \\ -1 & -2 & 1 & 1 & 15 & -2 \end{array}\right] \xrightarrow[5R_1+R_2]{-R_1+R_3;\ R_1+R_4} \left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & 5 & 0 \\ 0 & 0 & 8 & 8 & 80 & -8 \\ 0 & 0 & 1 & -4 & -10 & 14 \\ 0 & 0 & 2 & 2 & 20 & -2 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{4}R_2+R_4]{-\frac{1}{8}R_2+R_3;\ \frac{1}{8}R_2} \left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & 5 & 0 \\ 0 & 0 & 1 & 1 & 10 & -1 \\ 0 & 0 & 0 & -5 & -20 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-\frac{1}{5}R_3]{R_3+R_2;\ -R_2+R_1}$$

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 & 6 & 2 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_3+R_1} \left[\begin{array}{rrrrr|r} \underline{1} & 2 & 0 & 0 & -5 & 1 \\ 0 & 0 & \underline{1} & 0 & 6 & 2 \\ 0 & 0 & 0 & \underline{1} & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So the system is consistent and, since x_2 and x_5 are free variables, there are infinitely many solutions.

b) From (a) we have the system $Ax = 0$ is equivalent to that with the coefficient matrix $\left[\begin{array}{ccccc|c} \underline{1} & 2 & 0 & 0 & -5 & 0 \\ 0 & 0 & \underline{1} & 0 & 6 & 0 \\ 0 & 0 & 0 & \underline{1} & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$. So we have the solution

$$x_h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5x_5 - 2x_2 \\ x_2 \\ -6x_5 \\ -4x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 5 \\ 0 \\ -6 \\ -4 \\ 1 \end{bmatrix}.$$

The dimension of the solution space is 2.

Find x_p from $\left[\begin{array}{ccccc|c} \underline{1} & 2 & 0 & 0 & -5 & 1 \\ 0 & 0 & \underline{1} & 0 & 6 & 2 \\ 0 & 0 & 0 & \underline{1} & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

$$x_h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 + 5x_5 - 2x_2 \\ x_2 \\ 2 - 6x_5 \\ -3 - 4x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 5 \\ 0 \\ -6 \\ -4 \\ 1 \end{bmatrix}. \quad \square$$

0.3 Determinants

Exercise 0.3.1 Evaluate the following determinants.

$$A = \begin{vmatrix} 1 & 2 & 0 & 4 & 2 \\ 2 & 3 & 4 & 5 & 6 \\ 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 1 & 2 & 0 & 4 & 1 \end{vmatrix} \quad \text{and} \quad B = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 0 & 4 & 0 \\ 5 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 4 \end{vmatrix}.$$

Solution: $A = (-1)^{2+3} \cdot 4 \cdot \begin{vmatrix} 1 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 2 & 4 & 1 \end{vmatrix} = (-4) \cdot 3 \cdot \begin{vmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix} =$

$= (-12) \cdot \left(2 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}\right) = (-12) \cdot (2 \cdot (-1) - 1 \cdot 0) = 24.$

$B = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 5 & 4 & 3 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 3 \\ 5 & 3 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2 \cdot (-12) \cdot (-2) = 48.$ □

Exercise 0.3.2 Compute the adjoint (adjugate) of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 4 & 3 & 0 \end{bmatrix}$.

1-st solution: $a_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} = -9$; $a_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 0 & 3 \\ 4 & 0 \end{vmatrix} = 12$; $a_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 0 & 2 \\ 4 & 3 \end{vmatrix} = -8$; $a_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} = 3$; $a_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} = -4$; $a_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = 1$; $a_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$; $a_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = -3$; $a_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$.

$$A_0 = \begin{bmatrix} -9 & 12 & -8 \\ 3 & -4 & 1 \\ 1 & -3 & 2 \end{bmatrix}; \qquad \text{adj}(A) = A_0^T = \begin{bmatrix} -9 & 3 & 1 \\ 12 & -4 & -3 \\ -8 & 1 & 2 \end{bmatrix}.$$

