CHAPTER 4

Infinitesimals in Vector Lattices

BY

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Infinitesimals in Vector Lattices

Infinitesimal analysis has lavishly contributed to various areas of mathematics since 1961 when A. Robinson published his famous paper [23]. The complete list of applications is very huge even as regards functional analysis. We give below some of them pertinent to the theory of vector lattices.

The first natural question is as follows: Where does infinitesimal analysis apply effectively? Clearly the methods of infinitesimal analysis are not a panacea and so they must fail in solving some problems. Furthermore, what new possibilities are open up by infinitesimal analysis when it applies? The book [17] offers a partial answer to this question. Roughly speaking, the nonstandard methods prove fruitful whenever the problem under consideration deals with such concepts as compactness or ultrafilter. On the other hand, these methods are often inapplicable to purely algebraic problems.

The main topic of further research is the theory of vector lattices and operators in them. We hope to convince the reader that vector lattice theory is a natural area for applying infinitesimal methods.

Vector lattices with some norm or other specific structure were studied by many authors (see, for example, [5, 11], and [18]) in the context of infinitesimal analysis. Our aim is to further pave the infinitesimal approach to vector lattices. In the sequel we proceed in the wake of the articles [6–10] with due modification, considering only real vector lattices.

The structure of this chapter is as follows. Section 4.0 is an introduction to Robinsonian infinitesimal analysis and its applications to normed spaces. In the end of the section, we make a short introduction to the theory of lattice normed spaces (for more information on this topic we refer the reader to the recent papers [13–16]).

Sections 4.1 and 4.2 deal mainly with an infinitesimal approach to representing vector lattices. It turns out that infinitesimal analysis provides new more natural and subtle possibilities for representing vector lattices as function spaces. We show below that construction of a representing topological space for a vector lattice is possible on using members of the lattice (more precisely, of a nonstandard enlargement of it) rather than ultrafilters or prime ideals (the latter inherent in Robinson's construction).

In Sections 4.3–4.7, we deal with infinitesimal interpretations of the basic concepts of the theory of vector lattices. We also consider the extension problem for a \( \star \)-invariant homomorphism over a vector lattice or a Boolean algebra. Some types of elements of a nonstandard enlargement of a vector lattice are defined: limited or finite elements, \((r)\)- and \((o)\)-infinitesimals, prenearstandard and nearstandard elements, etc. We obtain some easily applicable nonstandard criteria for a vector lattice to be Archimedean, Dedekind complete, atomic, etc., and present a nonstandard construction of a Dedekind completion of an Archimedean vector lattice.
Using the concepts of Sections 4.3–4.7, we follow the Luxemburg scheme in Sections 4.8–4.11 for defining and studying two nonstandard hulls of a vector lattice: the order regular hulls. An elementary theory of these hulls is given in Sections 4.8 and 4.9. In Section 4.10 we introduce and study the concept of the nonstandard hull of a lattice normed space and that of the space associated with the order hull of a decomposable lattice normed space.

Section 4.11 discusses a nonstandard construction of an order completion of a decomposable lattice normed space. The scheme rests on embedding such a space into the associated Banach–Kantorovich space.

Throughout the sequel, we use the terminology and notations regarding vector lattices and operators from the books [1, 21, 24], and [28]. Lattice normed spaces are dealt with only in Sections 4.10 and 4.11. Therein we appeal to the terminology and notations of [13–16]. In the current chapter, we use some (standard and nonstandard) results on Boolean algebras and measure spaces which can be found in [2, 4], and [25]. For Robinsonian nonstandard analysis and its applications we refer the reader to [2, 12, 20] and [27]. Other explanations are to be made on the route.

4.0. Preliminaries

We start with a brief introduction to Robinsonian infinitesimal analysis and recall several facts of it without proofs. The level of formal requirements is chosen so as to avoid overloading the text with unnecessary details. We will mostly follow the books [2, 12] and [20]. We then recall some well-known results on applications of infinitesimal analysis to the theory of normed spaces and operators in them. For more applications we refer the reader to [2, 12] and [27]. In the end of the Section, we touch the theory of lattice normed spaces. Our exposition of this part rests on [13–16]. The concept of a quotient of a lattice normed space and Proposition 4.0.14 are new (cf. [9, Lemma 0.5.7]).

4.0.1. Let $S$ be a set. A superstructure over $S$ is the set $V(S) := \bigcup_n V_n(S)$, with $V_n(S)$ defined by recursion:

$$V_1(S) := S,$$

$$V_{n+1}(S) := V_n(S) \cup \{X : X \subseteq V_n(S)\}.$$ 

Superstructures are fragments of the von Neumann universe, providing a basis for various mathematical theories in dependence on the choice of the basic set. For example, superstructures over the reals serve the needs of calculus. When working with the superstructure over some set $S$, we suppose throughout that $\mathbb{R} \subseteq S$.

We need some formal language $L$. The alphabet of $L$ contains
(1) variables: small and capital letters with possible indices;
(2) the symbols = and ∈ for equality and membership;
(3) symbols for propositional connectives and quantifiers;
(4) auxiliary symbols.

Atomic formulas of the language \( L \) are expressions of the form \( x = y \) or \( x \in y \). Arbitrary formulas are obtained from atomic formulas by applying propositional connectives and bounded quantifiers for sets (i.e. the prefixes \( \forall x \in y \) and \( \exists x \in y \)).

Given an arbitrary set \( S \) on which some (partial) operations and relations are defined, we introduce some language \( L_V(S) \) of the superstructure \( V(S) \). To make the presentation easier, we construct the language \( L_V(S) \) in a simple particular case.

Let \( S = E \cup \mathbb{N} \), with \( E \) a lattice. The set \( E \) is equipped with the operations \( \wedge, \vee \), and the relation \( \preceq \); the set \( \mathbb{N} \) of naturals is equipped with the operations \(+, \cdot\), and the relation \( \leq \). In this case, the language \( L_V(S) \) is obtained from \( L \) by enriching the alphabet with the symbols \( \wedge, \vee, +, \cdot \) and \( \preceq \) for operations and \( \leq \) for predicates. The list of atomic formulas extends to include the expressions of the form \( t_1 \vee t_2 = t_3 \), \( t_1 \cdot t_2 = t_3 \), \( t_1 \leq t_2 \), etc., where \( t_1, t_2, \) and \( t_3 \) are arbitrary terms of \( L \). Every formula of the language \( L_V(S) \) is naturally interpreted in the superstructure \( V(S) \). For example, the formula \( \Psi(x, y, z) := x + y \leq z \) is true for a triple \((a, b, c)\) of elements of \( V(S) \) if and only if \( a, b, c \in \mathbb{N} \) and \( a + b \leq c \). It is clear that interpretation of arbitrary formulas of \( L_V(S) \) involves no difficulty either.

In what follows, we choose the basic set \( S \) depending on the context. This set will be assumed to contain various objects: real and complex numbers, vector spaces, vector lattices, etc. Our presentation is structured so that it may be translated into the formal language \( L_V(S) \) if need be. Throughout the article, the word "interpretation" means the natural interpretation in the respective superstructure.

4.0.2. Let \( S \) be a set equipped with some operations and relations (not necessarily defined everywhere). Then there exist an enlargement \( *S \) of \( S \) and embedding \( * : V(S) \hookrightarrow V(*S) \) satisfying the following principles:

**Extension Principle.** The set \( *S \) is a proper enlargement of \( S \). Moreover, \( *S \) is equipped with the same set of operations and relations as \( S \). In addition, \( *x = x \) for every element \( x \in S \).

**Transfer Principle.** Let \( \psi(x_1, x_2, \ldots, x_n) \) be a formula of \( L_V(S) \), and let \( A_1, A_2, \ldots, A_n \) be elements of the superstructure \( V(S) \). Then the assertion

\[
\psi(A_1, A_2, \ldots, A_n)
\]

about elements of \( V(S) \) is true if and only if the assertion

\[
\psi(*A_1, *A_2, \ldots, *A_n)
\]
about elements of $V(^*S)$ is true.

Construction of the enlargement $^*S$ and embedding $*: V(S) \hookrightarrow V(^*S)$ with the required properties can be found, for example, in [2]. For convenience, we suppose that $*$ is the identical embedding and so $V(S) \subseteq V(^*S)$.

**Definition 1.** The superstructure $V(^*S)$ is called a nonstandard enlargement of $V(S)$ if the embedding $V(S) \subseteq V(^*S)$ satisfies the transfer and extension principles.

Dealing with some superstructure in the sequel, we will not specify the basic set over which it is constructed. This set is chosen to be sufficiently substantive.

**Definition 2.** Let $^*M$ be a nonstandard enlargement of a superstructure $M$. An element $x \in ^*M$ is called:

1. **standard** if $x = ^*B$ for some $B \in M$;
2. **internal** if $x \in ^*B$ for some $B \in M$;
3. **external** if $x \notin ^*B$ for every $B \in M$.

Note that every standard set is internal and every element of an internal set is internal too. The following is easy from the transfer principle:

**Internal Definition Principle.** Let $\psi(x, x_1, x_2, \ldots, x_n)$ be a formula of the language $L_M$, and let $A, A_1, A_2, \ldots, A_n$ be internal sets. Then the set $\{x \in A : \psi(x, A_1, A_2, \ldots, A_n)\}$ is internal too.

It is well known that a nonstandard enlargement $^*M$ of the superstructure $M$ may be chosen so that the following principle holds:

**General Saturation Principle.** For each family $\{X_\gamma\}_{\gamma \in \Gamma}$ of internal sets which has standard cardinality (i.e., $\text{card}({\Gamma}) < \text{card}(M)$) and enjoys the finite intersection property, the condition $\bigcap_{\gamma \in \Gamma} X_\gamma \neq \emptyset$ is valid.

In the sequel, we deal only with nonstandard enlargements satisfying the general saturation principle. These nonstandard enlargements are called polysaturated.

**4.0.3.** Let $X$ be an element of a superstructure $M$. We denote by $\mathcal{F}(X)$ the family of all finite subsets of $X$. Recall that elements of $^*\mathcal{F}(X)$ are exactly the subsets $A \subseteq ^*X$ for which there exist an internal function $f$ and element $\nu \in ^*\mathbb{N}$ such that $\text{dom}(f) = \{1, \ldots, \nu\}$ and $\text{im}(f) = A$. These subsets of $^*X$ are called hyperfinite and denoted, for example, by $\{x_n\}_{n=1}^\nu$ in analogy with finite families.

**Lemma.** Let $X \in M$ be an infinite set and $\nu \in ^*\mathbb{N} \setminus \mathbb{N}$. Then there exists an internal function $f$ such that $\text{dom}(f) = \{1, \ldots, \nu\}$ and $X \subseteq \text{im}(f) \subseteq ^*X$. In other words, there is a hyperfinite set $\{x_n\}_{n=1}^\nu$ with $X \subseteq \{x_n\}_{n=1}^\nu \subseteq ^*X$. 
< Let $\Psi$ be the set of all functions $\psi$ such that $\text{dom}(\psi) \subseteq \mathbb{N}$ and $\text{im}(\psi) \subseteq X$. Given $\pi \in \mathcal{P}(X)$, we assign

$$A_\pi := \{ \varphi \in {}^*\Psi : \text{dom}(\varphi) = \{1, \ldots, \nu\} \& \pi \subseteq \text{im}(\varphi) \}.$$ 

Since $X$ is infinite, every set $A_\pi$ is not empty. By the internal definition principle, the sets $A_\pi$ are internal. They form a family with the finite intersection property. Since $\text{card}(\mathcal{P}(X)) < \text{card}(M)$, by the general saturation principle we have $f \in \bigcap_{\pi \in \mathcal{P}(X)} A_\pi$ for some $f \in {}^*\Psi$. It is easy to see that $f$ is an internal function with $\text{dom}(f) = \{1, \ldots, \nu\}$ and $X \subseteq \text{im}(f) \subseteq {}^*X$. >

4.0.4. Lemma. Suppose that $X \in M$ and $X \subseteq M$. Then, for every $\nu \in {}^*\mathbb{N}\setminus\mathbb{N}$, we have $\text{card}(X) < \text{card}(\nu)$. 

< Consider the set $\mathcal{P}(X)$ of all subsets of $X$. Since $\mathcal{P}(X) \in M$, by the preceding lemma, $\mathcal{P}(X) \subseteq \{x_n\}_{n=1}^{\nu}$, where $\{x_n\}_{n=1}^{\nu}$ is a hyperfinite subset of $^*\mathcal{P}(X)$. Then $\text{card}(X) < \text{card}(\mathcal{P}(X)) \leq \text{card}(\nu)$. >

4.0.5. Let $\Theta$ be a directed set which is an element of the basic superstructure $M$. Denote the set $\{ \xi \in {^*}\Theta : (\forall \tau \in \Theta) \xi \geq \tau \}$ by $^a\Theta$. Elements of $^a\Theta$ are called (infinitely) remote.

Lemma. For every directed set $\Theta \in M$, there is at least one remote element $\alpha \in {^a}\Theta$.

< In the case when $\Theta$ is a finite set there is nothing to prove. So, we assume that $\Theta$ is infinite. By Lemma 4.0.3, there is a hyperfinite set $A$ such that $\Theta \subseteq A \subseteq {^*}\Theta$. Since $^*\Theta$ is an internal directed set, there is an element $\alpha \in {^*}\Theta$ satisfying $\alpha \geq \tau$ for all $\tau \in A$. It is clear that $\alpha \in {^a}\Theta$. >

4.0.6. The main tool for applying nonstandard analysis to normed spaces is the following simple construction discovered by W. A. J. Luxemburg [20]. Let $X$ be a normed space. Consider the two external subspaces

$$\text{Fin}(^*X) := \{ x \in {}^*X : (\exists r \in \mathbb{R})\|x\| \leq r \},$$ 

$$\mu(^*X) := \{ x \in {}^*X : (\exists r \in \mathbb{R})(\forall n \in \mathbb{N}) \|nx\| \leq r \}$$ 

in $^*X$. Elements of $\text{Fin}(^*X)$ are called (norm) limited (or finite in norm) and elements of $\mu(^*X)$ are called infinitesimal. Obviously, $\mu(^*X)$ is a subspace of $\text{Fin}(^*X)$. Thus we can take the quotient space

$$^X := \text{Fin}(^*X)/\mu(^*X)$$ 

under the norm $\|\lambda x\| := \text{st}(\|\lambda x\|)$. In the wake of W. A. J. Luxemburg, we call $^X$ the nonstandard hull of $X$. Define the mapping $\tilde{\mu}_X : X \to ^X$ as

$$\tilde{\mu}_X(x) := [x] \quad (x \in X).$$ 

It is easy that $\tilde{\mu}_X$ is an embedding of $X$ into $^X$. The following is well known:
**Proposition.** For every normed space $X$, the quotient $\tilde{X}$ is a Banach space and the map $\tilde{\mu}_X$ is onto if and only if $X$ is finite-dimensional.

4.0.7. A nonstandard construction of a norm completion of a normed space lies very closely to the construction of the nonstandard hull of the space under study. Let $X$ be a normed space. Consider the external subspace

$$\text{pns}(*X) := \{x \in *X : (\forall n \in \mathbb{N})(\exists y \in X) n\rho(x - y) \leq 1\}$$

of $*X$.

**Proposition.** The quotient normed space $\text{pns}(*X)/\mu(*X)$ is a norm completion of $X$ under the embedding $\tilde{\mu}_X$.

4.0.8. Let $A$ be a subset in a normed space $X$. We have a simple and useful criterion for boundedness of $A$.

**Proposition.** The following are equivalent:

1. $A$ is a norm bounded set;
2. $*A \subseteq \text{Fin}(*X)$.

4.0.9. Let $X$ and $Y$ be normed spaces and let $T : X \to Y$ be a linear operator. The next well-known proposition is immediate from 4.0.8.

**Proposition.** The following are equivalent:

1. $T$ is a bounded operator;
2. $*T(\text{Fin}(*X)) \subseteq \text{Fin}(*Y)$;
3. $*T(\mu(*X)) \subseteq \mu(*Y)$;
4. $*T(\mu(*X)) \subseteq \text{Fin}(*Y)$.

Thus the operator $\tilde{T} : \tilde{X} \to \tilde{Y}$, acting as $\tilde{T}([x]) := [Tx]$ for all $x \in \text{Fin}(*X)$, is well defined and bounded together with $T$. This operator is the nonstandard hull of $T$.

Now we briefly present necessary facts from the theory of lattice normed spaces and dominated operators. Our exposition follows [13-16].

4.0.10. A lattice normed space is a triple $(X, p, E)$ with $X$ a vector space, $E$ a vector lattice, and $p$ a mapping $X \to E_+$ such that

1. $p(x) = 0 \iff x = 0$;
2. $p(\lambda x) = |\lambda|p(x)$ ($\lambda \in \mathbb{R}$, $x \in X$);
Theorem. The mapping $p$ is called an $E$-valued norm on $X$. The lattice norm $p$ is called decomposable ($(d)$-decomposable) if, for all $e_1, e_2 \in E_+$ (for all disjoint $e_1, e_2 \in E_+$) and every $x \in X$, the condition $p(x) = e_1 + e_2$ implies existence of $x_1, x_2 \in X$ such that $x_1 + x_2 = x$ and $p(x_k) = e_k$ for $k = 1, 2$. A lattice normed space with decomposable ($(d)$-decomposable) norm is called decomposable ($(d)$-decomposable).

We say that a sequence $(x_n)_{n \in \mathbb{N}}$ in $(X, p, E)$ $(r)$-converges to $x \in X$ if there exist a sequence $(\varepsilon_n) \subseteq \mathbb{R}$, $\varepsilon_n \downarrow 0$, and an element $u \in E$ such that $p(x_n - x) \leq \varepsilon_n u$ for all $n \in \mathbb{N}$. By definition, a sequence $(x_n) \subseteq X$ is $(r)$-Cauchy if there is a sequence $(\varepsilon_n) \subseteq \mathbb{R}$, $\varepsilon_n \downarrow 0$, such that $p(x_k - x_m) \leq \varepsilon_n u$ for all $k, m, n \in \mathbb{N}$ such that $k, m \geq n$. A lattice normed space $X$ is called $(r)$-complete if every $(r)$-Cauchy sequence in $X$ $(r)$-converges to some element of $X$. The following assertion is a consequence of [13, 1.1.3] and the Freudenthal Spectral Theorem.

**Proposition.** A $(d)$-decomposable $(r)$-complete lattice normed space is decomposable.

**4.0.11.** A net $(x_\alpha)_{\alpha \in \Delta}$ in a lattice normed space $X$ is called $(o)$-convergent to $x \in X$ if there exists a decreasing net $(e_\gamma)_{\gamma \in \Gamma}$ in $E$ such that $e_\gamma \downarrow 0$ and, for every $\gamma \in \Gamma$, there is a subscript $\alpha(\gamma)$ for which $a(x_\alpha - x) \leq e_\gamma$ whenever $\alpha \geq \alpha(\gamma)$. A net $(x_\alpha)_{\alpha \in \Delta}$ is called $(o)$-Cauchy if the net $(x_\alpha - x_\beta)_{(\alpha, \beta) \in \Delta \times \Delta}$ $(o)$-converges to zero. A lattice normed space is called $(o)$-complete if every $(o)$-Cauchy net in it $(o)$-converges to an element of the space. A decomposable $(o)$-complete lattice normed space is said to be a Banach–Kantorovich space.

Consider a decomposable lattice normed space $(X, p, E)$. Let $(\pi_\xi)_{\xi \in \Xi}$ be some partition of unity in the Boolean algebra $\mathcal{B}(E)$ and let $(x_\xi)_{\xi \in \Xi}$ be a family of elements in $X$. If there exists an $x \in X$ satisfying the condition

$$p(x - x_\xi) = 0 \quad (\xi \in \Xi),$$

then such an element $x$ is uniquely determined. It is called the mixing of $(x_\xi)$ by $(\pi_\xi)$ and denoted by $\text{mix}(\pi_\xi x_\xi)_{\xi \in \Xi}$ or simply $\text{mix}(\pi_\xi x_\xi)$. A lattice normed space $(X, p, E)$ is said to be $(d)$-complete if the mixing $\text{mix}(\pi_\xi x_\xi) \in X$ exists for every partition of unity $(\pi_\xi) \subseteq \mathcal{B}(E)$ and every norm bounded family $(x_\xi) \subseteq X$.

**Proposition** [13, Theorem 1.3.2]. A decomposable lattice normed space is $(o)$-complete if and only if it is $(r)$- and $(d)$-complete.

**4.0.12.** Let $(X, p, E)$ be a decomposable lattice normed space whose norm lattice $E$ is Dedekind complete. Then there is a unique lattice normed space $(X', p', E)$ to within isometric isomorphism with the following properties (see [13, Theorem 1.3.8]).
(1) \((X', p', E)\) is a Banach–Kantorovich space;

(2) there exists a linear embedding \(\iota : X \to X'\) such that \(p'(\iota(x)) = p(x)\) for all \(x \in X\);

(3) \(X'\) is the least \((o)\)-complete lattice normed subspace of \(X'\) that contains \(\iota(X)\).

The space \((X', p', E)\) is an \((o)\)-completion of the lattice normed space \((X, p, E)\).

Consider some properties of lattice normed spaces connected with decomposability and \((r)\)-completeness.

4.0.13. Lemma. Let \((X, p, E)\) be a decomposable lattice normed space and let \(I\) be an ideal in \(E\). Then, for arbitrary elements \(x, y \in X, q \in E_+,\) and \(\eta \in I\) satisfying the condition \(p(x - y) \leq q + \eta\), there is an element \(y' \in X\) such that

(1) \(p(x - y') \leq q;\)

(2) \(p(y' - y) \in I.\)

\(<\) Clearly, \(p(x - y) \leq q + \eta_+\). By the Riesz decomposition property, there are elements \(a_1, a_2 \in E\) such that

\[
0 \leq a_1 \leq q, 0 \leq a_2 \leq \eta_+, a_1 + a_2 = p(x - y).
\]

Since \(p\) is decomposable, we may find elements \(z_1, z_2 \in X\) with

\[
p(z_1) = a_1, p(z_2) = a_2, z_1 + z_2 = x - y.
\]

It is easy to see that the element \(y' := y + z_2\) satisfies (1) and (2). \(>)

4.0.14. Let \((X, p, E)\) be an arbitrary lattice normed space. Take an ideal \(I\) in \(E\) and consider the quotient space \(X'\) of \(X\) by the subspace \(X(I) := \{x \in X : p(x) \in I\}\). Given \(x \in X, (e \in E),\) assign \([x] := x + X(I)\) (respectively, \([e] := e + I \in E/I\)). Define the mapping \(p' : X' \to E/I\) by the rule

\[
p'([x]) := [p(x)] \quad (x \in X).
\]

It is easy that the mapping \(p'\) is defined correctly and presents an \(E/I\)-valued norm on the space \(X'\). The so-obtained lattice normed space \((X', p', E/I)\) is called the quotient space of \(X\) by the ideal \(I\) of the norm lattice \(E\).

Proposition. Let \((X, p, E)\) be a decomposable \((r)\)-complete lattice normed space, and let \(I\) be an ideal of the norm lattice \(E\). Then the quotient space \((X', p', E/I)\) is decomposable and \((r)\)-complete.
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< Verify that the norm \( p' \) is decomposable. Let \( p'([x]) = [e_1] + [e_2] \), where \([e_1],[e_2] \in (E/I)_+\). We may assume that \( e_1, e_2 \geq 0 \). There is an element \( \eta \in E \) such that \( p(x) = e_1 + e_2 + \eta \). By Lemma 4.0.13, there is an element \( x' \in X \) such that \( p(x - x') \leq e_1 + e_2 \) and \( p(x') \in I \). In particular,

\[
p(x - x') = e_1 + e_2 + \eta' \tag{1}
\]

for some \( \eta' \in I \). Applying the Riesz decomposition property to the inequality \( p(x - x') \leq e_1 + e_2 \), we find elements \( e_1', e_2' \in E \) for which

\[
p(x - x') = e_1' + e_2', \quad 0 \leq e_k' \leq e_k \quad (k = 1, 2). \tag{2}
\]

Then \([e_k'] = [e_k] \ (k = 1, 2)\). Indeed, supposing that for instance \([e_1 - e_1'] > 0\), in view of (1) and (2) we obtain a contradiction:

\[
0 \leq [e_2 - e_2'] \\
= [p(x - x') - e_1 - \eta' - e_2'] = [e_1' - e_1 - \eta'] \\
= [e_1' - e_1] < 0.
\]

Since the norm \( p \) is assumed decomposable, it follows from (2) that there exist \( y_1, y_2 \in X \) such that \( x - x' = y_1 + y_2 \) and \( p(y_k) = e_k' \ (k = 1, 2)\). Then \( p'([y_k]) = [p(y_k)] = [e_k'] = [e_k] \) for \( k = 1, 2 \), and

\[
[y_1] + [y_2] = [x - x'] = [x],
\]

as required.

