$u_7$-TOPOLOGIES IN LOCALLY SOLID RIESZ SPACES

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2. Introduction

- All Riesz spaces in the talk are supposed to be real and Archimedean.
- Let \( \mathcal{C} \) be a **convergence** on a Riesz space \( X \) which agrees with linear and lattice operations in \( X \) in the following sense:

\[
x_\alpha \equiv x \Rightarrow x_\alpha \xrightarrow{\mathcal{C}} x;
\]  
and

\[
X \ni x_\alpha \xrightarrow{\mathcal{C}} x, \ X \ni y_\alpha \xrightarrow{\mathcal{C}} y, \mathbb{R} \ni r_\alpha \rightarrow r
\]  
imply

\[
r_\alpha \cdot x_\alpha + y_\alpha \xrightarrow{\mathcal{C}} r \cdot x + y; \quad (2)
\]

\[
r_\alpha \cdot x_\alpha \wedge y_\alpha \xrightarrow{\mathcal{C}} r \cdot x \wedge y. \quad (3)
\]

- As basic examples of such a convergence, we keep in mind **\( \alpha \)-convergence**, **\( ru \)-convergence**, **\( \tau \)-convergence** (if \( X \) is a locally solid Riesz space with its locally solid topology \( \tau \)), and **\( m \)-convergence** (if \( X = (X, m) \) is a multi-normed Riesz space).
• We say that $x_\alpha \xrightarrow{uc} x$ in $X$ if $|x_\alpha - x| \wedge u \to 0$ for any $u \in X_+$, where "$u \to$" stands for unbounded $c$-convergence.

• It is immediate to see that $uc$-convergence is also agreed with linear and lattice operations in $X$, and $uuc$-convergence coincides with $uc$-convergence.

• $uo$-Convergence was first introduced for sequences on $\sigma$-Dedekind complete Riesz spaces by Nakano (1948) under the name individual convergence.

• Notice that, in $L_1[0,1]$, the $uo$-convergence of sequences is equivalent to the almost everywhere convergence.

• The name unbounded order convergence was first proposed by DeMarr (1964).

• Recently, various types of unbounded convergences have been studied intensively by many authors.
• *uo-Convergence* was investigated in papers [20, 17, 11, 12, 13, 14, 18] of Gao, Kaplan, Troitsky, Wickstead, and Xanthos.

• The name **unbounded order convergence** was first proposed by DeMarr (1964).

• Recently, various types of **unbounded convergences** have been studied intensively by many authors.

• *uo-Convergence* was investigated in papers [20, 17, 11, 12, 13, 14, 18] of Gao, Kaplan, Troitsky, Wickstead, and Xanthos.

• *un-Convergence*, where $n$ stands for **norm convergence**, was investigated by Chen, Deng, Kandić, Li, Marabeh, O’Brein, and Troitsky in [10, 16, 15].

• *um-Convergence*, where $m$ stands for **lattice multi-norm convergence**, was studied by Dabboorasad, Emelyanov, Marabeh, and Zabeti in [21, 8].
• **$u\tau$-Convergence** was studied by Dabboorasad, Emelyanov, Marabeh, Taylor, and Vural in [7, 19, 9].

• **$up$-Convergence**, where $p$ stands for lattice-valued norm, was contributed by Aydin, Emelyanov, Erkursun-Ozcan, and Marabeh in [3, 4].

• While some of convergences are **topological**, others are not.

• It was shown in [6, Thm.1], that $\alpha$-convergence in a locally solid Riesz space $X$ is topological iff $\dim(X) < \infty$.

• From the other hand, $uo$-convergence is always topological if $X$ is discrete [6, Thm.2].

• Notice also that $ru$-convergence is topological iff $X$ has a strong order unit [7, Thm.5].