2-nd solution: $adj(A) = |A| \cdot A^{-1}$, $|A| = 2 \cdot \begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} - 3 \cdot \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = -8+3 = -5.$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-4R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & -1 & -4 & -4 & 0 & 1 \end{array}\right] \xrightarrow[-R_2]{R_2 \leftrightarrow R_3}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 4 & 0 & -1 \\ 0 & 2 & 3 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 4 & 0 & -1 \\ 0 & 0 & -5 & -8 & 1 & 2 \end{array}\right] \xrightarrow{\frac{4}{5}R_3+R_2}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -12/5 & 4/5 & 3/5 \\ 0 & 0 & -5 & -8 & 1 & 2 \end{array}\right] \xrightarrow[-R_2+R_1]{\frac{1}{5}R_3+R_1}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9/5 & -3/5 & -1/5 \\ 0 & 1 & 0 & -12/5 & 4/5 & 3/5 \\ 0 & 0 & -5 & -8 & 1 & 2 \end{array}\right] \xrightarrow{-\frac{1}{5}R_3}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9/5 & -3/5 & -1/5 \\ 0 & 1 & 0 & -12/5 & 4/5 & 3/5 \\ 0 & 0 & 1 & 8/5 & -1/5 & -2/5 \end{array}\right] \Rightarrow A^{-1} = \frac{1}{5}\begin{bmatrix} 9 & -3 & -1 \\ -12 & 4 & 3 \\ 8 & -1 & -2 \end{bmatrix}$$

$$\Rightarrow adj(A) = |A| \cdot A^{-1} = (-5) \cdot \frac{1}{5}\begin{bmatrix} -9 & 3 & 1 \\ 12 & -4 & -3 \\ -8 & 1 & 2 \end{bmatrix}. \square$$

Exercise 0.3.3 Given that $\begin{vmatrix} 1 & a & 1 \\ b & 1 & b \\ a & 2 & b \end{vmatrix} = 9$, compute the following determinants

$$A = \begin{vmatrix} a+1 & a+2 & b+1 \\ b & 1 & b \\ 2a & 4 & 2b \end{vmatrix} \quad \text{and} \quad B = \begin{vmatrix} x^2 & ax^3 & x^4 \\ bx & x^2 & bx^3 \\ ax & 2x^2 & bx^3 \end{vmatrix}.$$

Solution: $A = 2 \cdot \begin{vmatrix} a+1 & a+2 & b+1 \\ b & 1 & b \\ a & 2 & b \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & a & 1 \\ b & 1 & b \\ a & 2 & b \end{vmatrix} = 2 \cdot 9 = 18.$

$B = x \cdot x^2 \cdot x^3 \cdot \begin{vmatrix} x & ax & x \\ b & 1 & b \\ a & 2 & b \end{vmatrix} = x^6 \cdot x \cdot \begin{vmatrix} 1 & a & 1 \\ b & 1 & b \\ a & 2 & b \end{vmatrix} = 9x^7. \square$

Exercise 0.3.4 Compute the following determinant $\begin{vmatrix} 1 & 2 & -3 & 4 & 5 & 6 \\ 2 & 3 & 0 & 7 & -8 & 9 \\ 3 & -2 & 1 & 10 & 9 & 8 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 2 & 5 & 5 \end{vmatrix}$.

Solution: $\left|\begin{array}{ccc|ccc} 1 & 2 & -3 & 4 & 5 & 6 \\ 2 & 3 & 0 & 7 & -8 & 9 \\ 3 & -2 & 1 & 10 & 9 & 8 \\ \hline 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 2 & 5 & 5 \end{array}\right| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 0 \\ 3 & -2 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 & 5 & 5 \\ 0 & 3 & 5 \\ 0 & 0 & 7 \end{vmatrix} \cdot (-1) =$

$= (-42) \cdot \left((-3) \cdot \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \right) = (-42) \cdot ((-3) \cdot (-13) + (-1)) =$ $-42 \cdot 38 = -1596.$ □

Exercise 0.3.5 Use Cramer's rule to solve for z: $\begin{cases} 2x - y - z = 0 \\ 2x - y + 4z = -1 \\ -x + 2y + z = 2 \end{cases}$.