We now verify that \( X' \) is \((r)\)-complete. Take some \((r)\)-Cauchy sequence \((x_i) \subseteq X'\). There exist \( e \in E \) and \((x_i(n)) \subseteq (x_i)\), for which

\[
p'(x_i(k) - x_i(m)) \leq 2^{-n}[e] \tag{3}
\]

whenever \( k, m \), and \( n \) are such that \( k, m \geq n \). Choose elements \( \kappa_n \in X \) so that \( x_i(n) = [\kappa_n] \). Then, by (3), there are \( \eta_{k,m} \in I \) with

\[
p(\kappa_k - \kappa_m) \leq 2^{-n}e + \eta_{k,m} \tag{4}
\]

for all \( k, m, n \in \mathbb{N} \) such that \( k, m \geq n \). By induction, construct a sequence \((\kappa'_n) \subseteq X \) that satisfies the conditions

\[
p(\kappa'_n - \kappa'_{n+1}) \leq 2^{-n}e; \tag{5}
\]
for all $n \in \mathbb{N}$. Assign $x'_1 := x_1$ and assume that the elements $x'_j$ are already defined for $j \leq n$. From (4) it follows that

$$p(x'_n - x_{n+1}) \leq p(x_n - x_{n+1}) - p(x_{n, n+1} - p(x_n - x'_n)).$$

Since by the induction hypothesis $p(x_n - x'_n) \in I$, we may apply Lemma 4.0.13 to the elements

$$x'_n, x_{n+1} \in X, \quad 2^{-n} \in E, \quad \eta := \eta_{n, n+1} + p(x_n - x'_n).$$

In result, we come to an element $x'_{n+1}$ satisfying conditions (5) and (6). It follows from (5) that the sequence $(x'_n)$ is $(r)$-Cauchy in $X$. Consequently, it $(r)$-converges to some element $x_0 \in X$. Then, as is easy to see, the sequence $(x_i(n)) = ([x_n])$ $(r)$-converges in the norm $p'$ to $[x_0] \in X'$. Since the initial sequence $(x_i)$ is $(r)$-Cauchy in the norm $p'$, we obtain $x_i \to (r) [x_0]$. ♦

4.0.15. Assume that $E$ and $F$ are some Dedekind complete vector lattices, while $(X, a, E)$ and $(Y, b, F)$ are decomposable LNSs. A linear operator $T : E \to F$ is called \textit{dominated} if there exists an order bounded linear operator $S : E \to F$ such that

$$|Tx| \leq U(|x|) \quad (x \in X).$$

The operator $S$ is called a \textit{dominant} of $T$. The least dominant of an operator $T$ is denoted by $|T|$.

By $M(X, Y)$ we will denote the vector space of all dominated operators from an LNS $X$ into an LNS $Y$. The mapping $T \mapsto |T|$ ($T \in M(E, F)$) satisfies all axioms of an $E$-valued norm from 4.0.10. Consequently, $M(E, F)$ is also an LNS with norm lattice the Dedekind complete vector lattice $L_b(E, F)$ (see, for example, [28, Theorem 83.4]) of all order bounded linear operators from $E$ to $F$.

4.1. Saturated Sets of Indivisibles

Here we deal with lattices with zero and present some elementary facts about them, standard and nonstandard. We prove that a nonstandard enlargement of a lattice with zero contains a saturating family of indivisible elements.

4.1.1. Let $L$ be an ordered set whose order is denoted by $\geq$. We write $x > y$ whenever $x \geq y$ and $x \neq y$. The set $L$ is called a \textit{lattice} if every two-element subset $\{x, y\}$ in $L$ has the least upper bound $x \vee y := \sup\{x, y\}$ and the greatest lower bound $x \wedge y := \inf\{x, y\}$. If a lattice contains the smallest (largest) element then
this element is called zero (unity) and denoted by 0 (respectively, 1). We always assume that each lattice under consideration has some zero. Elements \( x \) and \( y \) of a lattice are disjoint if \( x \wedge y = 0 \). A lattice is called distributive if every triple \( x, y, z \) of its elements satisfies \( x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \) and \( x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \).

A distributive lattice \( L \) with zero 0 and unity 1 is called a Boolean algebra if every element \( x \in L \) possesses the complement, i.e., a (unique) element \( x' \in L \) such that \( x \wedge x' = 0 \) and \( x \vee x' = 1 \).

Let \( L \) be a lattice. An element \( e \in L \) is said to be a weak (order) unity if \( e \wedge x > 0 \) for every \( x \in L \), \( x > 0 \). We call \( y \in L \) a pseudocomplement of an element \( x \in L \) if \( x \wedge y = 0 \) and \( x \vee y \) is a weak unity. The lattice \( L \) is called a pseudocomplemented lattice if, for each \( x \in L \), there is at least one pseudocomplement in \( L \). An example of a pseudocomplemented lattice is a Boolean algebra. A less trivial example is the lattice of all nonnegative continuous functions on an arbitrary metric space. Also, we will use the following notation: Given an element \( \kappa \) of a nonstandard enlargement \( *L \), we denote by \( U(\kappa) := \{ x \in E : x \geq \kappa \} \) the set of standard upper bounds of \( \kappa \) and by \( L(\kappa) := \{ y \in E : \kappa \geq y \} \), the set of standard lower bounds of \( \kappa \).

4.1.2. Let \( L \) be a distributive lattice. An ideal of the lattice \( L \) is a nonempty set \( I \subseteq L \) such that \( x, y \in I \) implies \( x \vee y \in I \), and \( z \in I \) if \( z \leq v \) for some \( v \in I \). An ideal \( P \) of \( L \) is called prime if, for every \( x, y \in L \), the condition \( x \wedge y = 0 \) implies that either \( x \in P \) or \( y \in P \). A prime ideal \( P \) is called minimal if, for every prime ideal \( P_1 \subseteq L \), the condition \( P_1 \subseteq P \) implies \( P_1 = P \). Every subset \( S \subseteq L \) such that \( 0 \notin S \) and \( x, y \in S \) implies \( x \wedge y \in S \) is called a lower sublattice. A lower sublattice \( S \) is called maximal or an ultrafilter if for every lower sublattice \( S_1 \subseteq L \) the condition \( S \subseteq S_1 \) implies \( S_1 = S \). The following assertion is well-known [21, Theorem 5.4].

**Lemma.** Let \( L \) be a distributive lattice and let \( P \) be some prime ideal in \( L \). Then \( L \setminus P \) is a lower sublattice of \( L \). Furthermore, \( L \setminus P \) is an ultrafilter if and only if \( P \) is a minimal prime ideal.

4.1.3. Consider a nonstandard enlargement \( *L \) of a lattice \( L \). By the transfer principle, \( *L \) has the same zero element 0 as the initial lattice \( L \). We give two important definitions.

**Definition 1.** An element \( \kappa \in *L \) is called indivisible if \( \kappa > 0 \) and, for every \( x \in L \), either \( x \geq \kappa \) or \( x \wedge \kappa = 0 \).

**Definition 2.** A subset \( \Lambda \) of the lattice \( *L \) is called saturating if \( \Lambda \) is internal and, for every \( x \in L \), \( x > 0 \), there is some \( a \in \Lambda \) such that \( x \geq a > 0 \).

The following simple lemma is a key for many of our further results.
Lemma. A nonstandard enlargement of an arbitrary lattice contains a hyperfinite saturating set of disjoint indivisible elements.

Let $L$ be a lattice. We denote by $\mathcal{F}$ the family of all finite subsets of $L \setminus \{0\}$ and put

$$\mathcal{L}_\pi := \{X \in \mathcal{F} : (\forall y \in \pi)(\forall x \in X)(y \geq x \text{ or } y \land x = 0)$$

$$\& (\forall y \in \pi)(\exists x \in X)(y \geq x)$$

$$\& (\forall x \in X)(\forall z \in X)(x \neq z \rightarrow x \land z = 0)\}$$

for all $\pi \in \mathcal{F}$. Show that the sets $\mathcal{L}_\pi$ are nonempty. Take an arbitrary $\pi \in \mathcal{F}$. For every element $x \in \pi$, there is a set $B_x$ such that $x \in B_x \subseteq \pi$, $\inf B_x > 0$, and $\inf(B_x \cup \{y\}) = 0$ for all $y \in \pi \setminus B_x$. It is easy to verify that the set $\{\inf B_x : x \in \pi\}$ belongs to $\mathcal{L}_\pi$. The fact that the sets $\mathcal{L}_\pi$ are nonempty and the condition $\mathcal{L}_\pi \cap \mathcal{L}_\gamma = \mathcal{L}_{\pi \cup \gamma}$ implies that the family $\{\mathcal{L}_\pi\}_{\pi \in \mathcal{F}}$ enjoys the finite intersection property. All elements of this family are internal sets by construction. Thus, by the general saturation principle, there is a $\Lambda \in \bigcap_{\pi \in \mathcal{F}} \mathcal{L}_\pi$. It is easy to see that $\Lambda$ is a desired saturating family of disjoint indivisible elements.

4.1.4. Let $L$ be a lattice. By Lemma 4.1.3, there exists a saturating set of indivisible elements in the lattice $^*L$. Let $\Lambda$ be such a set. Denote $\Lambda^x := \{x \in \Lambda : x \geq x\}$ for all $x \in L$. It is easy to see that, in the case when $L$ possesses a weak unity $e$, the family $\{\Lambda^x : x \in L\}$ is an open base for a topology $\tau$ on $\Lambda$. This topology is called canonical on $\Lambda$.

Theorem. Let $\Lambda$ be a saturating set of indivisible elements in a nonstandard enlargement of a lattice with weak unity and let $\tau$ be the canonical topology on $\Lambda$. Then $(\Lambda, \tau)$ is a compact space.

Assume that $(\Lambda, \tau)$ is not compact. In this case we may extract a subset $\{\Lambda^x\}_{x \in X}$ from the base $\{\Lambda^x\}_{x \in L}$ of the topology $\tau$ such that $\Lambda = \bigcup_{x \in X} \Lambda^x$ and, for every finite subset $\pi$ of $X$, the following holds:

$$A_\pi := \Lambda \setminus \bigcup_{x \in \pi} \Lambda^x \neq \emptyset.$$ 

It is easy to verify that $\{A_\pi : \pi \in \mathcal{P}_{\text{fin}}(X)\}$ is a family of internal sets with the finite intersection property. By the general saturation principle, there is an element $\kappa \in \bigcap\{A_\pi : \pi \in \mathcal{P}_{\text{fin}}(X)\}$. Then $\kappa \in \Lambda \setminus \bigcup_{x \in X} \Lambda^x$, a contradiction with the fact that $\{\Lambda^x\}_{x \in X}$ is an open covering of $\Lambda$. Thus, $(\Lambda, \tau)$ is a compact space.

\[\square\]
We introduce some equivalence relation $\sim$ in the lattice $^*L$ by putting $x_1 \sim x_2$ whenever the inequalities $x \geq x_1$ and $x \geq x_2$ are equivalent for all $x \in L$. Suppose that a lattice $L$ possesses a weak unity $e$. Take a saturating set $\Lambda$ of indivisible elements in $^*L$ (such a set exists by Lemma 4.1.3). Let $\tau$ be the canonical topology on $\Lambda$. The topological space $(\Lambda, \tau)$ is compact by Theorem 4.1.4. Its quotient space by the equivalence $\sim$ is a compact $T_0$-space. We denote this quotient space by $\tilde{\Lambda}$. It is clear that the sets of the type

$$\tilde{\Lambda}^x := \{[x] \in \tilde{\Lambda} : x \geq x\} \quad (x \in L)$$

form an open base for the quotient topology (throughout, we denote by $[x]$ the coset containing an element $x \in \Lambda$).

Take a Boolean algebra $B$ as $L$ and consider a saturating set $\Upsilon$ of indivisible elements in $^*B$. It is easy to see that $\Upsilon$ is a totally disconnected compact Hausdorff space, while the mapping associating with each element $b \in B$ the subset $\Upsilon^b$ of the space $\Upsilon$ is a Boolean isomorphism of $B$ onto the algebra $\text{clop}(\Upsilon)$ of clopen (closed and open) subsets of the compact Hausdorff space $\Upsilon$. Thus we obtained the following

**Theorem.** Let $B$ be a Boolean algebra and let $\Upsilon$ be a saturating set of indivisible elements in a nonstandard enlargement of $B$. Then the corresponding topological space $\Upsilon$ is the Stone space of $B$.

4.1.6. The following theorem describes connection between the properties of a lattice and the corresponding topological space.

**Theorem.** Let $L$ be a distributive lattice with weak unity. Then the following are equivalent:

1. $L$ is a pseudocomplemented lattice;
2. The topological space $\tilde{\Lambda}$ is totally disconnected for every saturating set $\Lambda$ of indivisible elements in $^*L$;
3. The topological space $\tilde{\Lambda}$ satisfies the $T_1$-separation axiom for every saturating set $\Lambda$ of indivisible elements in $^*L$;
4. The set $\{x \in L : x \wedge x = 0\}$ is a minimal prime ideal in $L$ for every indivisible element $x \in L$.

$\langle (1) \to (2) \rangle$: It is easy to see that if condition (1) holds, then the base $\{\tilde{\Lambda}^x\}_{x \in L}$ for the topology of $\tilde{\Lambda}$ consists of clopen sets. Indeed, given an $x \in L$, we have $\tilde{\Lambda}^x \cup \tilde{\Lambda}^y = \tilde{\Lambda}$ and $\tilde{\Lambda}^x \cap \tilde{\Lambda}^y = \emptyset$, where $y$ is some pseudocomplement of $x$.

$\langle (2) \to (3) \rangle$: Obvious.
(3)→(4): Let condition (3) be satisfied. Take some indivisible element \( \kappa \in \mathscr{L} \) and consider the set \( I_\kappa := \{ x \in L : x \land \kappa = 0 \} \). It is easy to see that \( I_\kappa \) is a prime ideal in the lattice \( L \). Indeed, by distributivity, it follows from \( x, y \in I_\kappa \) that \( (x \lor y) \land \kappa = (x \land \kappa) \lor (y \land \kappa) = 0 \), and consequently \( x \lor y \in I_\kappa \). If \( x \land y \in I_\kappa \) then either \( x \in I_\kappa \) or \( y \in I_\kappa \) (otherwise, since the element \( \kappa \) is indivisible, we would have \( x \geq \kappa \) and \( y \geq \kappa \)). It remains to verify that the ideal \( I_\kappa \) is minimal.

Take an arbitrary prime ideal \( P \subseteq I_\kappa \). Assume that \( y \in I_\kappa \setminus P \) for some element \( x \in L \). Then \( x \land y > 0 \) for every \( x \in L \setminus I_\kappa \). Indeed, otherwise it would be valid that \( x \land y = 0 \), and hence we would have either \( x \in P \) or \( y \in P \), which is impossible. By Lemma 4.1.3, there exists a saturating set of indivisible elements in the lattice \( \mathscr{L} \). Let \( \Lambda' \) be such a set. Assign \( \Lambda := \Lambda' \cup \{ \kappa \} \). Then \( \Lambda \) is also a saturating set of indivisible elements, and \( \kappa \in \Lambda \). As was mentioned above, \( x \land y > 0 \) for every \( x \in L \setminus I_\kappa \). Using this observation, it is easy to show that \( \{ \Lambda^{x\land y} \} \) is a system of internal sets with the finite intersection property. Applying the general saturation principle, we find an element \( \delta \in \Lambda \) such that

\[ \delta \in \cap \{ \Lambda^{x\land y} : x \in L \setminus I_\kappa \}. \]

The indivisible element \( \delta \) satisfies the condition \( \delta \leq y \). At the same time, \( \kappa \land y = 0 \) because \( y \in I_\kappa \). Consequently, \( \delta \not\leq \kappa \). By condition (3), the topological space \( \hat{\Lambda} \) satisfies the T1-separation axiom. Therefore, there is \( z \in L \) for which \( [\kappa] \in \mathbb{A}^z \) and \( [\delta] \notin \mathbb{A}^z \). Then the relations \( \kappa \leq z \) and \( z \land \delta = 0 \) are valid. The first of them implies \( z \in L \setminus I_\kappa \), which contradicts the second relation. The obtained contradiction shows that \( I_\kappa \setminus P = \emptyset \). Since the choice of the prime ideal \( P \) satisfying the condition \( P \subseteq I_\kappa \) was arbitrary, \( I_\kappa \) is a minimal prime ideal.

(4)→(1): Suppose now that condition (4) is satisfied. Show that every element of the lattice \( L \) has a pseudocomplement. Take an arbitrary \( a \in L \). By Lemma 4.1.3, there exists some saturating set \( \Lambda \) of indivisible elements in \( \mathscr{L} \). Consider the topological space \( (\Lambda, \tau) \), where \( \tau \) is the canonical topology in \( \Lambda \). Let \( \kappa \in \Lambda \setminus \Lambda^a \). By hypothesis, the set

\[ I_\kappa := \{ x \in L : x \land \kappa = 0 \} \]

is a minimal prime ideal of \( L \). By the choice of \( \kappa \), we have \( a \land \kappa = 0 \), and so \( a \notin L \setminus I_\kappa \). Since \( L \setminus I_\kappa \) is an ultrafilter of the lattice \( L \) by Lemma 4.1.2, there exists an element \( y(\kappa) \in L \setminus I_\kappa \) such that \( y(\kappa) \land a = 0 \). In other words, \( \kappa \in \Lambda^{y(\kappa)} \) and \( \Lambda^a \cap \Lambda^{y(\kappa)} = 0 \). The family \( \{ \Lambda^{y(\kappa)} \} \) is an open covering of the closed set \( \Lambda \setminus \Lambda^a \) in the space \( (\Lambda, \tau) \). By Theorem 4.1.4, it contains a finite subcovering \( \{ \Lambda^{y(\kappa)} \} \). The element \( b := \bigvee_{k=1}^n y(\kappa_k) \) satisfies the conditions

\[ \Lambda^b \cap \Lambda^a = \emptyset, \quad \Lambda^b \cup \Lambda^a = \Lambda \]

and, consequently, it is the desired pseudocomplement of \( a \). \( \triangleright \)
4.1.7. Let $L$ be a distributive lattice. If $L$ is pseudocomplemented then, by Theorem 4.1.6, for every indivisible element $\kappa \in L^*$, there exists a respective minimal prime ideal $I_\kappa := \{x \in L : x \wedge \kappa = 0\}$. The converse is true in a more general setting:

**Lemma.** Let $I$ be a minimal prime ideal in a distributive lattice $L$. Then there is an indivisible element $\kappa \in L^*$ such that $I = \{x \in L : x \wedge \kappa = 0\}$.

Observe that the subset $U := L \setminus I$ in $L$ is directed downwards. By Lemma 4.0.5, there exists a remote element $\kappa \in U$. An easy check shows that $\kappa$ is an indivisible element in the lattice $L^*$ and $I = \{x \in L : x \wedge \kappa = 0\}$.  

4.1.8. Let $L$ be a distributive lattice. Denote by $\mathcal{M}$ the set of all minimal prime ideals in $L$. The set $\mathcal{M}$ is equipped with the canonical topology generated by the open base of all sets of the form

$$\mathcal{M}^u := \{P \in \mathcal{M} : u \notin P\} \quad (u \in L)$$

(see, for example, [21, Section 7]).

**Theorem.** Let $L$ be a pseudocomplemented distributive lattice. Then, for every saturating set $\Lambda$ of indivisible elements of $L^*$, the mapping $\varphi_\Lambda$, defined by the rule

$$\varphi_\Lambda([\kappa]) := \{x \in L : x \wedge \kappa = 0\} \quad ([\kappa] \in \Lambda),$$

is a homeomorphism of the topological space $\Lambda$ onto $\mathcal{M}$.

Let $\Lambda$ be a saturating set of indivisible elements in the lattice $L^*$. By Theorem 4.1.6, the mapping $\varphi_\Lambda$ ranges in the space $\mathcal{M}$. The mapping $\varphi_\Lambda$ is injective. Indeed, take arbitrary elements $\kappa_1, \kappa_2 \in \Lambda$ such that $\kappa_1 \neq \kappa_2$. Then

$$U(\kappa_1) \neq U(\kappa_2), \varphi_\Lambda([\kappa_1]) \neq \varphi_\Lambda([\kappa_2]),$$

since

$$\varphi_\Lambda([\kappa_1]) = L \setminus U(\kappa)$$

for all $\kappa \in \Lambda$. Show that $\varphi_\Lambda(\Lambda) = \mathcal{M}$. Take an arbitrary $P \in \mathcal{M}$. As is easy to verify, $\{A^x\}_{x \in L \setminus P}$ is a system of internal sets with the finite intersection property. Thus, by the general saturation principle we may find $\kappa \in \bigcap_{x \in L \setminus P} \Lambda^x$. Then

$$\varphi_\Lambda([\kappa]) = \{x \in L : x \wedge \kappa = 0\} = \{x \in L : \kappa \notin \Lambda^x\} = L \setminus \{x \in L : \kappa \in \Lambda^x\} \subseteq L \setminus (L \setminus P) = P$$
and so \(\varphi_\Lambda([\kappa]) = P\), because the ideal \(P\) is minimal. It remains to verify that \(\varphi_\Lambda\) is a homeomorphism. This readily follows on observing that

\[
\varphi_\Lambda(\hat{\Lambda}^x) = \{\varphi_\Lambda([\kappa]) : \kappa \leq x\} = \{P \in \mathcal{M} : x \notin P\} = \mathcal{M}^x. \>
\]

4.1.9. Let \(\Lambda_1\) and \(\Lambda_2\) be saturating sets of indivisible elements in a distributive pseudocomplemented lattice \(L\). Then, by the preceding theorem, the mapping \(\psi := \varphi_\Lambda^{-1} \circ \varphi_\Lambda\) is a homeomorphism of the topological space \(\hat{\Lambda}_1\) onto \(\hat{\Lambda}_2\). Note that this homeomorphism can be defined explicitly as follows:

\[
\psi([\kappa_1]) := \{\kappa_2 \in \Lambda_2 : U(\kappa_1) \geq \kappa_2\}
\]

for every element \(\kappa_1 \in \Lambda_1\). Thus, the topological space \(\hat{\Lambda}\) is uniquely determined to within a homeomorphism by the distributive pseudocomplemented lattice \(L\) and does not depend on the choice of a saturating set \(\Lambda\) of indivisible elements.

4.2. Representation of Archimedean Vector Lattices

In this section, we prove some nonstandard variant of the representation theorem for Archimedean vector lattices. We then give nonstandard proofs for the Brothers Kreĭn-Kakutani and Ogasawara-Vulikh representation theorems.

Throughout this section we suppose that \(E\) is an Archimedean vector lattice. The positive cone \(E^+\) of \(E\) is a distributive lattice with zero. Therefore, by Lemma 4.1.3, there exists a saturating set of indivisible elements in \(*E^+\). Here we fix such a set and denote it by \(\Lambda\) up to the end of this section.

4.2.1. Let \(e \in E\) and \(\kappa \in \Lambda\) be such that \(e \geq \kappa\). Given \(f \in E\), define the element \(f^\kappa\) of \(\mathbb{R}\) as

\[
f^\kappa := \inf \{\lambda \in \mathbb{R} : (\lambda e - f)_+ \geq \kappa\}. \quad (1)
\]

Granted \(f \in E\), let \(\mathcal{D}(f)\) stand for the subset \(\{\kappa \in \Lambda : |f^\kappa| < \infty\}\) of \(\Lambda\). We establish some properties of the mapping \(f \mapsto f^\kappa\).

**Lemma.** For every \(f, g \in E\) and every \(\alpha \in \mathbb{R}\), the following hold:

1. \(f^\kappa = \sup \{\lambda \in \mathbb{R} : (\lambda e - f)_- \geq \kappa\}\);
2. \((\alpha f)^\kappa = \alpha f^\kappa\);
3. \((f + g)^\kappa = f^\kappa + g^\kappa\) for all \(\kappa \in \mathcal{D}(f) \cap \mathcal{D}(g)\);
\( (f \wedge g)^\langle \varkappa \rangle = \min \{ f^\langle \varkappa \rangle, g^\langle \varkappa \rangle \} \)

and \( (f \lor g)^\langle \varkappa \rangle = \max \{ f^\langle \varkappa \rangle, g^\langle \varkappa \rangle \} \).

\(<1(1)\): Denote the right side of the equality under proof by \( f^\langle \varkappa \rangle \). We consider only the case in which both \( f^\langle \varkappa \rangle \) and \( f^\langle \varkappa \rangle \) are finite. Let \( \alpha > f^\langle \varkappa \rangle \). Then \( (\alpha e - f)_+ \geq \varkappa \), and so \( (\alpha e - f)_- \wedge \varkappa = 0 \). Consequently, \( \alpha \geq f^\langle \varkappa \rangle \), which implies \( f^\langle \varkappa \rangle \geq f^\langle \varkappa \rangle \), since the choice of the number \( \alpha > f^\langle \varkappa \rangle \) is arbitrary. Conversely, suppose that \( \alpha > f^\langle \varkappa \rangle \). Then \( (\alpha e - f)_- \not\geq \varkappa \), and hence \( (\alpha e - f)_- \wedge \varkappa = 0 \), because \( \varkappa \) is an indivisible element. At the same time, since \( e \geq \varkappa \), we have
\[
((\alpha + 1/n)e - f)_+ + (\alpha e - f)_- \geq (1/n)e \geq \varkappa
\]
for all natural \( n \). Hence
\[
((\alpha + 1/n)e - f)_+ \geq \varkappa \text{ and } \alpha + 1/n \geq f^\langle \varkappa \rangle
\]
for every \( \alpha > f^\langle \varkappa \rangle \) and \( n \in \mathbb{N} \), which is possible only if \( f^\langle \varkappa \rangle \geq f^\langle \varkappa \rangle \).

\(<2\): Omitting easy verification of the relation \( (\alpha f)^\langle \varkappa \rangle = \alpha f^\langle \varkappa \rangle \) with \( 0 \leq \alpha < \infty \), we show only that \( (-f)^\langle \varkappa \rangle = -f^\langle \varkappa \rangle \). Indeed, the required condition follows from the equalities
\[
(-f)^\langle \varkappa \rangle = \inf \{ \lambda : (\lambda e - f)_+ \geq \varkappa \}
\]
\[
= -\sup \{ \beta : (\beta e + f)_+ \geq \varkappa \} = -\sup \{ \beta : (\beta e - f)_- \geq \varkappa \} = -f^\langle \varkappa \rangle.
\]
The last equality is valid by assertion (1) proven above.