• It seems to be interesting to develop the general theory of $uc$-convergence in **Riesz convergence spaces** in the sense of the monograph [5] of Beattie and Butzmann.
UC-convergence [unbounded Canadian convergence]

UFO, rare and dangerous animals, Beattie & Buzymann,

U.M.-convergence

no roads,
no census,
no safe camping

[U.M. convergence]

2500 m

UC-convergence

U.P.-convergence

restricted
camping,
no fuel/power
supply

1500 m

[camping area]

U.P.-convergence

U.M.-convergence

V.G. Troitsky
(2004)

convergences related to measure

convergence in measure

M. Nakane
(1948)

U.O.-convergence

500 m
• For example, it needs an investigation the question of finding conditions ensuring that \( uc \)-convergence is topological or even metrizable.

• Another natural question consists in finding conditions under which \( uc \)-convergence agrees with \( c \)-convergence.

• The main topic of this talk is related to the case when our \( c \)-convergence is already topological.

• However, in the end of the talk, we also touch several questions related to \( o \)-convergence, \( ru \)-convergence, \( m \)-convergence, and their unbounded versions in Riesz spaces, that could be not topological.
3. **Several results on $u\tau$-topology**

We begin with two equivalent definitions of $u\tau$-topology (cf. [7, 19, 9]). In the case, when $c$-convergence is topological, say $c$-convergence coincides with $\tau$-convergence produced by a locally solid topology $\tau$ on $X$, the uc-convergence is topological too, and a zero-base of the corresponding locally solid topology $u\tau$ can be taken as follows [7, 19]:

$$U_{w,V} = \{ x \in X : |x| \wedge w \in V \} \ (w \in X_+, V \in \tau(0)).$$

(4)

Notice also that, as the topology $\tau$ is generated by a family $R = \{\pi_i\}_{i \in I}$ of Riesz pseudo-semi-norms, the corresponding $u\tau$-topology can be equivalently defined [9] as the topology generated by the family $u(R) = \{\pi_{i,w}\}_{i \in I, w \in X_+}$ of the following Riesz pseudo-semi-norms:

$$\pi_{i,w}(x) := \pi_i(|x| \wedge w) \ (i \in I, w \in X_+, x \in X).$$

(5)

The following several results have been proven in [7].
The next theorem [7, Thm.1] should be compared with Lemma 2.1 in [16], where it was proved for Banach lattices.

**Theorem 1.** Let \((X, \tau)\) be a locally solid Riesz space, where \(\tau\) is generated by a family \((\pi_j)_{j \in J}\) of Riesz pseudo-semi-norms. Let \(j \in J\), \(w \in X_+\), and \(\varepsilon > 0\). Define a zero-neighborhood \(V_{j,w,\varepsilon} \in u\tau(0)\) as follows:

\[ V_{j,w,\varepsilon} := \{x \in X : \pi_j(|x| \wedge w) < \varepsilon\}. \]

Then, either \(V_{j,w,\varepsilon}\) is contained in \([-w, w]\), or \(V_{j,w,\varepsilon}\) contains a non-trivial ideal.

For the next result, we need to remind the following definition which is due to Schaefer.

**Definition 1.** An element \(0 \neq e \in X_+\) of a topological Riesz space \((X, \tau)\) is called a **quasi-interior point**, if the principal ideal \(I_e\) is \(\tau\)-dense in \(X\).

The next theorem [7, Thm. 2] extends Theorem 3.1 in [16], where it was proved for Banach lattices.
Theorem 2. Let $(X, \tau)$ be a sequentially complete locally solid Riesz space. Let $e \in X_+$. The following are equivalent:

(1) $e$ is a quasi-interior point;
(2) for every net $x_\alpha$ in $X_+$, if $x_\alpha \wedge e \xrightarrow{\tau} 0$ then $x_\alpha \xrightarrow{u\tau} 0$;
(3) for every sequence $x_n$ in $X_+$, if $x_n \wedge e \xrightarrow{\tau} 0$ then $x_n \xrightarrow{u\tau} 0$.

Now, we need the following generalization of quasi-interior points.

Definition 2. A pairwise disjoint system $Q = \{e_\gamma\}_{\gamma \in \Gamma}$ of non-zero positive elements of a topological Riesz space $(X, \tau)$ is said to be a topological orthogonal system, if the ideal $I_Q$ generated by $Q$ is $\tau$-dense in $X$.