Solution: The augmented matrix is $\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 2 & -1 & 4 & -1 \\ -1 & 2 & 1 & 2 \end{array}\right]$.

$$z = \frac{\begin{vmatrix} 2 & -1 & 0 \\ 2 & -1 & -1 \\ -1 & 2 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & -1 \\ 2 & -1 & 4 \\ -1 & 2 & 1 \end{vmatrix}} = \frac{2 \cdot \begin{vmatrix} -1 & -1 \\ 2 & 2 \end{vmatrix} + (-1)(-1) \cdot \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}}{2 \cdot \begin{vmatrix} -1 & 4 \\ 2 & 1 \end{vmatrix} + (-1) \cdot (-1) \cdot \begin{vmatrix} 2 & 4 \\ -1 & 1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}} =$$

$$= \frac{3}{3 \cdot (-9) + 6 - 3} = \frac{3}{-15} = -\frac{1}{5}. \quad \square$$

Exercise 0.3.6 Given that $\begin{vmatrix} a & b & c & d \\ b & c & 0 & b \\ c & 0 & b & c \\ d & b & a & d \end{vmatrix} = 6$, compute the determinant

$$D = \begin{vmatrix} a+c & 3bx & c & d \\ d+a & 3bx & a & d \\ c+b & 0 & b & c \\ bx^2 & 3cx^3 & 0 & bx^2 \end{vmatrix}.$$

Solution: $D = \begin{vmatrix} a & 3bx & c & d \\ d & 3bx & a & d \\ c & 0 & b & c \\ bx^2 & 3cx^3 & 0 & bx^2 \end{vmatrix} = 3x \cdot \begin{vmatrix} a & b & c & d \\ d & b & a & d \\ c & 0 & b & c \\ bx^2 & 3cx^3 & 0 & bx^2 \end{vmatrix} = 3x^3 \cdot \begin{vmatrix} a & b & c & d \\ d & b & a & d \\ c & 0 & b & c \\ b & c & 0 & b \end{vmatrix} = [R_2 \leftrightarrow R_4] = -3x^3 \begin{vmatrix} a & b & c & d \\ b & c & 0 & b \\ c & 0 & b & c \\ d & b & a & d \end{vmatrix} = -3x^3 \cdot 6 = -18x^3.$ □

Exercise 0.3.7 Compute the determinants of the following matrices

$$A = \begin{bmatrix} 2 & 1 & 2 & -1 & 1 & 2 \\ 1 & 2 & 1 & 1 & -1 & 3 \\ 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 7 & 5 & 2 & -1 \\ 0 & 0 & 3 & 2 & 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 3 & 2 & 4 \\ 2 & -1 & 4 & 7 & 2 & 1 \\ 0 & 1 & 2 & -5 & 6 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix}.$$

Solution:

$$|A| = \left|\begin{array}{cc|cc|cc} 2 & 1 & 2 & -1 & 1 & 2 \\ 1 & 2 & 1 & 1 & -1 & 3 \\ \hline 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ \hline 0 & 0 & 7 & 5 & 2 & -1 \\ 0 & 0 & 3 & 2 & 1 & 2 \end{array}\right| = |A| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} \cdot \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 3 \cdot 5 \cdot 5 = 75.$$

$$B \xrightarrow[R_1 \leftrightarrow R_4]{R_2 \leftrightarrow R_5; R_3 \leftrightarrow R_6} \begin{bmatrix} 2 & -1 & 4 & 7 & 2 & 1 \\ 0 & 1 & 2 & -5 & 6 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 3 & 2 & 4 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_6} \begin{bmatrix} 2 & -1 & 4 & 7 & 2 & 1 \\ 0 & 1 & 2 & -5 & 6 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 4 \\ 0 & 0 & 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} =$$

C.