\(<3\): Let \( \varkappa \in \mathcal{D}(f) \cap \mathcal{D}(g) \). Observe that the conditions \( (\lambda e - f)_+ \geq \varkappa \) and \( (\beta e - g)_+ \geq \varkappa \) imply
\[
((\lambda + \beta)e - (f + g))_+ = ((\lambda e - f)_+ + (\beta e - g)_+)
\]
\[
\geq ((\lambda e - f) \wedge (\beta e - g))_+ = (\lambda e - f)_+ \wedge (\beta e - g)_+ \geq \varkappa.
\]
The following inequality is easy from this remark:
\[
f^\langle \varkappa \rangle + g^\langle \varkappa \rangle = \inf \{ \lambda : (\lambda e - f)_+ \geq \varkappa \} + \inf \{ \beta : (\beta e - g)_+ \geq \varkappa \}
\]
\[
\geq \inf \{ \gamma : (\gamma e - (f + g))_+ \geq \varkappa \} = (f + g)^\langle \varkappa \rangle.
\]
Replacing \( f \) by \( -f \) and \( g \) by \( -g \) and applying (2), we obtain the reverse inequality. Thus we have \( f^\langle \varkappa \rangle + g^\langle \varkappa \rangle = (f + g)^\langle \varkappa \rangle \), as required.

\(<4\): It suffices to prove that \( (f \wedge g)^\langle \varkappa \rangle = \min \{ f^\langle \varkappa \rangle, g^\langle \varkappa \rangle \} \). Since
\[
(\lambda e - (f \wedge g))_- = l((f \wedge g) - \lambda e)_+ = ((\lambda e) \wedge (g - \lambda e))_+
\]
\[
= (f - \lambda e)_+ \wedge (g - \lambda e)_+ = (\lambda e - f)_- \wedge (\lambda e - g)_-,
\]
the condition \( (\lambda e - (f \wedge g))_- \geq \varkappa \) is valid if and only if \( (\lambda e - f)_- \geq \varkappa \) and \( (\lambda e - g)_- \geq \varkappa \). The required assertion follows now from (1). \( \triangleright \)
4.2.2. Let \( \kappa \in \Omega \) and \( e \in E \) be such that \( e \geq \kappa \). Consider the mapping \( h_{\kappa} : E \to \mathbb{R} \) assigning to each \( f \in E \) the element \( f^+(\kappa) \) defined by (1). By the preceding lemma, the restriction of the mapping \( h_{\kappa} \) onto the vector sublattice \( E_{\kappa} := \{ x \in E : |h_{\kappa}(x)| < \infty \} \) of the lattice \( E \) is an \( \mathbb{R} \)-valued Riesz homomorphism on \( E_{\kappa} \).

Let \( h \) be an arbitrary \( \mathbb{R} \)-valued Riesz homomorphism on \( \Omega \). By the general saturation principle, there exists an element \( \kappa \in \Omega \) satisfying the condition \( \kappa \leq x \) for every \( x \in \Omega_+ \) whenever \( h(x) > 0 \). Take an arbitrary \( e \in \Omega_+ \) with \( h(e) = 1 \). Clearly, \( h = h_{\kappa} \). In other words, each real-valued Riesz homomorphism on the vector lattice \( E \) may be represented as \( h_{\kappa} \).

4.2.3. Take some maximal family \( (e_{\sigma})_{\sigma \in S} \) of disjoint nonzero positive elements in a vector lattice \( E \). Put

\[
0^\Omega := \{ \kappa \in \Omega : (\exists \sigma \in S) \, \kappa \leq e_{\sigma} \}
\]

and consider the family of subsets

\[
0^\Omega^x := \{ \kappa \in \Omega : \kappa \leq x \} \quad (x \in \Omega_S)
\]

of the set \( 0^\Omega \), where \( \Omega_S \) is the union of the order intervals \( I_{\sigma} = [0, e_{\sigma}] \) for all \( \sigma \in S \). It is easy to see that \( \{ 0^\Omega^x \}_{x \in \Omega_S} \) is a base for some topology \( \tau \) on \( 0^\Omega \). Throughout this section, we denote by \( (0^\Omega, \tau) \) the corresponding topological space. It is easy to verify that the condition

\[
f^+(\kappa) := \inf \{ \lambda \in \mathbb{R} : (\lambda e_{\sigma(\kappa)} - f)_+ \geq \kappa \}
\]

soundly defines an \( \mathbb{R} \)-valued function on \( (0^\Omega, \tau) \). We now state the main result of the section:

**Theorem.** Under the above assumptions, the function \( f^+ \) belongs to \( C_\infty(0^\Omega) \) for every \( f \in E \). Moreover, the mapping assigning to each element \( f \in E \) the function \( f^+ \) is a Riesz isomorphism of the vector lattice \( E \) onto the vector sublattice \( f^+(E) \) of the space \( C_\infty(0^\Omega) \).

We show that the functions defined by (3) are continuous in the topology of \( 0^\Omega \). Take an arbitrary \( f \in E \). It suffices to establish the continuity of the function \( f^+ \) on the subspaces \( 0^\Omega^{e_{\sigma}} \) \( (\sigma \in S) \) of \( 0^\Omega \). Fix some \( \sigma \in S \) and let \( e := e_{\sigma} \). Consider the sets

\[
P_\lambda := \{ \kappa \in 0^\Omega^e : (\lambda e - f)_+ \geq \kappa \},
\]

\[
N_\lambda := \{ \kappa \in 0^\Omega^e : (\lambda e - f)_- \geq \kappa \}
\]
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for all \( \lambda \in \mathbb{R} \). Then \( \{P_\lambda\}_{\lambda \in \mathbb{R}} \) is an increasing family of open subsets of \( ^0\Lambda^e \), while \( \{N_\lambda\}_{\lambda \in \mathbb{R}} \) is a decreasing family; moreover, \( P_\lambda \cap N_\lambda = \emptyset \) for all \( \lambda \). In addition, since

\[
(se - f)_+ + (te - f)_+ \geq (s - t)e
\]

for arbitrary \( s, t \in \mathbb{R}, s > t \), we have \( P_s \cup N_t = ^0\Lambda^e \). Hence,

\[
\text{cl} P_t \subseteq ^0\Lambda^e \setminus N_t \subseteq P_s = \text{int} P_s \quad (s > t).
\]

By the definition of \( f^\wedge \), the following holds for all \( \varkappa \in ^0\Lambda^e \):

\[
f^\wedge(\varkappa) = \inf \{ \lambda \in \mathbb{R} : \varkappa \in P_\lambda \}.
\]

It is easy to verify that conditions (4) and (5) imply continuity of \( f^\wedge \) on \( ^0\Lambda^e \), as required.

Now we show that functions of the form \( f^\wedge \) are finite on dense subsets of the space \( ^0\Lambda \). As in the proof of continuity, we confine exposition to considering the functions on the subspaces \( ^0\Lambda^e \), where \( e \) is some element of the family \( (e_\sigma)_{\sigma \in S} \). Thus, we must prove that the set \( \mathcal{B}(f) \) is dense in \( ^0\Lambda^e \) for every \( f \in E\). Take an arbitrary \( f \in E \) (we may assume that \( f \geq 0 \)) and suppose that an element \( u \in E, 0 < u \leq e \), satisfies \( f^\wedge(\varkappa) = \infty (\varkappa \in ^0\Lambda^u) \). Then \( u = 0 \). Indeed, the condition \( (\lambda e - f)_+ \geq \varkappa \) fails for all \( \varkappa \in ^0\Lambda^e \) and \( \varkappa \leq u \). Since elements of the set \( ^0\Lambda^e \) are indivisible; therefore,

\[
\varkappa \wedge (\lambda e - f)_+ = 0 \text{ for all } \varkappa \in ^0\Lambda^u, \quad \lambda \in \mathbb{R},
\]

The set \( \Lambda \) is saturating, and so (6) implies that the elements \( u \) and \( (\lambda e - f)_+ \) of the lattice \( E \) are disjoint for all \( \lambda \in \mathbb{R} \). Consequently,

\[
u \wedge (e - (1/n)f)_+ = 0 \quad (n \in \mathbb{N}).
\]

It follows that

\[
u \wedge e = u \wedge \sup_{E}(e - (1/n)f)_+ : n \in \mathbb{N} = 0,
\]

because the lattice \( E \) is Archimedean. At the same time, \( u \leq e \). Hence, \( u = 0 \). Thus, the set \( \{f^\wedge = \infty\} \) does not contain any nonempty open subset of the space \( \Lambda \).

By Lemma 4.2.1, the mapping \( f \mapsto f^\wedge \) is a Riesz homomorphism of the vector lattice \( E \) onto the vector sublattice \( f^\wedge(E) \) of the space \( C_\infty(0\Lambda) \).

To complete the proof of the theorem, it remains to establish that this mapping is injective. To this end, it suffices to verify that the conditions \( f \in E_+ \) and \( f^\wedge = 0 \)
imply \( f = 0 \). Let an element \( f \in E_+ \) satisfy \( f^\wedge(\lambda) = 0 \) for all \( \lambda \in \Lambda^0 \). Choose an arbitrary \( \sigma \in S \). Then

\[
\inf \{ \lambda : (\lambda e_\sigma - f)_+ \geq \lambda \} = 0 \quad (\lambda \leq e_\sigma),
\]

and so \( (f - (1/n)e_\sigma)_+ \wedge \lambda = 0 \) for all \( n \in \mathbb{N} \) and \( \lambda \leq e_\sigma \). Since the set \( \Lambda \) is saturating, it follows that

\[
(f - (1/n)e_\sigma)_+ \wedge e_\sigma = 0 \quad (n \in \mathbb{N}).
\]

The vector lattice \( E \) is Archimedean. Therefore, the last relation implies

\[
e_\sigma \wedge f = e_\sigma \wedge \sup \{ (f - (1/n)e_\sigma)_+ : n \in \mathbb{N} \} = \sup \{ (f - (1/n)e_\sigma)_+ \wedge e_\sigma : n \in \mathbb{N} \} = 0.
\]

Thus, since the choice of \( \sigma \in S \) was arbitrary, the element \( f \) is disjoint from every element of the family \( (e_\sigma)_{\sigma \in S} \), which is possible (since this family is maximal) only if \( f = 0 \). So, the mapping \( f \mapsto f^\wedge \) is injective. The proof of the theorem is complete. \( \triangleright \)

4.2.4. Define some equivalence relation \( \mathcal{R} \) on \( \Lambda \) as follows: \( \lambda_1 \mathcal{R} \lambda_2 \) means that \( f^\wedge(\lambda_1) = f^\wedge(\lambda_2) \) for every \( f \in E \). By Theorem 4.1.4, \( \Lambda \) is a compact topological space. It follows immediately that the quotient space \( \Lambda_{\mathcal{R}} \) of \( \Lambda \) by \( \mathcal{R} \) is compact too. This quotient space is Hausdorff by construction. Given \( \lambda \in \Lambda \), denote by \( (\lambda) \) the coset of \( \lambda \) in the space \( \Lambda_{\mathcal{R}} \). It is easy to see that the formula

\[
\varphi(f)((\lambda)) := f^\wedge(\lambda) \quad (f \in E, \quad \lambda \in \Lambda)
\]

soundly defines the mapping \( \varphi : E \to C_\infty(\Lambda_{\mathcal{R}}) \), where \( C_\infty(\Lambda_{\mathcal{R}}) \) is the space of extended continuous functions on the compact Hausdorff space \( \Lambda_{\mathcal{R}} \). The following lemma is a consequence of Theorem 4.2.3 and the definition of \( \varphi \).

**Lemma.** The mapping \( \varphi \) is a Riesz isomorphism of the vector lattice \( E \) onto the vector sublattice \( \varphi(E) \) of \( C_\infty(\Lambda_{\mathcal{R}}) \). Furthermore, \( \varphi(E) \) separates points of \( \Lambda_{\mathcal{R}} \), and \( \varphi \) maps the element \( e \) to the identically one function.

4.2.5. Let \( E \) be a relatively uniformly complete Archimedean vector lattice with a strong unity \( e \). Then, by the preceding lemma, \( E \) is Riesz isomorphic to the vector sublattice \( \varphi(E) \) of the space \( C(\Lambda_{\mathcal{R}}) \) of continuous functions on the compact Hausdorff space \( \Lambda_{\mathcal{R}} \); moreover, \( \varphi(E) \) separates points of \( \Lambda_{\mathcal{R}} \) and contains all constant functions. Since \( E \) is relatively uniformly complete, the sublattice \( \varphi(E) \) is uniformly closed in \( C(\Lambda_{\mathcal{R}}) \). Applying the Stone theorem, we obtain \( \varphi(E) = C(\Lambda_{\mathcal{R}}) \). Thus, we have
Theorem (S. Kakutani; M. G. Kreĭn and S. G. Kreĭn). For every relatively uniformly complete Archimedean vector lattice $E$ with a strong unity $e$, there exists a compact Hausdorff space $Q$ such that $E$ is Riesz isomorphic to the vector lattice $C(Q)$. Moreover, such an isomorphism may be constructed so as to send the element $e$ to the identically one function.

4.2.6. We also give a sketch of a nonstandard proof of the Ogasawara–Vulikh Theorem.

Theorem (T. Ogasawara; B. Z. Vulikh). For every Dedekind complete vector lattice $E$ with unity $e$, there is an extremally disconnected compact Hausdorff space $Q$ such that $E$ is Riesz isomorphic to an order dense ideal $E'$ of the Dedekind complete vector lattice $C_\infty(Q)$. Moreover, some isomorphism may be constructed so that $C(Q) \subseteq E'$ and the identically one function corresponds to $e$.

Let $E$ be a Dedekind complete vector lattice with unity $e$. Take $\Lambda_\mathcal{E}$ as the compact Hausdorff space $Q$. We first verify that $\Lambda_\mathcal{E}$ is extremally disconnected. It suffices to show that the closure of the union of every family of sets in some base for the topology on $\Lambda_\mathcal{E}$ is open. Consider the base $\{\Lambda_\mathcal{E}^x\}_{x \in E^+}$ of the topology of the space $\Lambda_\mathcal{E}$ constituted by the sets

$$\Lambda_\mathcal{E}^x := \{x \in \Lambda_\mathcal{E} : x \leq x\}.$$ 

Take an arbitrary family $\{\Lambda_\mathcal{E}^x\}_{x \in A}$ of sets in this base. The closure of the union $\bigcup_{x \in A} \Lambda_\mathcal{E}^x$ is open, because it coincides with the set $\Lambda_\mathcal{E}^y$ where $y$ is the band projection of $e$ onto the band generated by the set $A$. The space $C_\infty(\Lambda_\mathcal{E})$ of extended continuous functions on the extremally disconnected compact Hausdorff space $\Lambda_\mathcal{E}$ is a Dedekind complete vector lattice (see [21, Theorem 47.4]). By Lemma 4.2.1, the mapping $\varphi$ defined by (7) is a Riesz isomorphism of $E$ onto the point-separating vector sublattice $\varphi(E)$ of $C_\infty(\Lambda_\mathcal{E})$; furthermore,

$$\varphi(e)[x] = 1 \text{ for all } x \in \Lambda_\mathcal{E}.$$ 

To complete the proof, it remains to show that $\varphi(E)$ is an order-dense ideal in $C_\infty(\Lambda_\mathcal{E})$. The vector lattice $E$ is Dedekind complete. So, it is relatively uniform complete. According to the result of the preceding subsection, $\varphi(E_e) = C(\Lambda_\mathcal{E})$, where $E_e$ is the principal ideal in $E$ generated by the element $e$. Thus, the vector lattice $\varphi(E)$ contains the order-dense ideal $C(\Lambda_\mathcal{E})$ of $C_\infty(\Lambda_\mathcal{E})$. Therefore, to be an order-dense ideal in $C_\infty(\Lambda_\mathcal{E})$, together with each element $\varphi(x) \geq 0$ the set $\varphi(E)$ must contain all elements $f \in C_\infty(\Lambda_\mathcal{E})$ with $0 \leq f \leq \varphi(x)$. Take an arbitrary $x \in E^+$ and let a function $f \in C_\infty(\Lambda_\mathcal{E})$ be such that $0 \leq f \leq \varphi(x)$. Consider the elements

$$f_n := f \wedge n\varphi(e) \ (n \in \mathbb{N})$$
of the space $C_\infty(\Lambda G)$. It is clear that $f_n \in C(\Lambda G)$. Consequently, there are $y_n \in E$ with $f_n = \varphi(y_n)$. Since $\varphi$ is a Riesz isomorphism, $y_n \uparrow \leq x$. By Dedekind completeness of $E$, there exists $y = \sup E\{y_n : n \in N\}$. Obviously, we obtain $f = \varphi(y)$.

4.2.7. In conclusion, we show that in the case when $E$ is a lattice with the principal projection property, the equivalence relation $R$ can be described in a simpler manner.

**Lemma.** Suppose that a vector lattice $E$ has the principal projection property. Then for every $\kappa_1, \kappa_2 \in \Lambda$ we have $\kappa_1 R \kappa_2$ if and only if $\kappa_1$ and $\kappa_2$ have the same standard upper bounds in $E$.

Assume that the elements $\kappa_1$ and $\kappa_2$ have the same sets of standard upper bounds. Then $\kappa_1 R \kappa_2$ follows immediately from the definition of $R$. Conversely, assume that the sets $\{f \in E : f \geq \kappa_1\}$ and $\{f \in E : f \geq \kappa_2\}$ are distinct. For instance, take $x \in E_+$ so that $x \geq \kappa_1$ and $x \not\geq \kappa_2$. Then $x \wedge \kappa = 0$ because the element $\kappa$ is indivisible. Consider the band projection $pr_x(e)$ of the unity $e$ of the lattice $E$ onto the principal band generated by $x$. It is easy to see that $\tilde{g}(\kappa_1) = 1$ and $\tilde{g}(\kappa_2) = 0$. Consequently, $(\kappa_1, \kappa_2) \notin R$.

4.3. Order, Relative Uniform Convergence, and the Archimedes Principle

We now introduce some types of infinitesimal elements in a nonstandard enlargement of a vector lattice and use them for a nonstandard description of various kinds of convergence. Also we obtain a nonstandard criterion for a vector lattice to be Archimedean.

4.3.1. Let $E$ be a vector lattice. Given $\kappa \in \mathcal{L} E$, we consider the set $U(\kappa) := \{x \in E : x \geq \kappa\}$ of standard upper bounds of $\kappa$ and the set $L(\kappa) := \{y \in E : \kappa \geq y\}$ of standard lower bounds of $\kappa$. Define the following external subsets of some nonstandard enlargement $\mathcal{L} E$ of the vector lattice $E$:

\[
\text{fin}(\mathcal{L} E) := \{\kappa \in \mathcal{L} E : U(\kappa) \neq \emptyset\},
\]

\[
\alpha-pns(\mathcal{L} E) := \{\kappa \in \mathcal{L} E : \inf_{E} (U(\kappa) - L(\kappa)) = 0\},
\]

\[
\eta(\mathcal{L} E) := \{\kappa \in \mathcal{L} E : \inf_{E} U(\kappa) = 0\},
\]

\[
\lambda(\mathcal{L} E) := \{\kappa \in \mathcal{L} E : (\exists y \in E)(\forall n \in N) |n\kappa| \leq y\}.
\]

It is easy to see that $\text{fin}(\mathcal{L} E)$, $\alpha-pns(\mathcal{L} E)$, $\eta(\mathcal{L} E)$, and $\lambda(\mathcal{L} E)$ are vector lattices with respect to the lattice operations, addition, and multiplication by scalars in $\mathbb{R}$.
which are inherited from the standard vector lattice $^*E$. The elements of \( \text{fin}(^*E) \) we call \textit{finite} or \textit{limited}; the elements of \( \text{o-pns}(^*E) \) we call \textit{(o)-prenearstandard}; the elements of \( \eta(^*E) \) we call \textit{(o)-infinitesimal}; the elements of \( \lambda(^*E) \) we call \textit{(r)-infinitesimal}. The elements of \( E + \eta(^*E) \) (of \( E + \lambda(^*E) \)) we call \textit{(o)-nearstandard} (respectively, \textit{(r)-nearstandard}). Note some simple properties:

1. \( E \) is a vector sublattice in \( \text{o-pns}(^*E) \), while \( \text{o-pns}(^*E) \) is a vector sublattice in \( \text{fin}(^*E) \);
2. \( \eta(^*E) \) is an ideal both in \( \text{fin}(^*E) \) and \( \text{o-pns}(^*E) \);
3. \( E \cap \eta(^*E) = \{0\} \);
4. \( \lambda(^*E) \) is an ideal in \( \text{fin}(^*E) \).

4.3.2. There are simple nonstandard conditions for a monotone net to be order convergent.

Let \((x_\alpha)_{\alpha \in \Xi}\) be a decreasing or increasing net in a vector lattice \( E \). Then the following are equivalent:

1. \( x_\alpha \) converges to \( 0 \);
2. \( x_\beta \in \eta(^*E) \) for all \( \beta \in ^{o} \Xi \);
3. \( x_\beta \in \eta(^*E) \) for some \( \beta \in ^{o} \Xi \).

\( \downarrow \) We consider only the case of a decreasing net.

\( 1 \to 2 \): Assume that \( x_\alpha \downarrow 0 \). Then every remote element \( \beta \in ^{o} \Xi \) satisfies \( x_\alpha \geq x_\beta \geq 0 \) for all \( \alpha \in \Xi \). Consequently, \( \inf_E U(|x_\beta|) = 0 \) and \( x_\beta \in \eta(^*E) \).

\( 2 \to 3 \): This is an immediate consequence of Lemma 4.0.5.

\( 3 \to 1 \): Take an element \( \beta \in ^{o} \Xi \) for which \( x_\beta \in \eta(^*E) \). Since \( x_\alpha \downarrow 0 \), we have \( (x_\beta)_- \geq (x_\alpha)_- \geq 0 \) for all \( \alpha \in \Xi \). In view of \( x_\beta \in \eta(^*E) \), we have \( (x_\alpha)_- = 0 \) for all \( \alpha \in \Xi \). Consequently, \( x_\alpha \downarrow 0 \). Let \( y \in E_+ \) be an arbitrary element such that \( x_\alpha \downarrow y \). By the transfer principle, each \( \alpha \in ^* \Xi \) satisfies \( x_\alpha \geq y \). In particular, \( x_\beta \geq y \). This is possible only if \( y = 0 \). Hence \( x_\alpha \downarrow 0 \).

It is easy to see that \( 1 \to 2 \) and \( 2 \to 3 \) are true for an arbitrary (not necessarily monotone) net \( (x_\alpha)_{\alpha \in \Xi} \subseteq E \). But the implication \( 3 \to 1 \) may be false without the monotonicity condition. Indeed, let \( E := L_1[0,1] \). For each \( n \in \mathbb{N} \), we take elements \( f_1^n, f_2^n, \ldots, f_{2^n}^n \in E \) such that \( f_k^n \) is the equivalence class containing the characteristic function of the interval \( \left[ \frac{k-1}{2^n}, \frac{k}{2^n} \right] \). Arrange these elements in the sequence

\[
\begin{align*}
    f_1^1, f_2^1, f_3^2, f_4^2, \ldots, f_1^n, f_2^n, \ldots, f_{2^n}^n, \ldots
\end{align*}
\]

Obviously, \( 2 \) and \( 3 \) hold for this sequence, but it does not converge in order to any element of \( E \).
4.3.3. We now give nonstandard conditions under which a monotone sequence converges relatively uniformly.

Let \((x_n)\) be a decreasing or increasing sequence of elements in a vector lattice \(E\). Then the following are equivalent:

1. \(x_n \xrightarrow{(r)} 0\);
2. \(x_\nu \in \lambda(*E)\) for every \(\nu \in *\mathbb{N} \setminus \mathbb{N}\);
3. \(x_\nu \in \lambda(*E)\) for some \(\nu \in *\mathbb{N} \setminus \mathbb{N}\).

\(<\) We verify only \((2) \implies (1)\) in the case of a decreasing sequence. Let \(x_\nu \in \lambda(*E)\) for some \(\nu \in *\mathbb{N} \setminus \mathbb{N}\). It is clear that \(x_n \geq 0\) for all \(n \in \mathbb{N}\). Assume that the condition \(x_n \xrightarrow{(r)} 0\) is false. Then, for every \(d \in E\), there is a number \(n(d) \in \mathbb{N}\) such that \(n(d) \cdot x_k \not\leq d\) for all \(k \in \mathbb{N}\). By the transfer principle, \(n(d) \cdot x_k \not\leq d\) for all \(k \in *\mathbb{N}\). In particular, \(n(d) \cdot x_\nu \not\leq d\), contradicting to \(x_\nu \in \lambda(*E)\). So, \(x_n \xrightarrow{(r)} 0\).\(>\)

As in 4.3.2, observe that \((1) \implies (2)\) and \((2) \implies (3)\) hold for every sequence \((x_n) \subseteq E\). At the same time, the implication \((3) \implies (1)\) can be false without the monotonicity condition. This may be checked by considering the example in 4.3.2. Indeed, it is easy to see that the constructed sequence satisfies conditions (2) and (3) but not (1).

4.3.4. We now give nonstandard conditions for a vector lattice to be Archimedean. We start with proving one auxiliary assertion.

Lemma. Let \(u\) be an element of a vector lattice \(E\) and let \(\nu \in *\mathbb{N} \setminus \mathbb{N}\). Then either \(u = 0\) or \(\nu u \not\in \text{o-pns}(\eta\nu)\).

\(<\) Let \(u \neq 0\). Take arbitrary \(x \in L(|\nu u|)\) and \(y \in U(|\nu u|)\). Then \(x \leq |\nu u| \leq y\). By the transfer principle, we find \(m \in \mathbb{N}\) such that \(x \leq |m\nu| \leq y\). Comparing this inequality with the previous, we obtain \(|u| \leq |\nu u| - |m\nu| \leq y - x\). Since the choice of the elements \(x\) in \(L(|\nu u|)\) and \(y\) in \(U(|\nu u|)\) is arbitrary, we have \(U(|\nu u|) - L(|\nu u|) \geq |u| > 0\). Thus, \(|\nu u|\) and, consequently, \(\nu u\) do not belong to \(\text{o-pns}(\eta\nu)\).\(>\)

4.3.5. Theorem. For every vector lattice \(E\), the following are equivalent:

1. \(E\) is an Archimedean vector lattice;
2. \(\lambda(*E) \cap E = \{0\}\);
3. \(\lambda(*E) \subseteq \eta(*E)\);
4. \(\lambda(*E) \subseteq \text{o-pns}(\eta\nu)\);
5. the set \(\text{o-pns}(\eta\nu)\) is a relatively uniformly closed vector sublattice of \(\text{fin}(\eta\nu)\).
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(6) \( \eta(\ast E) \) is a relatively uniformly closed ideal of \( \text{fin}(\ast E) \).