The next theorem [7, Thm. 7] extends the above theorem to the case when our Riesz space $X$ is possessing just a topological orthogonal system.
Theorem 3. Let \((X, \tau)\) be a locally solid Riesz space, and \(Q = \{e_\gamma\}_{\gamma \in \Gamma}\) be a topological orthogonal system of \((X, \tau)\). Then \(x_\alpha \xrightarrow{u\tau} 0 \iff |x_\alpha| \wedge e_\gamma \xrightarrow{\tau} 0\) for every \(\gamma \in \Gamma\).

The next theorem [7, Prop.5] gives a sufficient condition for the metrizability of \(u\tau\)-topology.

Theorem 4. Let \((X, \tau)\) be a complete metrizable locally solid Riesz space. If \(X\) has a countable topological orthogonal system, then the \(u\tau\)-topology is metrizable.

- Notice that it is still unknown whether or not the converse of this theorem is true.

For further results on \(u\tau\)-topologies, I refer to our joint paper [7] with Dabboorasad and Marabel, and to the paper [19] of Taylor.
4. When uc-convergence coincides with c-convergence

- Clearly, c-convergence always implies uc-convergence, and they coincide, when the Riesz space X under the consideration is finite dimensional.
- Here we discuss for several types of c-convergence the question: whether or not c-convergence coincides with uc-convergence only if \( \dim(X) < \infty \)? To avoid trivial cases, suppose throughout this section that \( \dim(X) = \infty \).
- We begin with o-convergence.
- It is known that o-convergence in X is not topological [6, Thm. 1]. However, uo-convergence in X is topological, when X is discrete [6, Thm. 2].
- We do not know any examples of non-discrete X in which uo-convergence in X is topological.
- Since uo-convergence of a net \( x_\alpha \in X \) coincides with uo-convergence of \( x_\alpha \) in \( X^\delta \) by [13, Thm. 3.2], where \( X^\delta \) is the Dedekind completion of
$X$, we may also assume that $X$ is **Dedekind complete**.

- Accordingly to the remark above, we exclude the case when $X \equiv c_{00}(\Omega)$. Indeed, $u_0$-convergence in $c_{00}(\Omega)$ is topological, but $\alpha$-convergence is not.

- Therefore, we may suppose that, for some $u \in X_+$, there is a sequence of pair-wise disjoint vectors $X_+ \ni u_n \neq 0$ such that $u = \sup_n u_n$.

- Now, we construct the following net $(x_\Delta)_{\Delta \in \mathcal{P}_{\text{fin}}(\mathbb{N})}$:

$$x_\Delta := |\Delta| \cdot \sup_{n \not\in \Delta} u_n \quad (\Delta \in \mathcal{P}_{\text{fin}}(\mathbb{N})), \quad (6)$$

where $\mathcal{P}_{\text{fin}}(\mathbb{N})$ is ordered by inclusion.

- It can be easily seen that the net $x_\Delta$ is not eventually $\alpha$-bounded in $X$. Thus, $x_\Delta$ is not $\alpha$-convergent in $X$.

- From the other hand, $x_\Delta \overset{u_0}{\to} 0$ in $I_u$ and hence $x_\Delta \overset{u_0}{\to} 0$ in $X$.

- Thus, we get the following result which is included in [9].
Proposition 1. Let $X$ be an Archimedean Riesz space. Then $u_0$-convergence agrees with $o$-convergence in $X$ iff $\dim(X) < \infty$.

- The case of $m$-convergence and, therefore of $\tau$-convergence, is surprisingly different.
- Namely, for $X = c_{00}$ equipped with the multinorm $M = \{m_n\}_{n \in \mathbb{N}}$:

  $$m_k(x) := x_k \quad (x = (x_n)_{n \in c_{00}}),$$

we have that $x_\alpha \xrightarrow{um} 0$ iff $x_\alpha \xrightarrow{m} 0$ iff $x_\alpha \rightarrow 0$ coordinatewise.
REFERENCES (to be extended in future)


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