$|B| = (-1)^4 \cdot |C| = |C| = 2 \cdot 1 \cdot 3 \cdot 3 \cdot (-2) \cdot 2 = -72.$ □

Exercise 0.3.8 Let C be a 11×11 skew-symmetric matrix. Find $\det(C)$.

Solution: Since C is a skew-symmetric, $C = -C^T$ and $\det(C) = (-1)^{11} \cdot \det(C^T) = -\det(C)$. Hence $2\det(C) = 0$, so $\det(C) = 0$. □

Exercise 0.3.9 Let A be a 3×3 matrix and let B be obtained from A by applying the following elementary row operations: $2R_1 + R_2$, $2R_2$, $-R_2 + R_3$, and $R_1 \leftrightarrow R_3$. Evaluate the followings if $B = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$:

a) $\det(A)$.

b) $\det(3A^{-1}B^T)$.

c) $\operatorname{adj}(B)$.

d) Express the matrix $\operatorname{adj}(2A)$ in terms of A^{-1}.

Solution: a) $B = \mathcal{E}_4 \cdot \mathcal{E}_3 \cdot \mathcal{E}_2 \cdot \mathcal{E}_1 \cdot A$, where $\mathcal{E}_1 = 2R_1 + R_2$, $\mathcal{E}_2 = 2R_2$, $\mathcal{E}_3 = -R_2 + R_3$, and $\mathcal{E}_4 = R_1 \leftrightarrow R_3$. Thus $\det(B) = -2\det(A)$.

$\det(B) = \begin{vmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = (-1) \cdot (-2) \cdot 2 = 4 = -2\det(A)$ hence $\det(A) = -2$.

b) $\det(3A^{-1}B^T) = 3^3 \det(A^{-1}) \det(B^T) = 27 \cdot (\det(A))^{-1} \cdot \det(B) = 27 \cdot \frac{1}{-2} \cdot 4 = -54$.

c) $b_{11} = (-1)^{1+1} \cdot \begin{vmatrix} -2 & 0 \\ 1 & 2 \end{vmatrix} = -4$; $b_{12} = (-1)^{1+2} \cdot \begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} = 2$; $b_{13} = (-1)^{1+3} \cdot \begin{vmatrix} -1 & -2 \\ 0 & 1 \end{vmatrix} = -1$; $b_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} = 0$; $b_{22} = (-1)^{2+2} \cdot \begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} = -2$; $b_{23} = (-1)^{2+3} \cdot \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1$; $b_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 0 & 0 \\ -2 & 0 \end{vmatrix} = 0$; $b_{32} = (-1)^{3+2} \cdot \begin{vmatrix} -1 & 0 \\ -1 & 0 \end{vmatrix} = 0$; $b_{33} = (-1)^{3+3} \cdot \begin{vmatrix} -1 & 0 \\ -1 & -2 \end{vmatrix} = 2$.

So the cofactor matrix of B is $\begin{bmatrix} -4 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Thus

$$\operatorname{adj}(B) = \begin{bmatrix} -4 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} -4 & 0 & 0 \\ 2 & -2 & 0 \\ -1 & 1 & 2 \end{bmatrix}.$$

d) By using the fact $A \cdot \operatorname{adj}(A) = \det(A) \cdot I$ for every square matrix, we have $2A \cdot \operatorname{adj}(2A) = \det(2A) \cdot I \Rightarrow \operatorname{adj}(2A) = 1/2 \cdot 2^3 \cdot \det(A) \cdot A^{-1} \cdot I$. Thus

$$\operatorname{adj}(2A) = -8 \cdot A^{-1}. \quad \square$$

Exercise 0.3.10 Consider the following list of statements. In each case either prove the statement if it is true or give an example showing that it is false.