We will prove the theorem by the following scheme:

\[ (1) \rightarrow (2) \rightarrow (3) \rightarrow (4) \rightarrow (1) \quad \text{and} \quad (1) \rightarrow (5) \rightarrow (6) \rightarrow (1). \]

(1)\( \rightarrow \) (2): This is obvious.

(2)\( \rightarrow \) (3): Let \( \kappa \in \lambda(\ast E) \setminus \eta(\ast E) \). Then there exists a \( y \in E \) such that \( |n \kappa| \leq y \) for all \( n \in \mathbb{N} \), and therefore, \( \frac{1}{n} y \in U(|\kappa|) \) for all \( n \in \mathbb{N} \). Since \( \kappa \notin \eta(\ast E) \), there exists a \( z \in E \) satisfying \( 0 < z \leq U(|\kappa|) \). In particular, \( 0 < z \leq (1/n)y \) for all \( n \in \mathbb{N} \). So, \( 0 \neq z \in \lambda(\ast E) \cap E \), which contradicts (2).

(3)\( \rightarrow \) (4): This is true since \( \eta(\ast E) \subseteq o\text{-}pns(\ast E) \).

(4)\( \rightarrow \) (1): Take arbitrary elements \( u, v \in E \) such that \( 0 \leq nu \leq v \) \( (n \in \mathbb{N}) \) and let \( v \in \ast \mathbb{N} \setminus \mathbb{N} \). Then it is obvious that \( nu \in \lambda(\ast E) \), and, by hypothesis, \( nu \in o\text{-}pns(\ast E) \). Hence, by Lemma 4.3.4, we have \( u = 0 \). So, \( E \) is Archimedean.

(1)\( \rightarrow \) (5): Take a sequence \( (\kappa_n) \) of elements in the vector lattice \( o\text{-}pns(\ast E) \) which converges relatively uniformly to some element \( \kappa \in \text{fin}(\ast E) \). Show that \( \kappa \) belongs to \( o\text{-}pns(\ast E) \). We may suppose that \( (\kappa_n) \) converges \( e \)-uniformly to \( \kappa \) for some \( e \in E \). Then there is a sequence \( \varepsilon_n \downarrow 0 \) of real numbers such that \( |\kappa_k - \kappa| \leq \varepsilon_n e \) for all natural \( k \geq n \). For every \( n \in \mathbb{N} \), we have \( \kappa_n - \varepsilon_n e \leq \kappa \leq \kappa_n + \varepsilon_n e \), and so

\[
L(\kappa_n - \varepsilon_n e) \leq \kappa \leq U(\kappa_n + \varepsilon_n e). \tag{1}
\]

Given \( n \in \mathbb{N} \), assign

\[
\mathcal{E}_n := U(\kappa_n + \varepsilon_n e) - L(\kappa_n - \varepsilon_n e).
\]

The inclusion \( (\kappa_n) \subseteq o\text{-}pns(\ast E) \) implies, by (1), that

\[
\inf_{E} \mathcal{E}_n = 2\varepsilon_n e \quad (n \in \mathbb{N}). \tag{2}
\]

Since \( E \) is Archimedean, it follows from (2) that \( \inf_{E} \bigcup_{n=1}^{\infty} \mathcal{E}_n = 0 \), and hence \( \inf_{E}(U(\kappa) - L(\kappa)) = 0 \). We have used the inclusion \( \bigcup_{n=1}^{\infty} \mathcal{E}_n \subseteq U(\kappa) - L(\kappa) \) which ensues from (1). Thus, \( \kappa \in o\text{-}pns(\ast E) \). Consequently, \( o\text{-}pns(\ast E) \) is relatively uniformly closed in \( \text{fin}(\ast E) \).

(5)\( \rightarrow \) (6): Let \( \kappa_n \in \eta(\ast E) \) and \( \kappa_n \xrightarrow{(r)} \kappa \in \text{fin}(\ast E) \). Then \( \kappa \in o\text{-}pns(\ast E) \), by hypothesis. Check that \( \kappa \in \eta(\ast E) \). We may assume that \( \kappa_n \xrightarrow{(r)} \kappa \) \( d \)-uniformly for some \( d \in E \). This means that \( |\kappa_n - \kappa| \leq \varepsilon_n d \) for all \( n \in \mathbb{N} \) and some appropriate sequence \( (\varepsilon_n) \subseteq \mathbb{R} \), \( \varepsilon_n \downarrow 0 \). Assume that \( \kappa \notin \eta(\ast E) \). Take an arbitrary element
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Let $a \in E$ satisfying $U(|x|) \geq a \geq 0$. For each $n \in \mathbb{N}$, choose an arbitrary $u_n \in U(|x_n|)$. It is obvious that $u_n + \varepsilon_n d \geq |x_n| + \varepsilon_n d \geq |x|$. We thus have

$$U(|x_n|) + \varepsilon_n d \subseteq U(|x|)$$

and

$$U(|x_n|) + \varepsilon_n d \geq a.$$

Now it follows from $\inf_E U(|x_n|) = 0$ that $\varepsilon_n d \geq a$. This inequality is true for all natural $n$ and the sequence $(\varepsilon_n)$ decreasing to zero. Therefore, $d \geq ka$ for every $k \in \mathbb{N}$. Applying the transfer principle, we obtain $d \geq ka$ for all $k \in \ast \mathbb{N}$. Take some $\nu \in \ast \mathbb{N} \setminus \mathbb{N}$. It is easy to see that the sequence $(ka)_{k=1}^{\infty}$ of elements of $E$ converges $d$-uniformly to the element $\nu a$ of the vector lattice $\text{fin}(\ast E)$. Since, by hypothesis, $\text{o-pns}(\ast E)$ is a relatively uniformly closed vector sublattice of $\text{fin}(\ast E)$, we have $\nu a \in \text{o-pns}(\ast E)$. By Lemma 4.3.4, this implies $a = 0$. Thus, $\inf_E U(|x|) = 0$, and so $x \in \eta(\ast E)$, as required.

(6) $\rightarrow$ (1): Take arbitrary $u, v \in E$ satisfying $0 \leq nu \leq v$ for all $n \in \mathbb{N}$. Show that $u = 0$. Let $\nu \in \ast \mathbb{N} \setminus \mathbb{N}$. It is easy to see that a sequence $(x_n)$ with $x_n = 0$ for all $n \in \mathbb{N}$ converges $\nu$-uniformly to an element $\nu u$. By hypothesis, the ideal $\eta(\ast E)$ is relatively uniformly closed in $\text{fin}(\ast E)$, so $\nu u \in \eta(\ast E)$. Therefore, by Lemma 4.3.4, we have $u = 0$.

4.3.6. Theorem. For a vector lattice $E$ the following are equivalent:

(1) $E$ is an order separable Archimedean vector lattice in which order convergence and relative uniform convergence coincide for any sequence;

(2) $\eta(\ast E) = \lambda(\ast E)$.

$\Leftarrow$ (1) $\rightarrow$ (2): $E$ satisfies the inclusion $\lambda(\ast E) \subseteq \eta(\ast E)$ by Theorem 4.3.5. Prove the reverse inclusion. Take an arbitrary $x \in \eta(\ast E)$. Then $\inf_E U(|x|) = 0$. Since $U := U(|x|)$ is a downwards-directed set such that $U \downarrow 0$ and since $E$ is an order separable vector lattice, there is a sequence $(u_n) \subseteq U$ with $u_n \downarrow 0$. The condition (1) implies $u_n \xrightarrow{(r)} 0$. Then, by 4.3.3, $u_\nu \in \lambda(\ast E)$ for all $\nu \in \ast \mathbb{N} \setminus \mathbb{N}$. Consequently, $x \in \lambda(\ast E)$, because $|x| \leq u_n$ for all $n \in \ast \mathbb{N}$.

(2) $\rightarrow$ (1): $E$ is Archimedean by Theorem 4.3.5. Show that order convergence and relative uniform convergence coincide for every sequence in $E$. It is sufficient to prove that $u_n \downarrow 0$ implies $u_n \xrightarrow{(r)} 0$. Take an arbitrary sequence $(u_n) \subseteq E$ such that $u_n \downarrow 0$. Then, by 4.3.2, $u_\nu \in \eta(\ast E)$ for all $\nu \in \ast \mathbb{N} \setminus \mathbb{N}$. Hence $u_\nu \in \lambda(\ast E)$ for all $\nu \in \ast \mathbb{N} \setminus \mathbb{N}$. Now we see from 4.3.3 that $u_n \xrightarrow{(r)} 0$.

It remains to verify that the vector lattice $E$ is order separable. Take an arbitrary net $(x_\xi)_{\xi \in \Xi} \subseteq E$ such that $x_\xi \downarrow 0$. By Lemma 4.1.5, there exists remote element $r$ in the standard directed set $\ast \Xi$. Then, by 4.3.2, we have $x_r \in \eta(\ast E)$, and
so \( x_\tau \in \lambda(*E) \). In this case, there is \( d \in E \) satisfying \( nx_\tau \leq d \) for all \( n \in \mathbb{N} \). Assume that \((x_\xi)\) does not contain a subsequence convergent relatively uniformly to zero. Then there is a number \( n_0 \in \mathbb{N} \) such that \( n_0 x_\xi \nless d \) for all \( \xi \in \Xi \). By the transfer principle, \( n_0 x_\xi \nless d \) for every \( \xi \in *\Xi \). The contradiction with \( n_0 x_\tau \leq d \) ensures existence of a subsequence \((x_{\xi_n}) \subseteq (x_\xi)\) with \( x_{\xi_n} \rightarrow (r) 0 \). Since \( E \) is Archimedean, we have \( x_{\xi_n} \rightarrow (o) 0 \). Thus, \( E \) is order separable. \( \triangleright \)

### 4.4. Conditional Completion and Atomicity

In this section, we give a nonstandard construction of a Dedekind completion of an Archimedean vector lattice. Also, we give an infinitesimal interpretation for the property of a vector lattice to be atomic.

**4.4.1.** Let \( E \) be a vector lattice. Consider the quotient vector lattice \( \hat{E} := \operatorname{o-pns}(E)/\eta(E) \) and denote by \( \hat{\eta} \) the mapping \( x \mapsto [x] \), where \( x \in E \) and \( [x] \in \hat{E} \) is the coset containing \( x \).

**Theorem.** For every Archimedean vector lattice \( E \), the following hold:

1. \( \hat{E} \) is Dedekind complete;
2. \( \hat{\eta} \) is a Riesz isomorphism of the vector lattice \( E \) into the vector lattice \( \hat{E} \);
3. for every \( x \in E \)

   \[
   x = \sup_{\hat{E}} \{ y \in \hat{\eta}(E) : y \leq x \} = \inf_{\hat{E}} \{ y \in \hat{\eta}(E) : y \geq x \}.
   \]

In other words, the vector lattice \( \hat{E} \) is a Dedekind completion of \( E \).

We prove the theorem in four steps:

**Step 1.** Let \( E \) be an arbitrary vector lattice. Then, for every \( 0 < x \in \hat{E} \), there exists an element \( e \in E \) such that \( 0 < \hat{\eta}(e) \leq x \).

\(< \) Take an element \( x \in \hat{E}, x > 0 \). Let \( \kappa \in \operatorname{o-pns}(E) \) be such that \( \kappa > 0 \) and \( x = [\kappa] \). Then there exists an \( e \in E \) for which \( 0 < e \leq \kappa \). Indeed, in the other case, \( \sup_E L(\kappa) = 0 \). Consequently, \( \inf_E U(\kappa) = 0 \), since \( \kappa \in \operatorname{o-pns}(E) \). This contradicts \( [\kappa] = x \neq 0 \). So, \( e \) is a sought element. \( \triangleright \)

**Step 2.** Given \( x \in \hat{E} \), assign

\[
\hat{\mathcal{U}}(x) := \{ y \in \hat{\eta}(E) : y \geq x \}; \quad \hat{\mathcal{L}}(x) := \{ z \in \hat{\eta}(E) : x \geq z \}.
\]

Then we have

\[
x = \inf_{\hat{E}} \hat{\mathcal{U}}(x) = \sup_{\hat{E}} \hat{\mathcal{L}}(x).
\]
Let \( \kappa \in \alpha\text{-pns}(E) \) and let \( x = [\kappa] \). To prove the claim it suffices to check that every element \( y \in \bar{E} \) satisfying \( \bar{L}(x) \leq y \leq \bar{U}(x) \) is equal to \( x \). Take an arbitrary \( y \in \bar{E} \) such that \( \bar{L}(x) \leq y \leq \bar{U}(x) \). Assume \( |x - y| > 0 \). Since the \( \bar{\eta}(E) \) is minorant in \( \bar{E} \) by Step 1, there exists an \( e \in E \) satisfying
\[
\bar{U}(x) - \bar{L}(x) \geq |x - y| \geq \bar{\eta}(e) > 0. \tag{3}
\]
It is easy to see that
\[
\bar{\eta}(U(\kappa)) \subseteq \bar{U}(x) \quad \text{and} \quad \bar{\eta}(L(\kappa)) \subseteq \bar{L}(x).
\]
Now, the inequality (3) implies
\[
\bar{\eta}(U(\kappa) - L(\kappa)) \geq \bar{\eta}(e) > 0,
\]
and consequently \( U(\kappa) - L(\kappa) \geq e > 0 \). We obtained a contradiction to \( \kappa \in \alpha\text{-pns}(E) \). Hence, \( |x - y| = 0 \), and \( y = x \). \( \triangleright \)

**Step 3.** Given an Archimedean vector lattice \( E \), every nonempty subset \( D \) in \( \bar{\eta}(E) \) bounded above has a least upper bound in \( \bar{E} \).

\(< \) Let \( D \subseteq \bar{\eta}(E) \) is nonempty and bounded above. Then the subset \( \mathcal{D} := \bar{\eta}^{-1}(D) \) of \( E \) is bounded above in \( E \). Denote by \( U(\mathcal{D}) \) the set of all upper bounds of \( \mathcal{D} \) in \( E \). Since \( E \) is Archimedean, we have
\[
\inf_{\bar{E}}(U(\mathcal{D}) - \mathcal{D}) = 0. \tag{4}
\]
Applying the general saturation principle, find an element \( \delta \in \ast E \) such that
\[
\mathcal{D} \leq \delta \leq U(\mathcal{D}). \tag{5}
\]
From (4) and (5), it follows that \( \inf_{\bar{E}}(U(\delta) - L(\delta)) = 0 \). Hence \( \delta \in \alpha\text{-pns}(E) \). The element \( [\delta] \in \bar{E} \) is an upper bound of the set \( D = \bar{\eta}(\mathcal{D}) \). Show that \( [\delta] = \sup_{\bar{E}} D \).
Let \( y \in \bar{E} \) be some upper bound of \( D \) such that \( [\delta] \geq y \). By (5) we have
\[
0 \leq [\delta] - y \leq \bar{\eta}(U(\mathcal{D}) - \mathcal{D}). \tag{6}
\]
According to Step 1, the vector lattice \( \bar{\eta}(E) \) is minorant in \( \bar{E} \). Therefore, (4) and (6) imply \( y = [\delta] \). Thus, \( [\delta] = \sup_{\bar{E}} D \). \( \triangleright \)
**Step 4. Proof of the Theorem.**

< Assertion (2) is obvious. Assertion (3) is valid in view of Step 2. It is interesting to note that (2) and (3) hold in an arbitrary vector lattice. Verify the condition (1). Take an arbitrary nonempty subset \( A \) in \( E \) bounded above. Denote
\[
\mathcal{A} := \{ x \in E : (\exists a \in A) \hat{\eta}(x) \leq a \}.
\]
According to Step 3, the set \( \hat{\eta}(\mathcal{A}) \) has a least upper bound in \( \hat{E} \). Assign \( a := \sup_{E} \hat{\eta}(\mathcal{A}) \). It is easy to see that \( a = \sup_{E} A \). Thus, every nonempty subset in \( E \) bounded above has a least upper bound, as required.>

4.4.2. The preceding theorem easily implies the following assertion to which we prefer to give a simpler and more direct proof:

**Theorem.** For every Archimedean vector lattice \( E \), the following are equivalent:

1. The vector lattice \( E \) is Dedekind complete;
2. \( o\)-pns\((E) = E + \eta(E)\).

< (1)→(2): Obviously, \( E + \eta(E) \subseteq o\)-pns\((E)\). Show the reverse inclusion. Take an arbitrary \( \varkappa \in o\)-pns\((E) \). Then \( \varkappa \in \text{fin}(E) \). So, \( U(\varkappa) \) is nonempty. Therefore, \( L(\varkappa) \) is bounded above. Since \( E \) is Dedekind complete, \( L(\varkappa) \) has a least upper bound. Assign \( a := \sup_{E} L(\varkappa) \). It is easy to see that \( L(\varkappa) \leq a \leq U(\varkappa) \). Hence \( |\varkappa - a| \leq U(\varkappa) - L(\varkappa) \). Since \( \varkappa \in o\)-pns\((E) \), the last inequality implies that \( \inf_{E} U(|\varkappa - a|) = 0 \). We have \( \varkappa = a + (\varkappa - a) \) with \( a \in E \) and \( \varkappa - a \in \eta(E) \). Consequently \( \varkappa \in E + \eta(E) \).

(2)→(1): It suffices to show that every net \((u_{\varepsilon})_{\varepsilon \in \Xi} \subseteq E \) such that \( u_{\varepsilon} \uparrow \leq d \in E \) is order convergent. Assume that \( u_{\varepsilon} \uparrow \leq d \in E \). It is well known (see, for example, [21, Theorem 22.5]) that the following condition holds in an Archimedean vector lattice \( E \):

\[
\inf_{E} \{ y - u_{\varepsilon} : \varepsilon \in \Xi, y \in E, u_{\varepsilon} \uparrow \leq y \} = 0.
\]

Fix a remote element \( \tau \in a\Xi \). It is easy to see that \( \{ y \in E : u_{\varepsilon} \uparrow \leq y \} = U(\tau) \). Moreover, \( (u_{\varepsilon}) \subseteq L(\tau) \). Thus, (7) implies \( \inf_{E} \{ U(\tau) - L(\tau) \} = 0 \) or, in other words, \( u_{\tau} \in o\)-pns\((E) \). Then \( u_{\tau} \in E + \eta(E) \) by (2). Let \( u \in E \) be such that \( u_{\tau} - u \in \eta(E) \). Thus, 4.3.2 implies that the net \((u_{\varepsilon})\) converges in order to \( u \).>

4.4.3. Now we consider the property of a vector lattice to be atomic. Recall that a vector lattice \( E \) is **atomic** if \( E \) is Archimedean and for every \( 0 < x \in E \) there exists an atom \( a \in E \) such that \( 0 < a \leq x \). Also, we recall that, for every atom \( a \) in an Archimedean vector lattice and for each element \( 0 \leq x \leq a \), there is a real \( \alpha \) such that \( x = \alpha a \). We start with the following
Chapter 4

Let $\mathbf{E}$ be an atomic vector lattice. Then $\text{fin}(\mathbf{E}) = o\text{-pns}(\mathbf{E})$.

It suffices to verify that every element $\mathbf{x} \in \text{fin}(\mathbf{E})$, $\mathbf{x} \geq 0$, satisfies $\mathbf{x} \in o\text{-pns}(\mathbf{E})$. Let $\mathbf{x}$ be an arbitrary positive element of $\text{fin}(\mathbf{E})$. Assume that $U(\mathbf{x}) - L(\mathbf{x}) \geq x > 0$. Then, by hypothesis, there exists an atom $a \in \mathbf{E}$ such that $U(\mathbf{x}) - L(\mathbf{x}) \geq a > 0$. Take an element $u \in U(\mathbf{x})$. Since $E$ is an Archimedean vector lattice, there exists a number $n \in \mathbb{N}$ for which $na \not\geq u$. The element $a$ is an atom; so we have $u \wedge na = \alpha a$ and $\mathbf{x} \wedge na = \beta a$ for appropriate $\alpha, \beta \in [0, n]$. Assign $l' := \text{st}(\beta - 1/3) \cdot a$ and $u' := u - \text{st}(\alpha - \beta - 1/3) \cdot a$,

where $\text{st}$ is the taking of the standard part of a real. Then $u' \in U(\mathbf{x})$ and $l' \in L(\mathbf{x})$, but $u' - l' \not\geq a$; a contradiction. Consequently, $\inf_{\mathbf{E}}(U(\mathbf{x}) - L(\mathbf{x})) = 0$, and so $\mathbf{x} \in o\text{-pns}(\mathbf{E})$.

4.4.4. The condition $\text{fin}(\mathbf{E}) = o\text{-pns}(\mathbf{E})$ is not only necessary but also sufficient for a vector lattice $E$ to be atomic. To prove this, we need to introduce the concept of punch of a positive element of a vector lattice.

**Definition.** Let $E$ be a vector lattice and let $e \in E_+$. An element $\mathbf{x}$ of a nonstandard enlargement $*E$ of $E$ is said to be an $e$-punch if

1. $0 \leq \mathbf{x} \leq e$;
2. $\inf_{\mathbf{E}}\{y \in E : y \geq \mathbf{x}\} = e$;
3. $\sup_{\mathbf{E}}\{z \in E : \mathbf{x} \geq z\} = 0$.

We recall that an element $e$ of the vector lattice $E$ is called nonatomic if $|e| \wedge a = 0$ for any atom $a \in E$. Below we will need the following easy

**Remark.** For every nonatomic element $e \in E$, $e > 0$, and every natural number $n$, there is a family $\{e_k\}_{k=1}^n \subseteq E$ of disjoint elements satisfying $0 < e_k \leq e$ for all $k = 1, \ldots, n$.

**Lemma.** Let $E$ be an arbitrary Archimedean vector lattice. Then, for every $\nu \in *\mathbb{N}$ and every nonatomic element $e \in E$, $e \geq 0$, there exists a family $\{e_k\}_{k=1}^\nu \subseteq *E$ of disjoint $e$-punches.

Take an arbitrary $\nu \in *\mathbb{N}$, and let $e \geq 0$ be some nonatomic element in $E$. Since the assertion of the lemma is obvious for $e = 0$, we suppose that $e > 0$. Denote by $L$ the set of positive elements of the principal ideal $E_e$ generated by $e$ in $E$. It is obvious that $L$ is a lattice with zero. By Lemma 4.1.3, there exists a hyperfinite saturating family $\{x_n\}_{n=1}^\omega$ of disjoint indivisible elements in a nonstandard enlargement $*L$ of $L$. Clearly, every $n = 1, \ldots, \omega$ satisfies $0 < x_n \leq e$. Applying the transfer principle and remark before the lemma under proof, we can easily see
that, for each \( n = 1, \ldots, \omega \), there exists a hyperfinite family \( \{ \gamma_n^k \}_{k=1}^{\nu+1} \subseteq \ast E \) such that
\[
0 < \gamma_n^k \leq x_n \quad (k = 1, \ldots, \nu + 1); \quad \gamma_n^k \land \gamma_n^p = 0 \quad (k \neq p).
\]
Applying the transfer principle once again and using the fact that the vector lattice \( E \) is Archimedean, we find a hyperfinite family \( \{ \alpha_n^k \}_{n=1}^{\omega}; \{ \beta_n^k \}_{k=1}^{\nu} \subseteq \ast \mathbb{R} \) with the following properties:

1. \( 0 < \alpha_n^k \gamma_n^k \leq x_n \) for all \( n = 1, \ldots, \omega \) and \( k = 1, \ldots, \nu \);
2. the condition \( \alpha > \alpha_n^k \) implies that
\[
\alpha \gamma_n^k \leq x_n \text{ for all } n = 1, \ldots, \omega, \; k = 1, \ldots, \nu, \; \text{and } \alpha \in \ast \mathbb{R}.
\]
Put \( e_k := \bigvee_{\omega} \alpha_n^k \gamma_n^k \) for all \( k = 1, \ldots, \nu \). It is easy to verify that \( \{ e_k \}_{k=1}^{\nu} \) is the desired family of \( \nu \) disjoint \( \varepsilon \)-punches.

4.4.5. Theorem. For every vector lattice \( E \), the following are equivalent:

1. \( E \) is an atomic vector lattice;
2. \( \text{fin}(\ast E) = o\text{-pns}(\ast E) \).

\(< (1) \rightarrow (2) : \) This is established in Lemma 4.4.3.
\( (2) \rightarrow (1) : \) Let \( \text{fin}(\ast E) = o\text{-pns}(\ast E) \). In particular, \( o\text{-pns}(\ast E) \) is an \( (r) \)-closed vector sublattice of \( \text{fin}(\ast E) \). Therefore, \( E \) is Archimedean by Theorem 4.3.5. Verify that \( E \) is atomic. It is sufficient to show that \( E \) has no nonzero nonatomic elements. Take an arbitrary nonatomic element \( e \in E \). We may suppose that \( e \geq 0 \). By Lemma 4.4.4, there exists an \( e \)-punch \( \varkappa \in \ast E \). The element \( \varkappa \) satisfies \( \inf_E(U(\varkappa) - L(\varkappa)) = e \). At the same time, \( \varkappa \) is finite and, by hypothesis, \( \varkappa \) is an \( (o) \)-prenearstandard element of \( \ast E \). So, \( e = 0 \).

4.4.6. As an application of the last theorem, we establish some useful nonstandard criterion for a vector lattice to be atomic and Dedekind complete.

Theorem. For every vector lattice \( E \), the following are equivalent:

1. \( E \) is a Dedekind complete atomic vector lattice;
2. \( \text{fin}(\ast E) = E + \eta(\ast E) \).

\(< \) Observe that \( E + \eta(\ast E) \subseteq o\text{-pns}(\ast E) \subseteq \text{fin}(\ast E) \) and use Theorems 4.4.5 and 4.4.2.

We point out that, in the proof of Theorem 4.4.5, we used only the fact that there exists one \( e \)-punch for a nonatomic element \( e \in E \). We will need the Lemma 4.4.4 in full strength below, in the proof of a criterion for a vector lattice \( E \) to be isomorphic to the order hull of \( E \).
4.5. Normed Vector Lattices

In this section, we consider normed vector lattices and study some of their infinitesimal interpretations. Throughout the section we assume \((E, \rho)\) to be a normed vector lattice.

4.5.1. It is well known that, in a nonstandard enlargement of \(E\), together with \(\text{fin}(^*E)\), \(\alpha\)-pns\((^*E)\), \(\eta\)(\(^*E\)), and \(\lambda\)(\(^*E\)), we may also consider the following subsets:

\[
\text{Fin}(^*E) := \{ x \in ^*E : \rho(x) \in \text{fin}(^*\mathbb{R}) \};
\]
\[
\text{pns}(^*E) := \{ x \in ^*E : (\forall n \in \mathbb{N})(\exists y \in E) \ n\rho(x - y) \leq 1 \};
\]
\[
\mu(^*E) := \{ x \in ^*E : \rho(x) \approx 0 \}.
\]

It is easy to see that these are vector lattices over \(\mathbb{R}\) under the operations inherited from \(^*E\). Furthermore, \(\text{pns}(^*E)\) is a vector sublattice of \(\text{Fin}(^*E)\), while \(\mu(^*E)\) is an ideal in \(\text{pns}(^*E)\) as well as in \(\text{Fin}(^*E)\).

4.5.2. Let \(E\) be a vector lattice. If there exists a strong unity \(e \in E\) then we may introduce the Riesz norm \(\| \cdot \|_e\) on \(E\) by the well known formula

\[
\|x\|_e := \inf\{ \lambda \in \mathbb{R} : |x| \leq \lambda e \} \quad (x \in E).
\]

We prove the next

**Theorem.** Let \((E, \| \cdot \|)\) be a normed vector lattice. Then the following are equivalent:

1. \(E\) possesses a strong unity \(e\), and the norm \(\| \cdot \|_e\) is equivalent to \(\| \cdot \|\);
2. \(\text{Fin}(^*E) = \text{fin}(^*E)\);
3. \(\mu(^*E) \subseteq \text{fin}(^*E)\);
4. \(\mu(^*E) = \lambda(^*E)\);
5. \(\mu(^*E) \subseteq \eta(^*E)\);
6. \(\text{Fin}(^*E) = \text{fin}(^*E) + \mu(^*E)\).

\(<\) First of all, we prove the equivalence of conditions (2)–(5). To this end, it suffices to show that (2) \(\rightarrow\) (3) \(\rightarrow\) (4) \(\rightarrow\) (5) \(\rightarrow\) (3) and (4) \(\rightarrow\) (2). The implications (2) \(\rightarrow\) (3), (4) \(\rightarrow\) (5), and (5) \(\rightarrow\) (3) do not require checking.

(3) \(\rightarrow\) (4): Let \(\mu(^*E) \subseteq \text{fin}(^*E)\). To prove the implication, it is sufficient to verify the inclusion \(\mu(^*E) \subseteq \lambda(^*E)\). Take an arbitrary \(x \in \mu(^*E)\). Then \(\|\alpha x\| \approx 0\) with \(\alpha = \|x\|^{-1/2}\), and consequently \(\alpha x \in \mu(^*E) \subseteq \text{fin}(^*E)\). Thus, there is...
an element \( y \in E \) for which \(|\alpha x| \leq y \). Then \(|n\alpha x| \leq |\alpha x| \leq y \) for all \( n \in \mathbb{N} \), and so \( x \in \lambda(*E) \).

(4) \( \rightarrow \) (2): Let \( \mu(*E) = \lambda(*E) \). It is obvious that \( \text{Fin}(*E) \subseteq \text{Fin}(\lambda(*E)) \). Assume that the inclusion is proper. Then there is a \( x \in \lambda(*E) \) such that \( \|x\| = 1 \) and \(|x| \leq y \) for all \( y \in E \).

Consider the internal sets
\[
A^n_y := \{ r \in \mathbb{R}_+: n \leq r \land |x| \leq ry \}
\]
for \( y \in E_+ \) and \( n \in \mathbb{N} \). Since \(|x| \leq (n+1)y \) for every \( y \in E_+ \) and every \( n \in \mathbb{N} \), we have \( n+1 \in A^n_y \), and all these sets are nonempty. The family \( \{A^n_y\}_{y \in E_+} \) possesses the finite intersection property since
\[
A^{\max\{n,m\}}_{y \lor z} \subseteq A^n_y \cap A^m_z.
\]
By the general saturation principle, there is some \( r \in \mathbb{R}_+ \) satisfying
\[
r \in \cap \{A^n_y : y \in E_+, \, n \in \mathbb{N} \}.
\]
Then \( r \) is an infinite positive number such that \(|x| \leq ry \) for all \( y \in E_+ \). However, \((1/r)x \in \mu(*E)\), since \( \|(1/r)x\| = 1/r \approx 0 \). By hypothesis, \( \mu(*E) = \lambda(*E) \), therefore, \(|(1/r)x| \leq z \) for some \( z \in E_+ \), and so \(|x| \leq rz \), which contradicts \( r \in A^1_x \).

Thus, the equivalence of conditions (2)–(5) is established.

In order to finish the proof, we show that (1) \( \rightarrow \) (2) \( \rightarrow \) (6) \( \rightarrow \) (1). The implications (1) \( \rightarrow \) (2) and (2) \( \rightarrow \) (6) are obviously true.

(6) \( \rightarrow \) (1): Let \( \text{Fin}(\lambda(*E)) = \text{Fin}(\mu(*E)) \). First, we prove that the unit ball \( B := \{ x \in E : \|x\| \leq 1 \} \) of the vector lattice \( E \) is order bounded. Assume the contrary. Take an arbitrary \( x \in E_+ \). There is a \( y \in E_+ \) such that \( \|y\| = 1 \) and \( y \not\leq x \). Consider \( z = y - y \wedge x \). Then \( 0 < z \leq y \), and so \( 0 < \|z\| \leq 1 \). Show that
\[
tz \wedge x \leq y \quad (t \in \mathbb{R}_+).
\]
Let \( t \in \mathbb{R}_+ \). We represent \( x \) as \((x - x \wedge y) + (x \wedge y)\) and assign \( u = tz, \, v = x - x \wedge y, \) and \( w = x \wedge y \). It is clear that \( u, v, w \in E_+ \) and \( tz \wedge x = u \wedge (v + w) \). The easy relations
\[
\begin{align*}
u \wedge (v + w) &\leq w, \\
u \wedge (v + w) &\leq u \wedge (v + w) \leq u
\end{align*}
\]
imply the inequality
\[
u \wedge (v + w) \leq u \wedge v + u \wedge w.
\]
The elements \( u = tz \) and \( v = x - x \land y \) are disjoint because
\[
z \land v = (y - x \land y) \land (x - x \land y) = y \land x - x \land y = 0.
\]

Hence, by (9), we have
\[
tz \land x = u \land (v + w) \leq w \land v = tz \land x \land y \leq y.
\]

Inequality (8) is proven. Consider the element \( s = (2/\|z\|)z \) of the vector lattice \( E \).
It is clear that \( \|s\| = 2 \). Since \( s \land x \leq y \), we have \( \|s \land x\| \leq \|y\| = 1 \). Consequently, the internal set
\[
A_x := \{ s \in \cdot E_+ : \|s\| = 2 \& s \land x \in B \}
\]
is nonempty for all \( x \in E_+ \). Since \( A_x \land y \subseteq A_x \cap A_y \) \( (x, y \in E_+) \), the family \( \{A_x\}_{x \in E_+} \) possesses the finite intersection property. By the general saturation principle, there exists \( y_0 \in \cdot E_+ \) such that
\[
y_0 \in \cap \{A_x : x \in E_+\}.
\]

It is clear that \( \|y_0\| = 2 \). In particular, \( y_0 \in \text{Fin}(\cdot E) \). By assumption (6), \( y_0 \in \text{fin}(\cdot E) + \mu(\cdot E) \). Therefore, there are elements \( x_0 \in X_+ \) and \( h \in \mu(\cdot E) \), for which
\[
y_0 \leq x_0 + h.\]
Obviously, \( \|y_0 \land x_0\| \approx \|y_0\| \). At the same time, \( \|y_0\| = 2 \) and \( \|y_0 \land x_0\| \leq 1 \), since \( y_0 \in A_{x_0} \). The obtained contradiction shows that the unit ball of \( E \) is order bounded.

Choose an \( e \in E \) so that \( |x| \leq e \) for all \( x \in B \). Then
\[
|x| \leq \|x\|_e \cdot e \quad (x \in E).
\]

This implies that \( e \) is a strong order unity of \( E \). Moreover, \( \|x\|_e \leq \|x\| \quad (x \in E) \).
At the same time, \( \|x\| \leq c\|x\|_e \quad (x \in E) \) for \( c = \|e\|^{-1} \). Consequently, the norms \( \| \cdot \|_e \) and \( \| \cdot \| \) are equivalent. The implication (6) \( \Rightarrow \) (1) is established. The proof of the theorem is complete. \( \triangleright \)

4.5.3. Now we give a nonstandard condition for a norm to be order continuous.

**Theorem.** The norm \( \rho \) of a normed vector lattice \( (E, \rho) \) is order continuous if and only if \( \eta(\cdot E) \subseteq \mu(\cdot E) \).

\( \triangleright \) Assume that the norm \( \rho \) is order continuous. Take an arbitrary \( \kappa \in \eta(\cdot E) \). Since \( U(|\kappa|) \) is directed downwards and \( \inf \, U(|\kappa|) = 0 \), order continuity of the norm \( \rho \) implies
\[
\inf \{ \rho(u) : u \in U(|\kappa|) \} = 0.
\]
Then $\rho(x) \approx 0$. Since the choice of $x \in \eta(^*E)$ is arbitrary, it follows that $\eta(^*E) \subseteq \mu(^*E)$.

Now, let $\eta(^*E) \subseteq \mu(^*E)$. Assume that $\rho$ is not order continuous. In this case there are a net $(x_\xi)_{\xi \in \Theta} \subseteq E$, $x_\xi \downarrow 0$, and a number $0 < a \in \mathbb{R}$ such that $\rho(x_\xi) \geq a$ for all $\xi \in \Theta$. Take some remote element $\beta \in ^*\Theta$. Then, by 4.3.2, $x_\beta \in \eta(^*E)$. Thus $\rho(x_\beta) \approx 0$. On the other hand, by the transfer principle, $\rho(x_\beta) \approx a$ for all $\xi \in ^*\Theta$. The contradiction shows that the norm $\rho$ is order continuous. 

As an example of applying Theorem 4.5.2, we propose a nonstandard proof for the following well-known assertion:

Let a Banach lattice $E$ have an order continuous norm. Then $E$ is order separable and Dedekind complete. Moreover, order convergence in $E$ coincides with relative uniform convergence.

\( \Leftarrow \) In view of 4.4.2 and 4.3.6, it suffices to verify the relations

$$o\text{-}\text{pns}(*E) \subseteq E + \eta(*E) \text{ and } \eta(*E) \subseteq \lambda(*E).$$

Let $x \in o\text{-}\text{pns}(*E)$. Using the fact that the norm is order continuous, it is easy to see that $x \in \text{pns}(*E)$. According to Proposition 4.0.7, the Banach lattice $(E, \rho)$ satisfies $\text{pns}(*E) = E + \mu(*E)$. So, there is an $x \in E$ such that $x - x \in \mu(*E)$. Obviously, $L(x) \leq x \leq U(x)$. Since $x \in o\text{-}\text{pns}(*E)$, we have $x - x \in \eta(*E)$. Thus, $x \in E + \eta(*E)$.

Verify that $\eta(*E) \subseteq \lambda(*E)$. Let $x \in \eta(*E)$. Then $U(|x|) \downarrow 0$. By order continuity of $\rho$, for every $n \in \mathbb{N}$, there is an $u_n \in U(|x|)$ with $\rho(u_n) \leq 2^{-n}$. The sum $u := \sum_{n=1}^{\infty} u_n$ exists in the Banach lattice $E$. Obviously, $|nx| \leq u$ for all $n \in \mathbb{N}$. Thus $x \in \lambda(*E)$. 

4.5.4. Concluding this section, we will establish a nonstandard criterion for a normed vector lattice to be finite-dimensional.

Theorem. A normed vector lattice $(E, \rho)$ is finite-dimensional if and only if $\eta(*E) = \mu(*E)$.

\( \Leftarrow \) Necessity is obvious. To prove sufficiency, we let $\eta(*E) = \mu(*E)$. By Theorem 4.5.2, $E$ possesses a strong unity $e$. Moreover, the norm $\| \cdot \|_e$ is equivalent to the initial norm $\rho$. According to Theorem 4.5.3, $\rho$ is order continuous. Consequently, $\| \cdot \|_e$ is order continuous. Next, by Theorem 4.5.2, $\eta(*E) = \lambda(*E)$. Applying Theorem 4.3.6, we conclude that $E$ is order separable.

Assume that $\dim E = \infty$. It is easy to see that in this case there exists an infinite disjoint order basis $A \subseteq E_+$ such that $a \in A$ implies $\|a\|_e = 1$. Since $E$ is order separable, the set $A$ is at most countable, because it is order bounded in $E$ by some element $e$. Thus, we may suppose $A = \{a_n\}_{n=1}^{\infty}$. For each natural $n$, ...
define the element
\[ u_n := e - \left( \sum_{k=1}^{n} a_k \right) \land e. \]
It is easy to see that \( u_n \downarrow 0 \). Since the norm \( \| \cdot \|_e \) is order continuous, it follows that \( \|u_n\|_e \to 0 \). On the other hand, by the construction of the sequence \( (u_n) \) we have \( \|u_n\|_e \geq \|a_{n+1}\|_e = 1 \); a contradiction. Hence, \( \dim E < \infty \).  

4.6. Linear Operators Between Vector Lattices

In this section, we establish nonstandard criteria for linear operators in vector lattices to be order continuous and order bounded. These criteria are similar to those in 4.0.9. Below, the symbols \( E \) and \( F \) denote some vector lattices and \( T : E \to F \) is a linear operator.

4.6.1. We first prove one useful auxiliary assertion (see also 4.0.8):

Lemma. For every nonempty subset \( D \) of a vector lattice \( E \), the following are equivalent:

1. \( D \) is order bounded;
2. \( *D \subseteq \text{fin}(\text{fin}(E)) \).

\(<\) We need to prove only the implication (2) \(\Rightarrow\) (1). Let \( *D \subseteq \text{fin}(\text{fin}(E)) \). Assume that \( D \) is not contained in any order interval. Then, for every \( u \in E_+ \), there is \( d_u \in D \) satisfying \( (d_u - u)_+ > 0 \). By the general saturation principle, there exists some \( d \in *D \) such that \( (d - u)_+ > 0 \) for all \( u \in E_+ \). Then \( d \) satisfies \( d \in *D \setminus \text{fin}(\text{fin}(E)) \). This contradiction with \( *D \subseteq \text{fin}(\text{fin}(E)) \) shows that \( D \) is order bounded.  

4.6.2. Theorem. Let \( E \) and \( F \) be vector lattices, and let \( T : E \to F \) be a linear operator. Then the following are equivalent:

1. \( T \) is an order bounded operator;
2. \( *T(\text{fin}(\text{fin}(E))) \subseteq \text{fin}(\text{fin}(F)) \);
3. \( *T(\lambda(\text{fin}(E))) \subseteq \lambda(\text{fin}(F)) \);
4. \( *T(\lambda(\text{fin}(E))) \subseteq \text{fin}(\text{fin}(F)) \).

\(<\) (1) \(\Rightarrow\) (2): Obvious.
(2) \(\Rightarrow\) (3): Let \( *T(\text{fin}(\text{fin}(E))) \subseteq \text{fin}(\text{fin}(F)) \). Take an arbitrary \( x \in \lambda(\text{fin}(E)) \). Then, for some \( d \in E \), the condition \( |nx| \leq d \) holds for all \( n \in \mathbb{N} \) simultaneously. It is easy to see that in this case there is a \( \nu \in *\mathbb{N} \setminus \mathbb{N} \) such that \( |\nu x| \leq d \). Consequently, \( \nu x \in \text{fin}(\text{fin}(E)) \) and, by hypothesis, \( \nu^*T(x) = *T(\nu x) \in \text{fin}(\text{fin}(F)) \). This implies
\*T \in \lambda(\*F). The latter means that condition (3) is valid, since the choice of
the element \( \in \lambda(\*E) \) was arbitrary.

(3)→(4): Obvious.

(4)→(2): Assume that \( \*T \notin \text{fin}(\*F) \) for some \( \in \text{fin}(\*E) \). For every \( n \in \mathbb{N} \)
and every \( f \in F \), assign

\[
A_{n,f} := \{ k \in \*\mathbb{N} : k \geq n \& (|\*T(k^{-1})| - f) > 0 \}.
\]

The sets \( A_{n,f} \) are nonempty by the choice of \( \in \). By construction, they are internal,
comprising a system with the finite intersection property since

\[
A_{\max(n,p),\sup(f,g)} \subseteq A_{n,f} \cap A_{p,g}
\]

for arbitrary \( n, p \in \mathbb{N} \) and \( f, g \in F \). Applying the general saturation principle,
we find a \( \nu \in \cap_{n,f} A_{n,f} \). It is obvious that \( \nu \in \*\mathbb{N} \setminus \mathbb{N} \), and so \( |\nu^{-1}| \in \lambda(\*E) \).
By assumption, \( \*T(\lambda(\*E)) \subseteq \text{fin}(\*F) \). Therefore, there is a \( y \in F \) such that

\[
(|\*T(\nu^{-1})| - y) > 0,
\]

which is impossible since \( \nu \in A_{1,y} \). The obtained contradiction means that \( \*T(\text{fin}(\*E)) \subseteq \text{fin}(\*F) \).

(2)→(1): Take an arbitrary \( u \in E_+ \). By condition (2),

\[
\*(T([-u,u])) = \*T([-u,u]) \subseteq \text{fin}(\*F).
\]

Hence, by Lemma 4.6.1, the set \( T([-u,u]) \) is order bounded. \( \triangleright \)

4.6.3. Before stating a nonstandard criterion for a linear operator to be order
continuous, we find a connection between \((\sigma)\)-infinitesimal elements of an Archime-
dean vector lattice \( F \) and those of a Dedekind completion of \( F \).

**Lemma.** Let \( F \) be an Archimedean vector lattice and let \( F_1 \) be a Dedekind
completion of \( F \). Then \( \eta(\*F) = \*F \cap \eta(\*F_1) \).

\( \triangleright \) Given \( \in \*F \), assign

\[
U_F(\in) := \{ x \in F : x \geq \in \}, \quad U_{F_1}(\in) := \{ x \in F_1 : x \geq \in \}.
\]

Let \( \in \in \eta(\*F) \). Then \( \inf F_U_F(|\in|) = 0 \) and, since \( F_1 \) is a Dedekind completion
of \( F \), we have \( \inf F_1 U_{F_1}(|\in|) = 0 \). Furthermore, \( U_F(|\in|) \subseteq U_{F_1}(|\in|) \), which implies
\( \inf F_1 U_{F_1}(|\in|) = 0 \). Hence, \( \in \in \eta(\*F_1) \). At the same time, \( \in \in \*F \). Consequently,
\( \in \in \*F \cap \eta(\*F_1) \). Conversely, let \( \in \in \*F \cap \eta(\*F_1) \). Then \( \inf F_1 U_{F_1}(|\in|) = 0 \). Since
\( F_1 \) is a Dedekind completion of \( F \), it is easy to verify that \( \inf F_1 U_{F}(|\in|) = 0 \). This
immediately implies \( \inf F_U_F(|\in|) = 0 \). Thus, \( \in \in \eta(\*F) \). \( \triangleright \)
4.6.4. Theorem. Let $E$ and $F$ be Archimedean vector lattices, let $F_1$ be a Dedekind completion of $F$, and let $T : E \rightarrow F$ be a linear operator. Then the following are equivalent:

1. $T$ is an order continuous operator;
2. $T(\eta(E)) \subseteq \eta(F_1);
3. T(\eta(E)) \subseteq \eta(F).

\(<\ (1)-(2)\): Let $T$ be an order continuous operator in $L(E,F)$. Then it is easy to verify that $T$ is an order continuous operator in $L(E,F_1)$. Since $F_1$ is Dedekind complete, $|T|$ is defined, presenting an order continuous operator from $E$ into $F_1$.

In view of the inequality $|Tx| \leq |T||x| \ (x \in E)$, to verify the required implication it is sufficient to show

\[*T|(\eta(E)) \subseteq \eta(F_1).\]

Take an arbitrary $x \in \eta(E)$. Then, since $|T|$ is order continuous, $\inf_{E} U(|x|) = 0$ implies $\inf_{F_1} |T|(U(|x|)) = 0$. But

\[|T|(U(|x|)) \subseteq U(|T(x)|),\]

so $\inf_{F_1} U(|T(x)|) = 0$ and, consequently, $x \in \eta(F_1)$.

\((2)-(3)\): This follows readily from Lemma 4.6.3 since $T(\eta(E)) \subseteq T(F)$.  

\((3)-(1)\): Let $T(\eta(E)) \subseteq \eta(F)$. Since $E$ is Archimedean, $\lambda(E) \subseteq \eta(E)$ by Theorem 4.3.5. Consequently,

\[T(\lambda(E)) \subseteq \eta(F) \subseteq \eta(F).\]

By Theorem 4.6.2, this implies $T \in L_r(E,F)$. To verify order continuity, it remains to prove that $\inf_{F} |Tx_\xi| = 0$ for every net $x_\xi \downarrow 0$ in $E$. Take an arbitrary net $(x_\xi)_{\xi \in \Xi} \subseteq E$ such that $x_\xi \downarrow 0$. Assume that, for some element $f \in F$, $f > 0$, the condition $|Tx_\xi| \geq f$ holds for all $\xi \in \Xi$ simultaneously. Then, by the transfer principle, $|Tx_\xi| \geq f$ for all $\xi \in \Xi$. Let $\beta$ be some remote element of the directed set $\Xi$ (such an element exists by Lemma 4.0.5). According to the criterion established in 4.3.2, we have $x_\beta \in \eta(F)$. Then, by condition (3), $T x_\beta \in \eta(F)$, which contradicts $|T x_\beta| \geq f$. Thus, $\inf |Tx_\xi| = 0$ for every net $(x_\xi)$ decreasing to zero, and so the operator $T$ is order continuous. $\triangleright$
Infinitesimals in Vector Lattices

4.7. *-Invariant Homomorphisms

One of the important facts of nonstandard analysis is the assertion that each limited internal real number α ∈ fin(*R) is infinitely close to a unique standard real number st(α) called the standard part of α. The operation st of taking the standard part of a real is a Riesz homomorphism of the external vector space fin(*R) into R such that st(a) = a for all a ∈ R and st(α1) = st(α2) whenever α1 ≈ α2. This leads to the problem whether or not we may take the standard part of an element in a nonstandard enlargement of a vector lattice or a Boolean algebra. In other words: What conditions will guarantee existence of a Riesz or Boolean homomorphism keeping standard elements invariant and not distinguishing infinitely close elements? In this section, we discuss this question for nonstandard enlargements of vector lattices and Boolean algebras and establish that such an invariant homomorphism exists if and only if the vector lattice (Boolean algebra) in question is Dedekind complete (complete). In the end, we consider the structure of invariant homomorphisms on nonstandard enlargements of complete normed Boolean algebras and establish that, for atomless complete normed Boolean algebras, every invariant homomorphism is almost singular with respect to the measure, in the sense that the carrier of the homomorphism is contained in an internal set whose measure is arbitrarily small but nonzero. The considerations in this section rest on [10].

4.7.1. Let E be a vector lattice and let *E be a nonstandard enlargement of E. We may assume that E is a vector sublattice of *E.

Definition. A mapping ψ : fin(*E) → E is called a *-invariant Riesz homomorphism if ψ is a Riesz homomorphism such that ψ(x) = x for x ∈ E.

Henceforth we abbreviate a *-invariant Riesz homomorphism to a *-IRH. It is easy to see that the inequalities

\[ \sup_{E} \{ x ∈ E : x ≤ χ \} ≤ ψ(χ) ≤ \inf_{E} \{ y ∈ E : y ≥ χ \} \]  \hspace{1cm} (10)

are valid for every *-IRH ψ and every χ ∈ fin(*E) provided that the supremum and infimum exist on the right and left sides. In particular this implies

\[ (x - χ) ∈ η(*E) → ψ(χ) = x \quad (x ∈ E, \ χ ∈ fin(*E)). \]  \hspace{1cm} (11)

Theorem. Let E be a vector lattice. There exists a *-invariant Riesz homomorphism ψ on fin(*E) if and only if the vector lattice E is Dedekind complete. If E is atomic and Dedekind complete, then a *-IRH on fin(*E) is uniquely defined by

\[ ψ(χ) = \sup_{E} \{ x ∈ E : x ≤ χ \} = \inf_{E} \{ y ∈ E : y ≥ χ \} \quad (χ ∈ fin(*E)). \]  \hspace{1cm} (12)
Let $E$ be a Dedekind complete vector lattice. We apply the extension theorem of Bernau–Lipecki–Luxemburg–Schep (see, for example, [3, Theorem 2.1]) to the triple $(E, \text{fin}(E), E)$ and the identical Riesz homomorphism $\iota: E \to E$. Then we obtain the Riesz homomorphism $\psi: \text{fin}(E) \to E$ which extends $\iota$. Obviously, $\psi$ is a *-IRH on $\text{fin}(E)$.