i) If $\det(A) = 0$ then A has two equal rows.

ii) If R is the row reduced echelon form of A then $\det(R) = \det(A)$.

iii) $\det(A^T) = -\det(A)$.

iv) If $\det(A) = \det(B)$ and matrices A and B have the same size, then $A = B$.

v) If $\det(A) \neq 0$ and $AB = AC$ then $B = C$.

vi) $\det(I + A) = 1 + \det(A)$.

vii) If $\det(A) = 1$ then $\operatorname{adj}(A) = A$.

viii) There is no invertible 17×17 skew-symmetric matrix.

Solution: i) False. For example, $A = \begin{bmatrix} 1 & 1/2 \\ 4 & 2 \end{bmatrix}$ has no equal rows, but $\det(A) = 0$.

ii) False. An example: $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 3/2 \\ 0 & 5/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R$. We see that $\det(A) = 5$ but $\det(R) = 1$.

iii) False. An example: $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, but $\det(A^T) = \det(A) = 2$.

iv) False. Consider $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We have $\det(A) = \det(B) = 1$. but $A \neq B$.

v) True. If $\det(A) \neq 0$ then A is invertible, i.e. there exists A^{-1}. So if $AB = AC$, we can write $A^{-1}(AB) = A^{-1}(AC)$ that is equal to $IB = IC$ or $B = C$.

vi) False. Consider $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$.

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\right) = 6 \neq 3 = 1 + \det(A).$$

vii) False. An example: $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \neq \operatorname{adj}(A) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

viii) True. Let A is 17×17 skew-symmetric matrix. Then $A^T = -A$ and $\det(A) = \det(A^T) = \det(-A) = (-1)^{17} \cdot \det(A) = -\det(A)$. So $\det(A) = -\det(A)$ hence $\det(A) = 0$, that means A is not invertible. $\square$

Exercise 0.3.11 Calculate the determinant of A where

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 3 \\ 3 & 1 & 0 & 13 & 6 & 8 \\ 0 & 2 & -1 & 21 & 0 & 7 \\ 1 & 1 & 2 & 17 & -5 & 3 \end{bmatrix}.$$

Solution: $A \xrightarrow[R_3 \leftrightarrow R_4]{R_1 \leftrightarrow R_6; R_2 \leftrightarrow R_5} \begin{bmatrix} 1 & 1 & 2 & 17 & -5 & 3 \\ 0 & 2 & -1 & 21 & 0 & 7 \\ 3 & 1 & 0 & 13 & 6 & 8 \\ 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \end{bmatrix}$. Hence

$$|A| = (-1)^3 \cdot \left|\begin{array}{ccc|ccc} 1 & 1 & 2 & 17 & -5 & 3 \\ 0 & 2 & -1 & 21 & 0 & 7 \\ 3 & 1 & 0 & 13 & 6 & 8 \\ \hline 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \end{array}\right| = (-1) \cdot \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 3 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & 2 & 3 \\ 2 & -1 & 1 \\ 2 & 1 & 0 \end{vmatrix} =$$

$$= (-1) \cdot \left(1 \cdot \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}\right) \cdot \left((-2) \cdot \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix}\right) =$$

$$= (-1) \cdot (1 + 3 \cdot (-5)) \cdot (6 + 2 \cdot 5) = (-1) \cdot (-14) \cdot 16 = 224. \ \square$$

Exercise 0.3.12 Use Crammer's rule to solve for u: $\begin{cases} x + y &= 0 \\ y - u &= -1 \\ x + z &= 0 \\ x - y &= 1 \end{cases}.$

Solution: We have $A \cdot \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

$$|A| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} - \begin{vmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -2 \neq 0.$$

$$|A_u| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - \left((-1) \cdot \begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}\right) = 0 - (0 + 1) = -1.$$

$$u = \frac{|A_u|}{|A|} = \frac{-1}{-2} = 1/2. \quad \square$$