Suppose there is a *-IRH $\psi: \text{fin}(E) \to E$. Take an order bounded upwards-directed nonempty set $\mathcal{D} \subseteq E$. By the general saturation principle, in $^*\mathcal{D}$ there exists an element $\delta \in \text{fin}(E)$ satisfying $\delta \geq d$ for all $d \in \mathcal{D}$. Then, as is easy to see, $\psi(\delta) = \sup_{E} \mathcal{D}$. Since the set $\mathcal{D} \subseteq E$ was chosen arbitrarily, this implies that $E$ is a Dedekind complete vector lattice.

Now let $E$ be atomic and Dedekind complete. Take a *-IRH $\psi: \text{fin}(E) \to E$ and $\chi \in \text{fin}(E)$. By Theorem 4.4.6, $\text{fin}(E) = E + \eta(E)$. Consequently, there exists a unique $x \in E$ obeying the condition $(x - \chi) \in \eta(E)$ By (11), we obtain $\psi(\chi) = x$. Thus the *-IRH $\psi$ is defined uniquely and satisfies (12).

Note that uniqueness of a *-IRH on a Dedekind complete vector lattice $E$ implies that $E$ is atomic. For a proof of this assertion we refer the reader to [10, Theorem 2.1].

4.7.2. We now consider a similar problem for Boolean algebras. Let $B$ be a Boolean algebra and let $^*B$ be a nonstandard enlargement of $B$. We assume that $B$ is a Boolean subalgebra of $^*B$.

**Definition.** A mapping $h: ^*B \to B$ is a *-invariant Boolean homomorphism if $h$ is a Boolean homomorphism such that $h(b) = b$ for all $b \in B$.

For brevity, a *-invariant Boolean homomorphism will be called a *-IBH henceforth. It is easy to see that every *-IBH $h$ satisfies

$$
\sup_{B} \{x \in B : x \leq \beta\} \leq h(\beta) \leq \inf_{B} \{y \in B : y \geq \beta\}
$$

for all $\beta \in ^*B$ provided that the supremum on the left side and the infimum on the right side both exist. This implies in particular that $h(\beta) = 0$ for every element $\beta \in ^*B$ such that $\inf_{B} \{b \in B : b \geq \beta\} = 0$.

**Theorem.** There exists a *-invariant Boolean homomorphism $^*B \to B$ if and only if the Boolean algebra $B$ is complete. Moreover, a *-IBH $h: ^*B \to B$ is defined uniquely if the complete Boolean algebra $B$ is atomic. In this case

$$
h(\beta) = \sup_{B} \{x \in B : x \leq \beta\} = \inf_{B} \{y \in B : y \geq \beta\}
$$

for all $\beta \in ^*B$. 
Before proving the theorem, we present one nonstandard characterization of an atomic complete Boolean algebra which is due to H. Conshor. For this we need some notations. Let $B$ be a Boolean algebra. Given $\kappa \in *B$, consider the set $U(\kappa) := \{ x \in B : x \geq \kappa \}$ of standard upper bounds of $\kappa$ and the set $L(\kappa) := \{ y \in B : \kappa \geq y \}$ of standard lower bounds of $\kappa$. Define the external subsets of $*B$:

$$o\text{-}\text{pns}(B) := \{ \kappa \in B : \inf_{B} (U(\kappa) - L(\kappa)) = 0 \},$$

$$\eta(B) := \{ \kappa \in B : \inf_{B} U(\kappa) = 0 \}.$$

It can be shown that $o\text{-}\text{pns}(B)$ is a Boolean subalgebra of $*B$, and $\eta(B)$ is an ideal in $o\text{-}\text{pns}(B)$ and the quotient $o\text{-}\text{pns}(B)/\eta(B)$ is Boolean isomorphic to $B$ (see [4, Theorem 4.1]). From this and from [4, Theorem 4.3] we have immediately the next

**Lemma** (H. Conshor). For every Boolean algebra $B$, the following are equivalent:

1. $B$ is an atomic complete Boolean algebra;
2. $*B = B + \eta(B)$.

**Proof of the Theorem:**

Let $B$ be a complete Boolean algebra. We apply Sikorski's Extension Theorem (see, for example, [25, Theorem 33.1]) to the triple $(B, *B, B)$ and the identical Boolean homomorphism $\iota : B \to B$. We then obtain a Boolean homomorphism $h : *B \to B$ that extends $\iota$. Obviously, $h$ is a $*$-IH on $*B$.

We now show that the existence of a $*$-IH $h : *B \to B$ implies completeness of $B$. Let $h : *B \to B$ be a $*$-IH. Take an upwards-directed nonempty set $\mathcal{D} \subseteq B$. By the general saturation principle, there exists an element $\delta$ in $*\mathcal{D}$ satisfying $\delta \geq d$ for all $d \in \mathcal{D}$. Then, as is easy to verify, $h(\delta) = \sup_{\mathcal{D}} \mathcal{D}$. Since the upwards-directed nonempty set $\mathcal{D} \subseteq B$ was chosen arbitrarily, this implies that $B$ is a complete Boolean algebra.

Let $E$ be atomic and Dedekind complete. Then the proof of uniqueness of a $*$-IH $h : *B \to B$ is similar to the proof of uniqueness of a $*$-IH in 4.7.1. We must use the preceding lemma instead of Theorem 4.4.6 only. $\triangleright$

Note that uniqueness of a $*$-IH on a complete Boolean algebra $B$ implies that $B$ is atomic. For a proof of this assertion we refer the reader to [10, Theorem 1.1].

**4.7.3.** Let $B$ be a complete Boolean algebra. For convenience, given a family $(a_{\tau}) \subseteq B$, we denote $\sup_{\tau} a_{\tau}$ by $\bigoplus_{\tau} a_{\tau}$ whenever the elements $a_{\tau}$ are disjoint. A partition of an element $b \in B$ is a family $(b_{\tau}) \subseteq B$ such that $b = \bigoplus_{\tau} b_{\tau}$. Let $\mu : B \to \mathbb{R}_{+}$ be a mapping on $B$ satisfying the following conditions:
(1) $\mu(b) > 0 \iff b > 0$;

(2) The equality $\mu(\bigoplus_{n=1}^\infty a_n) = \sum_{n=1}^\infty \mu(a_n)$ is valid for every sequence $a_1, a_2, \ldots$ of disjoint elements of $B$.

Recall that a mapping $\mu$ with the above properties is a $\sigma$-additive measure and the pair $(B, \mu)$ is a complete normed Boolean algebra.

Let $(B, \mu)$ be a complete normed Boolean algebra and let $h : ^*B \to B$ be a $*$-IBH. Represent $B$ as a direct sum of atomic and atomless components: $B = B_a \bigoplus B_c$. Then $(B_a, \mu)$ and $(B_c, \mu)$ are complete normed Boolean algebras. The restriction of $h$ to $^*B_a$ is a $*$-invariant Boolean homomorphism preserving the measure $\mu$ in the sense that $\mu(h(\alpha)) = st(\mu(\alpha))$ for all $\alpha \in ^*B_a$. The restriction of $h$ to $^*B_c$ with respect to the measure $\mu$ has an opposite behavior. Namely, the following holds:

**Theorem.** Let $(B, \mu)$ be an atomless complete normed Boolean algebra and let $h : ^*B \to B$ be a $*$-invariant Boolean homomorphism. Then, for every real $\varepsilon > 0$, there exists $\chi_\varepsilon \in ^*B, \mu(\chi_\varepsilon) < \varepsilon$, satisfying the condition $h(b) = h(b \land \chi_\varepsilon)$ for all $b \in ^*B$.

Before proving the theorem, we establish one elementary property of atomless complete normed Boolean algebras.

**Lemma.** For every atomless complete normed Boolean algebra $(B, \mu)$ and every natural $n$, there exists a partition $(\chi_i)_{i=1}^n \subseteq B$ of $1 \cdot B$ such that $^*\mu(\chi_i \land d) = \frac{1}{n} \mu(d)$ for all $d \in B$, $i = 1, \ldots, n$.

$\langle$ Take an arbitrary hyperfinite partition $(e_k)_{k=1}^\nu$ of $1 \cdot B$ in the Boolean algebra $^*B$ which is refined into each finite standard partition. Existence of such a partition is easy on using the general saturation principle. Since $B$ is atomless and the measure $\mu$ is $\sigma$-additive, there exist partitions $e_k = \bigoplus_{i=1}^n e_i^k$ such that

$$^*\mu(e_i^k) = \frac{1}{n} ^*\mu(e_k)$$

for all $k = 1, \ldots, \nu$ and $i = 1, \ldots, n$. Put $\chi_i := \bigoplus_{k=1}^\nu e_i^k$. The family $(\chi_i)_{i=1}^n \subseteq ^*B$ is a required partition of unity. $\rangle$

**Proof of the Theorem:**

$\langle$ Take an $n \in \mathbb{N}$ such that $\frac{1}{n} \mu(1) < \varepsilon$. According to the preceding lemma, there exists a partition $(\chi_i)_{i=1}^n \subseteq ^*B$ of $1 \cdot B$ satisfying the condition

$$^*\mu(\chi_i \land d) = \frac{1}{n} \mu(d)$$

for arbitrary $d \in B$, $i = 1, \ldots, n$. In particular,

$$^*\mu(\chi_k \land h(\chi_m)) = \frac{1}{n} \mu(h(\chi_m))$$
for $k, m \in 1, \ldots, n$. Consider the element
\[ \chi_\varepsilon := \bigoplus_{k=1}^{n} \chi_k \wedge h(\chi_k). \]

Then
\[ *\mu(\chi_\varepsilon) = \sum_{k=1}^{n} *\mu(\chi_k \wedge h(\chi_k)) = \sum_{k=1}^{n} \frac{1}{n} \mu(h(\chi_k)) = \frac{1}{n} \sum_{k=1}^{n} \mu(h(\chi_k)) \]
\[ = \frac{1}{n} \mu\left( \bigoplus_{k=1}^{n} h(\chi_k) \right) = \frac{1}{n} \mu\left( h\left( \bigoplus_{k=1}^{n} \chi_k \right) \right) = \frac{1}{n} \mu(h(1_B)) = \frac{1}{n} \mu(1_B) < \varepsilon. \]

At the same time
\[ h(1_B \setminus \chi_\varepsilon) = h\left( \bigoplus_{m=1}^{n} \bigoplus_{k \neq m} \chi_k \wedge h(\chi_m) \right) = \bigoplus_{m=1}^{n} \bigoplus_{k \neq m} h(\chi_k \wedge h(\chi_m)) = 0, \]

since
\[ h(\chi_k \wedge h(\chi_m)) = h(\chi_k) \wedge h^2(\chi_m) \]
\[ = h(\chi_k) \wedge h(\chi_m) = h(\chi_k \wedge \chi_m) = h(0) = 0 \]
for $k \neq m$. Thus,
\[ h(b) = h(b \wedge 1_B) = h((b \wedge \chi_\varepsilon) \oplus (b \wedge (1_B \setminus \chi_\varepsilon))) \]
\[ = h(b \wedge \chi_\varepsilon) \oplus (h(b) \wedge h(1_B \setminus \chi_\varepsilon)) = h(b \wedge \chi_\varepsilon) \]
for all $b \in ^*B$. \(\triangleright\)

4.7.4. Consider the real-valued mappings $st \circ *\mu$ and $\mu \circ h$ defined on a Boolean algebra $^*B$, where $h$ is a $*$-IBH. Obviously, the mappings $st \circ *\mu$ and $\mu \circ h$ are finitely additive measures on $^*B$. Moreover, these mappings are $\sigma$-additive, since the condition $b = \bigoplus_{n=1}^{\infty} b_n$ imposed on elements of $^*B$ implies $b = \bigoplus_{n=1}^{m} b_n$ for some $m \in \mathbb{N}$. Thus, $st \circ *\mu$ and $\mu \circ h$ extend to $\sigma$-additive measures $\tilde{\mu}$ and $\tilde{\mu}_h$ on the $\sigma$-completion $^*B_{\sigma}$ of the Boolean algebra $^*B$. Observe that $\tilde{\mu}$ is the Loeb measure corresponding to the initial measure $\mu$.

**Theorem.** Let $(B, \mu)$ be an atomless complete normed Boolean algebra and let $h : ^*B \rightarrow B$ be a $*$-invariant Boolean homomorphism. Then the following hold:
(1) There exists an element \( x_n \in *B_\sigma \), \( \mu(x_n) = 0 \), such that the equality \( h(b) = 0 \) holds for every \( b \in *B \) satisfying the condition \( b \land x_n = 0 \);

(2) The carriers of the measures \( \mu \) and \( \mu_h \) are disjoint.

\(<(1)\): By Theorem 4.7.3, for every \( n \in \mathbb{N} \), there exists an element \( x_n \in *B \) such that \( \bar{\mu}(x_n) \leq \frac{1}{n} \) and \( h(b) = 0 \) for all \( b \in *B \), \( b \land x_n = 0 \). Put \( x_n = \bigwedge_{n=1}^{\infty} x_n \).

It is clear that \( x_n \in *B_\sigma \) and \( \bar{\mu}(x_n) = 0 \). Take an arbitrary \( b \in *B \), \( b \land x_n = 0 \).

Then there exists an \( n \in \mathbb{N} \) such that \( b \land x_n = 0 \). Thus, \( h(b) = 0 \), as required.

(2) ensues from (1). Indeed, the carriers of the measures \( \bar{\mu} \) and \( \bar{\mu}_h \) are disjoint elements \( x_n \) and \( 1 \setminus x_n \) of the Boolean algebra \( *B_\sigma \).

4.8. Order Hulls of Vector Lattices

In this section, we define the order hull of a vector lattice. Some properties of order hulls are established. In particular, the question about their \((r)\)- and \((o)\)-completeness is studied to some extent. Some conditions are found for the order hull of a vector lattice \( E \) to be isomorphic with the initial vector lattice \( E \) and (if \( E \) is a normed lattice) for the order hull of \( E \) to be isomorphic with the nonstandard hull of \( E \) regarded as a normed vector space.

4.8.1. Let \( E \) be a vector lattice. As it was mentioned in 4.3.1, the set \( \text{fin}(*E) \) of limited elements in \( *E \) is a vector lattice too, while the set \( \eta(*)E \) of \((o)\)-infinitesimal elements in \( *E \) is an ideal in \( \text{fin}(*E) \). Consider the quotient vector lattice

\[(o)-E := \text{fin}(*E)/\eta(*)E.\]

We call \((o)-E\) the order hull of \( E \) and denote by \( [x] \) the coset \( x + \eta(*)E \in (o)-E \) that contains \( x \in \text{fin}(*E) \). Define the mapping \( \eta_E : E \to (o)-E \) by

\[\eta_E(x) := [x] \ (x \in E).\]

Clearly, \( \eta_E : E \to (o)-E \) is a Riesz homomorphism. It will be denoted by \( \hat{\eta} \) if this does not lead to ambiguity.

4.8.2. Theorem. The set \( \hat{\eta}(E) \) is a complete vector sublattice of \((o)-E\).

Before proving, we give some explanations. Let \( L \) be a vector sublattice of a vector lattice \( M \). Recall that \( L \) is a complete vector sublattice of \( M \) if, for every nonempty \( D \subseteq L \) and every \( a \in L \), the condition \( \inf_L D = a \) implies \( \inf_M D = a \).

It is easy to see that \( L \) is a complete vector sublattice of \( M \) if and only if, for every nonempty \( D \subseteq L \), the condition \( \inf_L D = 0 \) implies \( \inf_M D = 0 \).

\(<\) Let \( D \subseteq E \) such that \( \inf_E D = 0 \). Show that \( \inf_{(o)-E} \hat{\eta}(D) = 0 \). Assume the contrary. Then for some \( x \in \text{fin}(*E) \) we have

\[\hat{\eta}(D) \geq [x] > 0.\]
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Since $x \notin \eta(^*E)$, there is an $a \in E$ such that

$$U(x) \geq a > 0.$$  

Take an arbitrary $d \in D$. Then $\hat{\eta}(d) \geq [x]$ and, consequently, $(x - d)_+ \in \eta(^*E)$. Thus, $\inf_E \mathcal{U} = 0$, where $\mathcal{U} := U((x - d)_+)$. Given $u \in \mathcal{U}$, note

$$d + u \geq d + (x - d)_+ \geq x.$$  

Hence, $d + \mathcal{U} \subseteq U(x)$, and so $d + \mathcal{U} \geq a$. Therefore,

$$d = \inf\limits_E (d + \mathcal{U}) \geq a.$$  

Since the last inequality is valid for all $d \in D$ and $\inf_E D = 0$, we have $a = 0$. A contradiction shows that $\inf\limits_{(o) \in E} \hat{\eta}(D) = 0$. 

4.8.3. Theorem. The order hull of a vector lattice $E$ is Archimedean if and only if $E$ is Archimedean. 

Necessity follows from the fact that each vector lattice may be embedded as a vector sublattice into its order hull. To prove sufficiency, we consider an Archimedean vector lattice $E$. By Theorem 4.3.5, $\eta(^*E)$ is a relatively uniform closed ideal in $\text{fin}(^*E)$. Then, according to the well-known theorem by A. I. Veksler [26] (see also [21, Theorem 60.2]), the quotient $(o)-E$ is Archimedean. 

4.8.4. Theorem. The order hull of a vector lattice is relatively uniformly complete. 

Let $E$ be a vector lattice. Since every quotient vector lattice of a relatively uniformly complete vector lattice is relatively uniform complete too (see, for example, [21, Theorem 59.4]), it is enough to establish relative uniform completeness of $\text{fin}(^*E)$. Take a relatively uniformly Cauchy sequence $(x_n)_{n=1}^\infty \subseteq \text{fin}(^*E)$. Then, there exist a sequence $(\varepsilon_n) \subseteq \mathbb{R}$, $\varepsilon_n \downarrow 0$, and an element $\delta \in \text{fin}(^*E)$ such that

$$|x_m - x_k| \leq \varepsilon_n \delta$$

for all $m, k, n \in \mathbb{N}$ whenever $m, k \geq n$. We extend $(x_n)_{n \in \mathbb{N}}$ to an internal sequence $(x_n)_{n \in \mathbb{N}} \subseteq ^*E$ and associate with each natural $k$ the internal set:

$$I_k := \{ m \in ^*\mathbb{N} : |x_m - x_k| \leq \varepsilon_m \delta \}.$$  

It is easy to see that the family $\{I_k\}_{k=1}^\infty$ has the finite intersection property. By the general saturation principle, there is a $\nu \in \bigcap_{k=1}^\infty I_k$. Then every $k \in \mathbb{N}$ satisfies

$$|x_k - x_\nu| \leq \varepsilon_k \delta.$$  

This implies $x_\nu \in \text{fin}(^*E)$ and $x_n \xrightarrow{(r)} x_\nu$. 

|
4.8.5. The matter with Dedekind completeness of the order hull of a lattice differs from that with relative uniform completeness. We show that the order hull of a Dedekind complete vector lattice containing nonatomic elements is not necessarily Dedekind complete (Theorem 4.8.7 establishes that the order hull of an atomic Dedekind complete vector lattice is Dedekind complete).

Recall that a Dedekind complete vector lattice $E$ is called regular (see, for example, [29]) if the following hold:

1. Order convergence and relative uniform convergence coincide for every sequence in $E$;
2. Each ideal with a countable order basis in $E$ is contained in some principal ideal;
3. $E$ is order separable.

As examples of regular vector lattices, we may take Banach lattices with order continuous norm and $L_p([0,1])$ with $0 < p < 1$.

**Theorem.** The order hull of a nonatomic regular vector lattice is not Dedekind complete.

Let $E$ be a nonatomic regular vector lattice. Then there is a nonatomic element $e \in E$, $e > 0$. Let $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ be some illimited natural number. By Lemma 4.4.4, there exists a family $\{e_n\}_{n=1}^\nu$ of disjoint e-punches. Assign $D := \{[e_n]\}_{n=1}^\infty$. Then $D$ is a nonempty and bounded above (for example, by the element $[e]$) subset of $({}o)-E$. We show that this subset fails to have a least upper bound in $({}o)-E$. By way of contradiction, assume

$$[x] = \sup_{({}o)-E} D$$

Then for all $k \in \mathbb{N}$, we have

$$(e_k - x)_+ \in \eta(\nu).$$

By Theorem 4.3.6, $E$ has the property $\eta(\nu) = \lambda(\nu)$ (here properties (1) and (3) from the definition of a regular vector lattice are used). Hence, $(e_k - x)_+ \in \lambda(\nu)$ holds for all $k \in \mathbb{N}$. By using (2), it is easy to see that there exists a $d \in E$ for which

$$(e_k - x)_+ \leq m^{-1}d \ (k,m \in \mathbb{N}).$$

Applying the general saturation principle, we find $\omega, \gamma \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that

$$\omega \leq \nu \text{ and } (e_\omega - x)_+ \leq \gamma^{-1}d.$$
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Then

\((e_\omega - x)_+ \in \lambda(^*E) = \eta(^*E)\)

and, consequently, \([e_\omega] \leq [x]\). At the same time, \([e_\omega] > 0\) (because \(\omega \leq \nu\) while \(e_\omega\) is an \(e\)-punch) and \([e_\omega] \land [e_k] = 0\) for all \(k \in \mathbb{N}\) (since \(e_\omega\) and \(e_k\) are disjoint). Hence

\([x] > [x] - [e_\omega] > [e_k]\) for every \(k \in \mathbb{N}\),

which contradicts the assumption \([x] = \sup_{(o)-E} D\). Thus, the order hull \((o)-E\) of \(E\) is not Dedekind complete. \(\triangleright\)

4.8.6. We establish one more property of order hulls concerning cardinality. We denote by \(\text{card}(A)\) the cardinality of a set \(A\).

**Lemma.** Assume that a vector lattice \(E\) is not Archimedean or not atomic. Then

\[\text{card}(E) < \text{card}((o)-E).\]

\(\triangleright\) Take an arbitrary \(\nu \in ^*\mathbb{N}\setminus \mathbb{N}.\) According to Lemma 4.0.4, \(\text{card}(E) < \text{card}(\nu)\). Therefore it is sufficient to establish that the order hull of \(E\) contains \(\nu\) distinct elements.

Assume first that \(E\) is not Archimedean. Then there are elements \(u, v \in E\) such that \(0 < nu \leq v\) for all natural \(n\). By the transfer principle, \(0 < nu \leq v\) holds for all \(n \in ^*\mathbb{N}\). In particular, \(nu \in \text{fin}(^*E)\) for all \(n \in ^*\mathbb{N}\). The inequality

\[0 < u \leq |nu - mu|\]

is valid for all \(n, m \in ^*\mathbb{N}\) such that \(n \neq m\). Therefore, \([nu] \neq [mu]\) whenever \(n, m \in ^*\mathbb{N}\) and \(n \neq m\). Thus, \([nu]_{n=1}^\nu\) is a family consisting of \(\nu\) distinct elements of \((o)-E\).

It remains to consider the case in which the vector lattice \(E\) is Archimedean but not atomic. Then, there is a nonatomic element \(e \in E, e > 0\). By Lemma 4.4.4, there exists a family \(\{e_k\}_{k=1}^\nu\) of disjoint \(e\)-punches in \(\text{fin}(^*E)\). It follows immediately that the elements \([e_k]\) of the order hull \((o)-E\) are distinct for \(k = 1, 2, \ldots, \nu\). \(\triangleright\)

In the rest of the section, we study the question about conditions under which the order hull of \(E\) coincides with \(E\) or with the nonstandard hull of \(E\) (if the lattice \(E\) is assumed to be normed and regarded as a normed vector space).

4.8.7. **Theorem.** For every vector lattice \(E\), the following are equivalent:

1. \(E\) is Riesz isomorphic to \((o)-E\);
2. \(E\) is an atomic Dedekind complete vector lattice;
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(3) \( \hat{\eta}_E \) is a Riesz isomorphism of \( E \) onto the order hull of \( E \).

< (1)→(2): The vector lattice \( E \) is Archimedean and atomic by Lemma 4.8.6. So, \( \text{fin}(E) = \sigma\text{-pns}(E) \) by Theorem 4.4.5. Then, \( (o)E = \sigma\text{-pns}(E)/\eta(E) \), and by 4.4.1(1), the vector lattice \( (o)E \) and, consequently, \( E \) is Dedekind complete.

(2)→(3): Since by Theorem 4.4.6 \( \text{fin}(E) = E + \eta(E) \), for every \( u \in (o)E \), there is an \( x \in E \) such that \( u = x + \eta(E) \). In particular, the image of the Riesz homomorphism \( \hat{\eta}: E \to (o)E \) coincides with \((o)E\). Thus, \( \hat{\eta} \) is a Riesz isomorphism.

The implication (3)→(1) is obvious. □

It is interesting to compare this theorem with Proposition 4.0.6.

4.8.8. Let \((E, \rho)\) be a normed vector lattice. Recall that, according to 4.0.6, we may arrange the quotient vector lattice

\[
\tilde{E} := \text{Fin}(E)/\mu(E)
\]

with the respective quotient norm. \( \tilde{E} \) is called the nonstandard hull of \((E, \rho)\). Obviously, \( \tilde{E} \) is a Banach lattice. Note that \( \tilde{E} \) depends not only on \( E \) but also on the choice of the norm \( \rho \). Denote the coset of an element \( x \in \text{Fin}(E) \) in the quotient vector lattice \( \tilde{E} \) by \( \langle x \rangle \) and consider the mapping \( \hat{\mu}: E \to \tilde{E} \) (cf. 4.0.6) such that \( \hat{\mu}(x) := \langle x \rangle \) for every \( x \in E \). It is easy to see that \( \hat{\mu} \) is a Riesz monomorphism.

**Theorem.** Let \((E, \rho)\) be a normed vector lattice. Then the following are equivalent:

1. There exists a Riesz isomorphism \( \pi \) of \((o)E\) onto \( \tilde{E} \) such that \( \pi \circ \hat{\eta} = \hat{\mu} \);
2. \( E \) is finite-dimensional.

< (1)→(2): Let \( \pi: (o)E \to \tilde{E} \) be a Riesz isomorphism such that \( \pi \circ \hat{\eta} = \hat{\mu} \). The ideal, generated by \( \hat{\eta}(E) \), coincides with \((o)E\) and, furthermore, \( \hat{\mu}(E) = \pi(\hat{\eta}(E)) \); therefore, the ideal, generated by \( \hat{\mu}(E) \), coincides with \( \tilde{E} \). Consequently,

\[
\text{Fin}(E) = \text{fin}(E) + \mu(E),
\]

which implies \( \mu(E) \subseteq \eta(E) \) by Theorem 4.5.2. According to Theorem 4.5.4, it remains to establish the reverse inclusion. Let \( x \in \eta(E) \). Then \( U := U(|x|) \) is directed downwards and \( U \downarrow 0 \). In this case \( \hat{\eta}(U) \downarrow 0 \) in \((o)E\) by Theorem 4.8.2. Hence, \( \pi \circ \hat{\eta}(U) \downarrow 0 \) in the vector lattice \( \tilde{E} \). In other words,

\[
\inf_{\tilde{E}} \hat{\mu}(U) = \inf_{E} \pi \circ \hat{\eta}(U) = 0.
\]

Since \( \hat{\mu}(U) \geq \langle x \rangle \geq 0 \), we have \( \langle x \rangle = 0 \) and, consequently, \( x \in \mu(E) \). Thus, \( \mu(E) = \eta(E) \), and so \( E \) is finite-dimensional by 4.5.4.

(2)→(1): This is obvious. □
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4.9. Regular Hulls of Vector Lattices

Here we define and study the regular hull of a vector lattice. We will obtain a criterion for a vector lattice to be isomorphic to its regular hull. Some related questions concerning regular hulls are discussed.

4.9.1. Let $E$ be a vector lattice. Like in the preceding section, we consider the quotient $(r)-E := \text{fin}(E)/\lambda(E)$ and call it the regular hull of $E$. We denote the coset $x + \lambda(E)$ where $x \in \text{fin}(E)$ by $(x)$, and define a mapping $\lambda_E : E \to (r)-E$ as follows:

$$\lambda_E(x) := (x) \quad (x \in E).$$

Obviously $\lambda_E$ is a Riesz homomorphism. We denoted it by $\lambda$ if this does not lead to ambiguity.

We present a criterion for a vector lattice to coincide with its regular hull. Recall that a vector lattice $E$ is called almost regular if $E$ is Dedekind complete and order separable, and if order convergence and relative uniform convergence are equivalent for every sequence in $E$.

Theorem. For every vector lattice $E$, the following are equivalent:

1. $\lambda : E \to (r)-E$ is a Riesz isomorphism of $E$ onto $(r)-E$;
2. $E$ is atomic and almost regular.

$\Leftarrow (1) \Rightarrow (2)$: Assume that $\lambda$ is a Riesz homomorphism of $E$ onto $(r)-E$. In particular, $\lambda$ is injective. Thus, obviously, $E$ is Archimedean, so $\lambda(E) \subseteq \eta(E)$, by Theorem 4.3.5. Since $\text{fin}(E) = E + \lambda(E)$, we have

$$\text{fin}(E) = E + \eta(E).$$

Therefore $E$ is an atomic Dedekind complete vector lattice by Theorem 4.4.6. In order to complete the proof of the implication $(1) \Rightarrow (2)$, in accordance with Theorem 4.3.6, it remains to check the inclusion $\eta(E) \subseteq \lambda(E)$. Take an arbitrary element $x \in \eta(E)$. Since $\eta(E) \subseteq E + \lambda(E)$ holds by (1), the element $x$ can be written as $x = e + x_1$, where $e \in E$ and $x_1 \in \lambda(E)$. Then

$$x = \lambda_1 \subseteq \lambda(E) \subseteq \eta(E).$$

Consequently, $e \in E \cap \eta(E) = \{0\}$, and hence $e = 0$. Finally, $x = x_1 \in \lambda(E)$.

$(2) \Rightarrow (1)$: Let $E$ be an almost regular vector lattice. Then, by Theorems 4.4.6 and 4.3.6,

$$\text{fin}(E) = E + \eta(E) = E + \lambda(E),$$

which immediately implies that $\lambda$ is onto. Moreover, since $E$ is Archimedean, $\lambda$ is an injection. Thus, $\lambda$ is a Riesz isomorphism of $E$ onto $(r)-E$. $\triangleright$
4.9.2. According to Theorem 4.3.6, the regular hull \((r)-E\) of an Archimedean order separable vector lattice \(E\) in which order convergence and relative uniform convergence are equivalent for every sequence coincides with the order hull \((o)-E\). We show that there are no other types of vector lattices with this property.

**Theorem.** For an arbitrary vector lattice \(E\), the following are equivalent:

1. There exists a Riesz isomorphism \(\pi\) of \((o)-E\) onto \((r)-E\) such that \(\pi \circ \widehat{\eta} = \widehat{\lambda}\);
2. \(\eta(\cdot E) = \lambda(\cdot E)\);
3. \(E\) is an order separable Archimedean vector lattice in which order convergence and relative uniform convergence are equivalent for every sequence.

By Theorem 4.3.6 it suffices to verify (1) \(\rightarrow\) (2).

Let \(\pi : (o)-E \rightarrow (r)-E\) be a Riesz isomorphism such that \(\pi \circ \widehat{\eta} = \widehat{\lambda}\). Take elements \(u, v \in E\) satisfying the condition \(0 \leq nu \leq v\) for all \(n \in \mathbb{N}\). It is easy to see that \(\pi \circ \widehat{\eta}(u) = \widehat{\lambda}(u)\), and hence \(\widehat{\eta}(u) = 0\). Since \(\widehat{\eta}\) is injection, the relation \(u = 0\) holds. Thus, \(E\) is Archimedean. The inclusion \(\lambda(\cdot E) \subseteq \eta(\cdot E)\) follows now.

To complete the proof, it remains to establish the reverse inclusion: \(\eta(\cdot E) \subseteq \lambda(\cdot E)\). Assume that there is a \(x \in \eta(\cdot E) \setminus \lambda(\cdot E)\). We may suppose \(x \geq 0\). Then \(\langle x \rangle > 0\). At the same time, the condition \(x \in \eta(\cdot E)\) implies \(\inf_{E} U(x) = 0\). Therefore, according to Theorem 4.8.2, \(\inf_{(o)-E} \widehat{\eta}(U(x)) = 0\). Since \(\pi\) is an isomorphism of \((o)-E\) onto \((r)-E\), we have

\[
\inf_{(r)-E} \widehat{\lambda}(U(x)) = \inf_{(o)-E} \pi \circ \widehat{\eta}(U(x)) = 0,
\]

which contradicts the condition \(\widehat{\lambda}(U(x)) \geq \langle x \rangle > 0\). Thus, we have \(\eta(\cdot E) \subseteq \lambda(\cdot E)\). The proof of the theorem is complete. \(\triangleright\)

4.9.3. We now discuss interrelation between the regular hull \((r)-E\) and the nonstandard hull \(\widehat{E}\) of a normed vector lattice \(E\). Namely, we find a condition for \((r)-E\) to coincide with \(\widehat{E}\). We use the notation and terminology of 4.4.2 and 4.8.8.

**Theorem.** Let \((E, \rho)\) be a normed vector lattice. Then the following are equivalent:

1. \(E\) possesses a strong order unity \(e\) such that the norm \(\rho\) is equivalent to the norm \(\| \cdot \|_{e}\);
2. \((r)-E = \widehat{E}\);
There exists a Riesz isomorphism $\varphi$ of $(r)-E$ onto $\tilde{E}$ such that $\varphi \circ \lambda = \mu$.

(1)$\rightarrow$(2): This is immediate from Theorem 4.5.2.
(2)$\rightarrow$(1): Obvious.
(3)$\rightarrow$(1): Let $\varphi : (r)-E \rightarrow \tilde{E}$ be a Riesz isomorphism such that $\varphi \circ \lambda = \mu$. By Theorem 4.5.2, we need to establish the relation $\text{Fin}(\lambda:\tilde{E}) = \text{fin}(\lambda:E) + \mu(\lambda:E)$. The inclusion $\text{fin}(\lambda:E) + \mu(\lambda:E) \subseteq \text{Fin}(\lambda:E)$ is obvious. For proving the reverse inclusion, take an arbitrary $x \in \text{Fin}(\lambda:E)$. Then $\langle x \rangle = \varphi(\langle x_1 \rangle)$ for some $x_1 \in \text{fin}(\lambda:E)$.

The inequality $|\langle x \rangle| \leq \langle x \rangle$ implies that the element $x$ can be written in the form $x = \xi_1 + \xi_2$, where $|\xi_1| \leq x$ and $\xi_2 \in \mu(\lambda:E)$. Thus, $x \in \text{fin}(\lambda:E) + \mu(\lambda:E)$. $\triangleright$

4.9.4. In contrast to 4.8.2, the image of the vector lattice $E$ under $\hat{\lambda}$ is not necessarily a complete vector sublattice of $(r)-E$. Indeed, consider the vector lattice $l_\infty$ of all bounded sequences in $\mathbb{R}$, and let $D$ be a subset of $l_\infty$ consisting of all sequences with the property that all but finitely many coordinates are equal to 1. Then $\inf_E D = 0$, but $\lambda(D) \geq \nu > 0$ for all $\nu \in \lambda:E$, where $\nu$ is the internal sequence in $\lambda:\mathbb{R}$ in which the only nonzero coordinate has index $\nu$ and equals 1.

4.9.5. Exactly as in the proof of Theorem 4.8.4 in which relative uniform completeness of order hulls was established, we can show that the regular hull of an arbitrary vector lattice is relatively uniformly complete. At the same time, since, by Theorem 4.3.6, the regular hull of a regular vector lattice coincides with its order hull, Theorem 4.8.5 shows that the regular hull of a nonatomic regular vector lattice is not Dedekind complete.

4.9.6. It follows from Theorems 4.3.6 and 4.3.3 that the regular hull of an order separable Archimedean vector lattice in which order convergence and relative uniform convergence are equivalent for every sequence is Archimedean too. Another case is described by the following

**Theorem.** Let $E$ be a vector lattice in which for every sequence $(x_n) \subseteq E^+$, there exists a sequence $(\lambda_n)$ of strictly positive reals such that the set $\{\lambda_n x_n\}$ is order bounded. Then $(r)-E$ is Archimedean.

We need to show that $\lambda(\lambda:E)$ is a relatively uniformly closed ideal in $\text{fin}(\lambda:E)$, by Theorem 4.3.5. To this, consider $0 \leq v_n \uparrow$ and $v_n \overset{(r)}{\rightarrow} v$, where $v_n \in \lambda(\lambda:E)$. It is sufficient to prove that $v \in \lambda(\lambda:E)$.
Since \( u_n \xrightarrow{(r)} u \), there exists a sequence \( (\varepsilon_n) \subseteq \mathbb{R}^+ \), \( \varepsilon_n \to 0 \), and an element \( d \in E^+ \) such that \( |u_n - v| \leq \varepsilon_n d \) for all \( n \in \mathbb{N} \). Since \( u_n \in \lambda(*E) \), there exists \( w_n \in E \) for which \( 0 \leq kv_n \leq w_n \) simultaneously for all \( k \in \mathbb{N} \). By hypothesis, take \( 0 < \lambda_n \in \mathbb{R} \) and \( w \in E \) such that \( \lambda_n w_n \leq w \) for all \( n \in \mathbb{N} \). Consequently,

\[
|v| \leq |u_n - v| + |v| \leq \varepsilon_n d + \max\{\varepsilon_n, 1/n\} \lambda_n w_n \leq \max\{\varepsilon_n, 1/n\}(d + w)
\]

for every \( n \in \mathbb{N} \). Therefore, we have \( v \in \lambda(*E) \), by using \( \varepsilon_n \to 0 \).

**Corollary.** The regular hull of a Banach lattice is Archimedean.

There are non-Archimedean vector lattices whose regular hulls are non-Archimedean either. To see this, we consider an example of the vector lattice \( L \) by T. Nakayama (see [21, Example 62.2]). The ideal

\[
I_0(L) := \{ x \in L : (\exists y \in L)(\forall n \in \mathbb{N}) |nx| \leq y \}
\]

is not relatively uniformly closed in \( L \). Hence, there are a sequence \( (x_n) \subseteq I_0(L) \), \( 0 \leq x_n \uparrow 1 \), and an element \( x \in L \) such that

\[
x_n \xrightarrow{(r)} x \notin I_0(L).
\]

Since \( I_0(L) = \lambda(*L) \cap L \), we have that the ideal \( \lambda(*L) \) is not relatively uniformly closed in \( \text{fin}(*L) \). Thus, by Veksler's Theorem (see [21, Theorem 60.2]), \( (r)-L \) is non-Archimedean. The question remains open whether the regular hull of an arbitrary Archimedean vector lattice is Archimedean.

### 4.10. Order and Regular Hulls of Lattice Normed Spaces

In this section we define and begin studying the order and regular hulls of lattice normed spaces.

#### 4.10.1. Let \( (\mathcal{X}, \alpha, *E) \) be some internal LNS normed by a standard lattice \( *E \). Consider the following external subspaces of the internal vector space \( \mathcal{X} \):

\[
\text{fin}(\mathcal{X}) := \{ x \in \mathcal{X} : \alpha(x) \in \text{fin}(*E) \},
\eta(\mathcal{X}) := \{ x \in \mathcal{X} : \alpha(x) \in \eta(*E) \},
\lambda(\mathcal{X}) := \{ x \in \mathcal{X} : \alpha(x) \in \lambda(*E) \}.
\]

The vector spaces \( \eta(\mathcal{X}) \) and \( \lambda(\mathcal{X}) \) are subspaces of \( \text{fin}(\mathcal{X}) \). Therefore, we may arrange the following quotients:

\[
(o)-\overline{\mathcal{X}} := \text{fin}(\mathcal{X})/\eta(\mathcal{X}),
(r)-\overline{\mathcal{X}} := \text{fin}(\mathcal{X})/\lambda(\mathcal{X}).
\]
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We denote by \([x]\) the coset \(x + \eta(\mathcal{X})\) in \((o) - \overline{\mathcal{X}}\) and by \(\langle x \rangle\) the coset \(x + \lambda(\mathcal{X})\) in \((r) - \overline{\mathcal{X}}\), where \(x \in \text{fin}(\mathcal{X})\). Given \(x \in \text{fin}(\mathcal{X})\), assign

\[
\overline{\alpha}(\langle x \rangle) := \alpha(x) + \eta(\mathcal{X}) \quad \overline{\alpha}(\langle x \rangle) := \alpha(x) + \lambda(\mathcal{X}).
\]

It is easy to see that the mappings \(\overline{\alpha} : (o) - \overline{\mathcal{X}} \to (o) - E\) and \(\overline{\alpha}(r) : (r) - \overline{\mathcal{X}} \to (r) - E\) are well defined.

**Definition.** We call the LNS \(((o) - \overline{\mathcal{X}}, \overline{\alpha}, (o) - E)\) \(((r) - \overline{\mathcal{X}}, \overline{\alpha}(r), (r) - E)\) the order hull (regular hull) of an internal LNS \((\mathcal{X}, \alpha, *E)\).

4.10.2. **Theorem.** Let \((\mathcal{X}, \alpha, *E)\) be an internal decomposable LNS with a standard norm lattice \(*E\). Then its order hull and regular hull are decomposable and \((r)\)-complete LNS.

\(<\) Consider the external LNS \((\text{fin}(\mathcal{X}), \alpha, \text{fin}(\mathcal{X}))\). The proof of \((r)\)-completeness of this LNS is almost the same as the proof of relative uniform completeness of the vector lattice \(\text{fin}(\mathcal{X})\) in 4.8.4 (it suffices to replace \(\text{fin}(\mathcal{X})\) by \(\text{fin}(\mathcal{X})\) and the modulus by the norm \(\alpha\)). Clearly, the norm \(\alpha\) is decomposable in \(\text{fin}(\mathcal{X})\). Since the order hull (regular hull) of \((\mathcal{X}, \alpha, *E)\) is the quotient of \((\text{fin}(\mathcal{X}), \alpha, \text{fin}(\mathcal{X}))\) by the ideal \(\eta(*E)\) (respectively, the ideal \(\lambda(*E)\)) of \(\text{fin}(\mathcal{X})\), we complete the proof by using Proposition 4.0.14. \(>\)

4.10.3. Let \((E, \| \cdot \|)\) be a normed vector lattice. A Dedekind completion \(\widehat{E} = \widehat{\eta}(E)\) of \(E\) is a normed vector lattice under the norm

\[
\|x\| := \inf\{\|e\| : e \in E \land \eta(e) \geq |x|\}. \quad (2)
\]

Now, the LNS \((o) - \overline{E}\) is a normed vector lattice under the norm \(|x| := \|p(x)\|\). We have the direct expression for \(|\cdot|\) as follows:

\[
|x| := \inf\{\|e\| : e \in E \land \eta(e) \geq |x|\} \quad (x \in (o) - \overline{E})
\]

which extends the norm (2) from \(\widehat{E}\) to \((o) - \overline{E}\). Note that the embeddings \(\eta : (E, \| \cdot \|) \hookrightarrow (\widehat{E}, \| \cdot \|)\) and \((\widehat{E}, \| \cdot \|) \subseteq ((o) - \overline{E}, |\cdot|)\) are isometric.

Recall that a normed vector lattice \((E, \| \cdot \|)\) satisfies the weak Riesz–Fisher condition if every sequence \((v_n) \subseteq E\) with the property \(\sum_{n=1}^{\infty} \|v_n\| < \infty\) is order bounded.

**Theorem.** The normed lattice \(((o) - \overline{E}, |\cdot|)\) is a Banach lattice if and only if \((E, \| \cdot \|)\) satisfies the weak Riesz–Fisher condition.
Suppose that \((E, \| \cdot \|)\) satisfies the weak Riesz–Fisher condition. Then \(\hat{E}\) is a Banach lattice under the norm (2) by [28, Theorem 101.6]. Applying [16, Theorem 4.1.2], we obtain from \((r)\)-completeness of the LNS \(((o)\overline{E}, p, \hat{E})\) with 
\[
p(x) = \inf_{\hat{E}} \{ \eta(e) : e \in E \& \eta(e) \geq |x| \},
\]
that \(((o)\overline{E}, |\cdot|)\) is a Banach lattice.

Conversely, suppose that \(((o)\overline{E}, |\cdot|)\) is a Banach lattice and take an arbitrary sequence \((v_n) \subseteq E\) such that \(\sum_{n=1}^{\infty} \|v_n\| < \infty\). Then
\[
\sum_{n=1}^{\infty} |\eta(|v_n|)| = \sum_{n=1}^{\infty} \|v_n\| < \infty.
\]
Consequently, there is an \(u \in (o)\overline{E},\)
\[
u = (o)\overline{E} - \sum_{n=1}^{\infty} |\eta(|v_n|)| \in (o)\overline{E}.
\]
Since \(\eta(E)\) is cofinal in \((o)\overline{E}\), there exists an element \(v \in E\) such that \(\eta(v) > u\). Obviously, \((v_n) \subseteq [-v, v]\). Thus, \((E, \| \cdot \|)\) satisfies the weak Riesz–Fisher condition. \(\triangleright\)

**4.10.4.** In the sequel, we assume that \(E\) is Archimedean. We consider the quotient
\[
\hat{E} := o\text{-pns}(^*E)/\eta(^*E)
\]
and recall that the vector lattice \(\hat{E}\) is a Dedekind completion of \(E\) by Theorem 4.4.1. We need some preliminary work. We start with a few lemmata:

**Lemma.** Let \(y \in o\text{-pns}(^*E)\). Then
\[
[y] = \inf_{\hat{E}} \eta(U(y)).
\]
\(<\) Since \(L(y) \leq y \leq U(y)\) and \(\hat{E}\) is Dedekind complete, the following holds:
\[
\sup_{\hat{E}} \eta(L(y)) \leq [y] \leq \inf_{\hat{E}} \eta(U(y)).
\]
Consequently,
\[
0 \leq \inf_{\hat{E}} \eta(U(y)) - [y] \leq \inf_{\hat{E}} \eta(U(y)) - \sup_{\hat{E}} \eta(L(y)) \leq \inf_{\hat{E}} \eta(U(y) - L(y)).
\]
Since \(y \in o\text{-pns}(^*E)\), we have \(\inf_{\hat{E}} (U(y) - L(y)) = 0\). Thus
\[
\inf_{\hat{E}} \eta(U(y) - L(y)) = 0,
\]
because \(\hat{E}\) is a Dedekind completion of the sublattice \(\eta(E)\). Therefore, the above-established inequality implies \([y] = \inf_{\hat{E}} \eta(U(y))\). \(\triangleright\)
4.10.5. **Lemma.** Each nonempty order bounded subset $\mathcal{D} \subseteq \widehat{E}$ possesses some supremum and infimum in $(\sigma)-E$. Moreover,

1. $\inf_{(\sigma)-E} \mathcal{D} = \inf_E \mathcal{D}$;
2. $\sup_{(\sigma)-E} \mathcal{D} = \sup_E \mathcal{D}$.

\( \leq \) (1) Suppose that $\mathcal{D} \subseteq \widehat{E}$ and $\mathcal{D} \neq \emptyset$. It is sufficient to show that $\inf_E \mathcal{D} = 0$ implies $\inf_{(\sigma)-E} \mathcal{D} = 0$. Take a $\kappa \in \text{fin}(E^*)$ such that $0 \leq \kappa$ and $[\kappa] \leq \mathcal{D}$. To complete the proof, it remains to establish that $[\kappa] = 0$ or, in other words, to verify the condition $\inf_E U(\kappa) = 0$. Assume that an element $a \in E$ satisfies the inequality

$$0 \leq a \leq U(\kappa)$$

and take an arbitrary $d \in \mathcal{D}$. Then $d = [\delta]$ for some $\delta \in \sigma\text{-pns}(E^*)$. It is obvious that

$$\kappa = \kappa \land \delta + (\kappa - \delta)_+ \leq U(\delta) + U((\kappa - \delta)_+).$$

Consequently,

$$U(\delta) + U((\kappa - \delta)_+) \subseteq U(\kappa).$$

From (3) and (4), it ensues that

$$0 \leq a \leq U(\delta) + U((\kappa - \delta)_+).$$

Using Lemma 4.1.2 and Dedekind completeness of $\widehat{E}$, we obtain from (5) that

$$0 \leq \hat{\eta}(a) \leq \inf_{\widehat{E}} \hat{\eta}(U(\delta)) + \inf_{\widehat{E}} \hat{\eta}(U((\kappa - \delta)_+)) = [\delta] + \inf_{\widehat{E}} \hat{\eta}(U((\kappa - \delta)_+:))$$

At the same time, $[(\kappa - \delta)_+)] = [\kappa] - d = d$. Hence, $\inf_E U((\kappa - \delta)_+) = 0$. Thus, we have $\inf_{\widehat{E}} \hat{\eta}(U((\kappa - \delta)_+)) = 0$. Now, (6) implies

$$0 \leq \hat{\eta}(a) \leq [\delta] = d.$$ 

Since $d \in \mathcal{D}$ is arbitrary; therefore, in view of $\inf_{\widehat{E}} \mathcal{D} = 0$, we deduce from (7) that $a = 0$, as required.

Assertion (2) ensues immediately from (a).

4.10.6. **Let** $x \in (\sigma)-E$. **Assign**

$$\mathcal{U}(x) := \{e \in E : \hat{\eta}(e) \geq x\},$$

$$\tilde{\mathcal{U}}(x) := \{y \in \widehat{E} : y \geq x\}.$$ 

It is clear that $\mathcal{U}(x)$ and $\tilde{\mathcal{U}}(x)$ are nonempty order bounded subsets of $E$ and $\widehat{E}$ respectively.
**Lemma.** For every \( x \in (o)-\tilde{E} \), the following hold:

1. \( \inf_{\tilde{E}} \hat{\mathcal{U}}(x) \in \hat{\mathcal{U}}(x) \);
2. \( \inf_{\tilde{E}} \hat{\mathcal{U}}(x) = \inf_{\tilde{E}} \hat{\eta}(\mathcal{U}(x)) \);
3. If an element \( \star \in \text{fin}(*E) \) satisfies \( x = [\star] \) then
   \[ \inf_{\tilde{E}} \hat{\mathcal{U}}(x) = \inf_{\tilde{E}} \hat{\eta}(\mathcal{U}(x)) = \inf_{\tilde{E}} \hat{\eta}(U(\star)) . \]

\(<1>(1): By Lemma 4.10.5, \( \mathcal{U}(x) \) has an infimum in \((o)-\tilde{E}\), and \( \inf_{(o)-\tilde{E}} \hat{\mathcal{U}}(x) = \inf_{\tilde{E}} \hat{\mathcal{U}}(x) \). Hence, from \( \mathcal{U}(x) \geq x \), it follows that \( \inf_{(o)-\tilde{E}} \hat{\mathcal{U}}(x) \geq x \). Then \( \inf_{\tilde{E}} \hat{\mathcal{U}}(x) \geq x \), as required.

\((2): Assign x_0 := \inf_{\tilde{E}} \hat{\mathcal{U}}(x) . From (1) it ensues that \( \mathcal{U}(x_0) \subseteq \mathcal{U}(x) \). Establish the reverse inclusion. Let \( z \in \mathcal{U}(x) \). Then \( \hat{\eta}(z) \geq x \), and hence \( \hat{\eta}(z) \in \hat{\mathcal{U}}(x) \). Consequently, \( \hat{\eta}(z) \geq x_0 \) which is equivalent to \( z \in \mathcal{U}(x_0) \). To complete the proof, it remains to see that the relation \( x_0 = \inf_{\tilde{E}} \hat{\eta}(\mathcal{U}(x_0)) \) is valid since \( \tilde{E} \) is an order completion of \( \hat{\eta}(E) \).

\((3): Let \( \star \in \text{fin}(\mathcal{E}) \) and \( x = [\star] \). Assign \( x_0 := \inf_{\tilde{E}} \hat{\mathcal{U}}(x) . According to (1), \( x_0 \geq x \). Take an element \( y \in \text{fin}(\mathcal{E}) \) such that \( x_0 = [y] \) and \( y \geq \star \). Then we have from Lemma 4.10.4

\[ x_0 = [y] = \inf_{\tilde{E}} \hat{\eta}(U(y)) \geq \inf_{\tilde{E}} \hat{\eta}(U(\star)). \]

Next, the obvious inclusion \( \hat{\eta}(U(\star)) \subseteq \hat{\mathcal{U}}([\star]) \) implies

\[ \inf_{\tilde{E}} \hat{\eta}(U(\star)) \geq \inf_{\tilde{E}} \hat{\mathcal{U}}([\star]) = x_0 . \]

From (8), (9), and (1) we now obtain the required result. \( \triangleright \)

**4.10.7.** Define the mapping \( p : (o)-\tilde{E} \to \tilde{E} \) as follows:

\[ p(x) := \inf_{\tilde{E}} \hat{\mathcal{U}}([x]) \ (x \in (o)-\tilde{E}). \]

We list some properties of this mapping.

**Theorem.** The mapping \( p \) is an \( \tilde{E} \)-valued norm on \((o)-\tilde{E}\) such that, for all \( x, y \in (o)-\tilde{E} \), the following hold:
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\( p(x) = \inf_{\mathcal{E}} \tilde{n}(\mathcal{U}(|x|)) \);

\( p(x) \geq |x| \);

\( |x| \geq |y| \) implies \( p(x) \geq p(y) \).

Moreover, an arbitrary sequence \( (x_n) \subseteq (o)-\mathcal{E} \) (r)-converges in the norm \( p \) to an element \( x_0 \in (o)-\mathcal{E} \) (is \( (r) \)-Cauchy in the norm \( p \)) if and only if it \( (r) \)-converges to \( x_0 \) in the vector lattice \((o)-\mathcal{E} \) (is \( (r) \)-Cauchy in \((o)-\mathcal{E} \)). The lattice normed space \(((o)-\mathcal{E}, p, \mathcal{E})\) is \((r)\)-complete and decomposable.

\(<\) From (10) it is straightforward that the mapping \( p \) satisfies conditions 4.0.10(2), and 4.0.10(3), and item (3) of the theorem. Item (2) of the theorem follows from 4.10.6(1). Now, conditions 4.0.10(1) ensue from (2). Hence, \( p \) is an \( \mathcal{E} \)-valued norm on \((o)-\mathcal{E}\). Condition (1) is a particular instance of 4.10.6(2).

If \( x_n \xrightarrow{(r)} x_0 \) in the norm \( p \) with regulator \( e \in \mathcal{E} \) then, in view of item (2) of the theorem, \( x_n \xrightarrow{(r)} x_0 \) in \((o)-\mathcal{E} \) with the same regulator. Conversely, if \( x_n \xrightarrow{(r)} x_0 \) in \((o)-\mathcal{E} \) with regulator \( d \in (o)-\mathcal{E} \) then \( x_n \xrightarrow{(r)} x_0 \) in the norm \( p \) with regulator \( p(d) \), in accordance with item (3) of the theorem. For \( (r) \)-Cauchy sequences the proof is essentially the same.

The quotient \((o)-\mathcal{E}\) is relatively uniformly complete in view of Theorem 4.8.4. So, as we showed, \(((o)-\mathcal{E}, p, \mathcal{E})\) is \((r)\)-complete. Now, to verify decomposability of \( p \), it is sufficient to establish \((d)\)-decomposability of \( p \) by Proposition 4.0.10. Let \( x \in (o)-\mathcal{E} \) and let \( e_1, e_2 \in \mathcal{E} \) be such that \( p(x) = e_1 + e_2 \) and \( e_1 \wedge e_2 = 0 \). Assign

\[
\begin{align*}
x_1 &= x_+ \wedge e_1 - x_- \wedge e_1; \\
x_2 &= x_+ \wedge e_2 - x_- \wedge e_2.
\end{align*}
\]

It is easy to see that \( p(x_1) = e_1, p(x_2) = e_2 \), and \( x = x_1 + x_2 \).

\[4.10.8.\] Consider the mapping \( p \circ \overline{\alpha} : (o) - \overline{\mathcal{E}} \to \hat{\mathcal{E}} \), where \( \overline{\alpha} : (o) - \overline{\mathcal{E}} \to (o) - \mathcal{E} \) is the \((o)\)-\( \mathcal{E} \)-valued norm defined in 4.10.1.

**Theorem.** The triple \(((o) - \overline{\mathcal{E}}, p \circ \overline{\alpha}, \hat{\mathcal{E}})\) is a decomposable \((r)\)-complete LNS.

\(<\) It is easy to see that \( p \circ \overline{\alpha} \) is an \( \hat{\mathcal{E}} \)-valued norm in \((o) - \overline{\mathcal{E}} \). Take an arbitrary sequence \( (x_n) \subseteq (o) - \overline{\mathcal{E}} \) that is \((r)\)-Cauchy in the norm \( p \circ \overline{\alpha} \) with regulator \( e \in \mathcal{E} \). In view of Theorem 4.10.7 (item (2)), it is \((r)\)-Cauchy in the norm \( \overline{\alpha} \) with the same regulator. Consequently, by Theorem 4.10.2, there is an element \( x_0 \in (o) - \overline{\mathcal{E}} \) such that \( x_n \xrightarrow{(r)} x_0 (e) \) in the norm \( \overline{\alpha} \). Now, from Theorem 4.10.7 (item (3)) it ensues that \( x_n \xrightarrow{(r)} x_0 \) in the norm \( p \circ \overline{\alpha} \) with regulator \( p(e) = e \). Thus, every sequence \( (x_n) \subseteq (o) - \overline{\mathcal{E}} \), that is \((r)\)-Cauchy in the norm \( p \circ \overline{\alpha} \), is \((r)\)-convergent in \((o) - \overline{\mathcal{E}} \) with the same regulator. Thus, \((o) - \overline{\mathcal{E}} \) is \((r)\)-complete in the norm \( p \circ \overline{\alpha} \).
In view of \((r)\)-completeness, to establish decomposability of the norm \(p \circ \overline{\alpha}\), it is sufficient to verify its \((d)\)-decomposability, by Proposition 4.0.10. Let \(x \in \langle o \rangle - \mathcal{X}\) and let \(e_1, e_2 \in \widehat{E}\) be such that \(p \circ \overline{\alpha}(x) = e_1 + e_2\) and \(e_1 \wedge e_2 = 0\). Then decomposability of the norm \(p\) implies that there are \(\alpha_1, \alpha_2 \in \langle o \rangle - \widehat{E}\) such that \(\overline{\alpha}(x) = \alpha_1 + \alpha_2\), \(p(\alpha_1) = e_1\), and \(p(\alpha_2) = e_2\). In view of Theorem 4.10.7 (item (2)), we have \(\alpha_1 \leq e_1\) and \(\alpha_2 \leq e_2\). Hence, from the conditions \(\alpha_1 + \alpha_2 = \overline{\alpha}(x) \geq 0\) and \(e_1 \wedge e_2 = 0\) it ensues that \(\alpha_1 \geq 0\) and \(\alpha_2 \geq 0\). It remains to use decomposability of the norm \(\overline{\alpha}\) for finding elements \(x_1, x_2 \in \langle o \rangle - \mathcal{X}\) such that \(x_1 + x_2 = x\), \(\alpha(x_1) = \alpha_1\), and \(\alpha(x_2) = \alpha_2\). It is clear that \(p \circ \overline{\alpha}(x_1) = e_1\) and \(p \circ \overline{\alpha}(x_2) = e_2\).

4.11. Associated Banach–Kantorovich Spaces

We give a nonstandard construction of an order completion of a decomposable LNS. The scheme rests on embedding the LNS into the associated Banach–Kantorovich space (BKS). We study extensions onto associated BKSs of internal dominated operators admitting standard \((o)\)-continuous dominants. Throughout the section we suppose that \((\mathcal{X}, a, E)\) and \((\mathcal{W}, b, F)\) are decomposable LNS in which the norm lattices \(E\) and \(F\) are Dedekind complete.

4.11.1. The lattice normed space \(((o) - \mathcal{X}, p \circ \overline{\alpha}, \widehat{E})\) defined in 4.10.8 is called associated with the order hull \(((o) - \mathcal{X}, \overline{\alpha}, \langle o \rangle - E)\) of the LNS \((X, a, E)\). We establish that this LNS is a Banach–Kantorovich space.

Since the vector lattice \(\widehat{E}\) is a Dedekind completion of \(E\), we have \(\widehat{E} \cong E\) under our assumptions. To be more precise, the mapping \(\hat{\eta} : E \to \widehat{E}\) is a Riesz isomorphism of \(E\) onto \(\widehat{E}\). Consider the mapping \(\rho_E : \langle o \rangle - \widehat{E} \to E\) defined by the rule

\[
\rho_E(x) := \inf_{E} \{ e \in E : \hat{\eta}(e) \geq |x| \} \quad (x \in \langle o \rangle - \widehat{E}).
\]

We have the following

**Lemma.** The mapping \(\rho_E\) is connected with the norm \(p : \langle o \rangle - \widehat{E} \to \widehat{E}\) by the relation \(\rho_E = \hat{\eta}^{-1} \circ p\). Moreover, for every \(x \in \text{fin}(*E)\), we have

\[
\rho_E([x]) = \inf_{E} \mathcal{W}([x]) = \inf_{E} \eta^{-1} \mathcal{W}(U([x])) = \hat{\eta}^{-1} \inf_{E} \hat{\eta}(U([x])) = \hat{\eta}^{-1} \inf_{E} \mathcal{U}(U([x])) = \hat{\eta}^{-1} \mathcal{U}(U([x]))
\]

\(< The first part of the lemma ensues from the definitions of \(p\) and \(\rho_E\), and the second from the relation

\[
\rho_E([x]) = \inf_{E} \mathcal{W}([x])
\]
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\[= \inf_{\mathcal{E}} U(\alpha) = \inf_{\mathcal{E}} \{e \in \mathcal{E} : e \geq \alpha\},\]

in which the second and the fourth equalities are valid because \(\hat{\eta}\) is a Riesz isomorphism, and the third in view of item (3) of Lemma 4.10.6. \(\triangleright\)

Theorem 4.10.8 and Lemma 4.11.1 imply the following

**Corollary.** The triple \(((\alpha), \rho, \alpha, E)\) is a decomposable \((\tau)\)-complete lattice normed space. Moreover, for each \(x \in \text{fin}(E)\), we have

\[\rho \circ \alpha((\alpha)) = \inf_{\mathcal{E}} \{e \in \mathcal{E} : e \geq \alpha(\alpha)\}.\]  (11)

4.11.2. Denote by \(\mathcal{B}(E)\) the family of all band projections in \(E\). Note that for each internal band projection \(\tau \in \mathcal{B}(E)\), there exists a unique band projection \(h(\tau)\) in \(\mathcal{X}\) satisfying the condition

\[\alpha(h(\tau)) = \tau \alpha(\alpha) \quad (\tau \in \mathcal{X}).\]

This property is easily obtainable from decomposability of the internal norm \(\alpha : \mathcal{X} \to \mathcal{E}\).

**Lemma.** For all \(\pi \in \mathcal{B}(E)\) and \(\alpha \in \text{fin}(\mathcal{X})\), we have

\[\pi \circ \rho \circ \alpha((\alpha)) = \rho \circ \alpha((h(\pi))\alpha)\]

Let \(x \in \text{fin}(\mathcal{X})\). Show that, for every \(\pi \in \mathcal{B}(E)\), the inequality

\[\pi \circ \rho \circ \alpha((\alpha)) \geq \rho \circ \alpha((h(\pi))\alpha)\]  (12)

holds. To this end, take an \(e \in \mathcal{E}, e \geq \alpha(\alpha)\). Then \(\pi e \geq \alpha(h(\pi))\). Let \(x \in \text{fin}(\mathcal{X})\). Then \(\pi \geq \alpha(h(\pi))\).

Applying (11), obtain

\[\pi(e) \geq \inf_{\mathcal{E}} \{f \in \mathcal{E} : f \geq \alpha(h(\pi))\} = \rho \circ \alpha((h(\pi))\alpha)\]

Since \(e \in \mathcal{E}, e \geq \alpha(\alpha)\), is taken arbitrarily, we obtain from order continuity of \(\pi\) and (11) that

\[\pi \circ \rho \circ \alpha((\alpha)) = \pi \inf_{\mathcal{E}} \{e \in \mathcal{E} : e \geq \alpha(\alpha)\}\]

= \inf_{\mathcal{E}} \{\pi e : e \in \mathcal{E} \& e \geq \alpha(\alpha)\} \geq \rho \circ \alpha((h(\pi))\alpha)\]

Inequality (12) is established.
Consider an arbitrary band projection $\pi \in \mathcal{B}(E)$ and denote by $\pi^\perp$ the complementary projection to $\pi$. Then, applying (12) to $\pi$ and $\pi^\perp$, we have

$$\rho_E \circ \alpha((x)) = \pi \circ \rho_E \circ \alpha((x)) + \pi^\perp \circ \rho_E \circ \alpha((x))$$

$$\geq \rho_E \circ \alpha((h^*(\pi)x)) + \rho_E \circ \alpha((h^*(\pi)x))$$

$$\geq \rho_E((\pi \circ \alpha(x) + \pi^\perp \circ \alpha(x))) = \rho_E \circ \alpha((x)).$$

Hence,

$$\pi \circ \rho_E \circ \alpha((x)) + \pi^\perp \circ \rho_E \circ \alpha((x))$$

$$= \rho_E \circ \alpha((h^*(\pi)x)) + \rho_E \circ \alpha((h^*(\pi)x)).$$

Consequently, in view of (12), $\pi \circ \rho_E \circ \alpha((x)) = \rho_E \circ \alpha((h^*(\pi)x))$, as required. \(\triangleright\)

**4.11.3. Lemma.** The associated lattice normed space \(((o)\cdot \mathcal{X}, \rho_E \circ \alpha, E)\) is disjointly complete.

Take an arbitrary partition of unity $(\pi_\xi)_{\xi \in \Xi} \subseteq \mathcal{B}(E)$ and a family $(x_\xi)_{\xi \in \Xi} \subseteq (o)\cdot \mathcal{X}$ bounded in the norm $\rho_E \circ \alpha$.

Suppose that, for $e \in E$, we have

$$\rho_E \circ \alpha(x_\xi) \leq e \quad (\xi \in \Xi). \quad (13)$$

Choose a $x_\xi \in \mathcal{X}$ such that $\langle x_\xi \rangle = x_\xi$ for all $\xi \in \Xi$. Using the definition of the norm $\alpha$, the relation $\rho_E = \eta^{-1} \circ \rho$, and item (2) of Theorem 4.10.7, we rewrite inequality (13) as $[\alpha(x_\xi)] \leq \eta(e)$. Consequently, for a suitable $\eta_\xi \in \eta(\mathcal{X})$, we have the inequality $\alpha(x_\xi) \leq e + \eta_\xi \quad (\xi \in \Xi)$. Hence, according to Lemma 4.0.13, there are elements $x'_\xi \in \text{fin}(\mathcal{X})$ for which

$$\alpha(x_\xi - x'_\xi) \in \eta(\mathcal{X}), \quad \alpha(x'_\xi) \leq e \quad (\xi \in \Xi). \quad (14)$$

Fix some $\nu \in \ast \mathbb{N} \setminus \mathbb{N}$ and denote by $\mathcal{F}$ the set of all internal mappings from $\ast \Xi$ into $\mathcal{X}$. Let Card stand for the internal cardinality. Given $\xi \in \Xi$, define the internal subset $A_\xi$ of $\mathcal{F}$ as follows:

$$A_\xi := \{ \varphi \in \mathcal{F} : \alpha \circ \varphi(\ast \Xi) \subseteq [-e, e] \land \varphi(\xi) = x'_\xi \land \text{Card}(\{\xi \in \ast \Xi : \varphi(\xi) \neq 0\}) \leq \nu \}.$$ 

It is easy to see that the family $(A_\xi)_{\xi \in \Xi}$ has the finite intersection property. Consequently, in view of the general saturation principle, there is an element $\varphi_0 \in \cap \{A_\xi : \xi \in \Xi\}$. Put

$$\Theta := \{ \xi \in \ast \Xi : \varphi_0(\xi) \neq 0 \}.$$
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Since Card(Θ) ≤ ν, the set Θ is hyperfinite. Furthermore,

Ξ ⊆ Θ ⊆ *Ξ and α(φ₀(ξ)) ≤ e (ξ ∈ Θ).

For convenience, we let x'ξ := φ₀(ξ) whenever ξ ∈ Θ. This does not cause difficulties, since φ₀(ξ) = x'ξ for all ξ ∈ Ξ by the choice of φ₀.

Thus, the family (x'ξ)ξ∈Ξ extends to a hyperfinite family (x'ξ)ξ∈Θ ⊆ X such that

α(x'ξ) ≤ e (ξ ∈ Θ).

(15)

Let (τξ)ξ∈Ξ := *(πξ)ξ∈Ξ be the nonstandard enlargement of the partition (πξ)ξ∈Ξ of unity. Then (τξ)ξ∈Ξ is an internal partition of unity in *B(E). Furthermore, τξ = *πξ for all ξ ∈ Ξ. The hyperfinite sum χ := Σξ∈Θ h(τξ)x'ξ is an element of the internal vector space X, where h(τξ)'s are the band projections defined in 4.11.2. Disjointness of the set of projections τξ, together with (15), implies |χ| ≤ e. In particular, χ ∈ fin(X).

For every ξ₀ ∈ Ξ, consider the following chain of equalities:

\[ \pi_{ξ₀} \circ ρ_E \circ \overline{α}(x_{ξ₀} - ⟨χ⟩) = \pi_{ξ₀} \circ ρ_E \circ \overline{α}(⟨x'_{ξ₀} - χ⟩) = ρ_E \circ \overline{α}(h(π_{ξ₀})(x'_{ξ₀} - χ)) = ρ_E \circ \overline{α}(⟨h(π_{ξ₀})\left( \sum_{ξ∈Θ\setminus\{ξ₀\}} h(τξ)x'ξ \right)⟩) = 0. \]

The first equality holds due to the choice of the elements xξ and (14). Validity of the second is ensured by Lemma 4.11.2. The third equality is valid by the choice of χ and the equality τξ₀ = *πξ₀ mentioned above. Finally, the last equality holds since τξ is disjoint. Therefore, πξ ∘ ρ_E ∘ \overline{α}(x_{ξ} - ⟨χ⟩) = 0 for every ξ ∈ Ξ, and hence ⟨χ⟩ = mix(π_{ξ}x_{ξ})ξ∈Ξ.

Thus, for every partition of unity (πξ)ξ∈Ξ ⊆ B(E) and every bounded in the norm ρ_E ∘ \overline{α} family (x_{ξ})ξ∈Ξ, the mixing mix(π_{ξ}x_{ξ}) ∈ ⟨o⟩-X exists. The proof of the lemma is complete.

4.11.4. We are in a position to state the main result of this section.

Theorem. The associated lattice normed space (⟨o⟩-X, ρ_E ∘ \overline{α}, E) is a Banach–Kantorovich space.

It follows immediately from Corollary 4.11.1 and Lemma 4.11.3 by using Proposition 4.0.11. □

Note that, whenever we take an internal normed space X as an internal decomposable LNS, the associated space coincides with the classical nonstandard hull.
Furthermore, from the above-established theorem, the well-known assertion ensues that the nonstandard hull of an internal normed space is a Banach space.

If we consider an internal lattice normed space \((*E, |·|, *E)\) then the associated space is the LNS \(((o)-E, \rho_E, E)\). From the definition of the mapping \(\rho_E : (o)-\bar{E} \to E\) it is clear that, for all \(x, y \in (o)-\bar{E}\), the condition \(|x| \leq |y|\) implies \(\rho_E(x) \leq \rho_E(y)\). So, we obtain that the associated LNS \(((o)-\bar{E}, \rho_E, E)\) is a Banach–Kantorovich lattice.

4.11.5. It is known that a norm completion of an normed space \(X\) can be obtained by taking the closure of the space in the nonstandard hull of \(X\). Similarly, as we show below, an \((o)\)-completion of a decomposable LNS can be constructed on embedding it into the associated BKS.

For simplicity, denote by \(((o)-\bar{X}, \rho_E \circ \bar{a}, E)\) the BKS associated with the order hull of \((X, a, E)\). Consider the mapping \(\hat{\eta} : X \to (o)-\bar{X}\) such that

\[
\hat{\eta}(x) := \langle x \rangle \quad (x \in X).
\]

It is easy to see that \(\hat{\eta}\) is an isometrically isomorphic embedding of the LNS \((X, a, E)\) into \(((o)-\bar{X}, \rho_E \circ \bar{a}, E)\). Denote by \(\bar{X}\) the set of limits of all \(\rho_E \circ \bar{a}\) convergent nets of elements of \(\hat{\eta}(X)\).

**Lemma.** For every element \(x \in (o)-\bar{X}\), the following are equivalent:

1. \(x \in \bar{X}\);
2. \(\inf_{y \in X} \rho_E \circ \bar{a}(x - \hat{\eta}(y)) = 0\).

\(< (1) \to (2): \) This is immediate from the definition of \(\bar{X}\).

\((2) \to (1): \) Let an element \(x \in (o)-\bar{X}\) satisfy condition (2). Show that \(x \in \bar{X}\). Define the relation \(\prec\) on \(X\) as follows:

\[y \prec z \iff \rho_E \circ \bar{a}(x - \hat{\eta}(y)) \geq \rho_E \circ \bar{a}(x - \hat{\eta}(z)).\]

The set \(X\) is directed upwards with respect to \(\prec\). Indeed, for all \(y, z \in X\), we have \(y, z \leq h(\pi)y + h(\pi^\perp)z\), where \(\pi \in B(E)\) is the band projection that satisfies the condition

\[\pi \circ \rho_E \circ \bar{a}(x - \hat{\eta}(y)) + \pi^\perp \circ \rho_E \circ \bar{a}(x - \hat{\eta}(z)) = \rho_E \circ \bar{a}(x - \hat{\eta}(y)) \wedge \rho_E \circ \bar{a}(x - \hat{\eta}(z)),\]

and \(h(\pi)\) and \(h(\pi^\perp)\) are the corresponding band projections in \((X, a, E)\). Consider the net \((\hat{\eta}(y))_{y \in (X, \prec)}\). From the definition of \((X, \prec)\), in view of the condition \(\inf_{y \in X} \rho_E \circ \bar{a}(x - \hat{\eta}(y)) = 0\), it follows that the net \((\hat{\eta}(y))_{y \in (X, \prec)}\) converges to \(x\) in \((o)-\bar{X}\). Since the net is constituted by elements of \(\hat{\eta}(X)\), we have \(x \in \bar{X}\).
4.11.6. Theorem. The triple \((\hat{X}, \rho_E \circ \overline{a}, E)\) is an \((\circ)\)-completion of the decomposable LNS \((X, a, E)\).

It is sufficient to verify properties 4.0.12 (1)–(3). Clearly, \((\hat{X}, \rho_E \circ \overline{a}, E)\) is a lattice normed space. Show that this space is \((\circ)\)-complete. Take an arbitrary \((\circ)\)-Cauchy net \((x_\xi)\). Then, by \((\circ)\)-completeness of the associated LNS, there is an element \(x \in (\circ)\overline{X}\) such that \(x = (\circ)\lim(x_\xi)\). Show that \(x \in \hat{X}\). The conditions \(x_\xi \in \hat{X}\) and \(x = (\circ)\lim(x_\xi)\) imply that

\[
\inf_{y \in \hat{X}} \rho_E \circ \overline{a}(x_\xi - \tilde{\eta}(y)) = 0, \quad \inf_{\xi} \rho_E \circ \overline{a}(x - x_\xi) = 0. \tag{17}
\]

Next, from (17) we obtain

\[
0 \leq \inf_{y \in \hat{X}} \rho_E \circ \overline{a}(x - \tilde{\eta}(y)) \\
\leq \inf_{\xi} \inf_{y \in \hat{X}} (\rho_E \circ \overline{a}(x - x_\xi) + \rho_E \circ \overline{a}(x_\xi - \tilde{\eta}(y))) \\
\leq \inf_{\xi} \rho_E \circ \overline{a}(x - x_\xi) + \inf_{\xi} \rho_E \circ \overline{a}(x_\xi - \tilde{\eta}(y)) = 0.
\]

Thereby, \(\inf_{y \in \hat{X}} \rho_E \circ \overline{a}(x - \tilde{\eta}(y)) = 0\) and, in view of Lemma 4.11.5, we have \(x \in \hat{X}\). Thus, every \((\circ)\)-Cauchy net \((x_\xi)\) in \(\hat{X}\) is \((\circ)\)-convergent. Hence, \(\hat{X}\) is \((\circ)\)-complete in the norm \(\rho_E \circ \overline{a}\). It is easy to verify that the norm \(\rho_E \circ \overline{a}\) is \((\delta)\)-decomposable on \(\hat{X}\). Consequently, taking \((\circ)\)-completeness and Proposition 4.0.10 into account, we obtain decomposability of the norm \(\rho_E \circ \overline{a}\) on \(\hat{X}\). Property 4.0.12 (1) is established. Property 4.0.12 (2) is obvious for the embedding \(\tilde{\eta} : X \to \hat{X}\).

To verify 4.0.12 (3), take some \(x' \in \hat{X}\) and \(e \in E_+\). Assign

\[
E' := \{ \pi \in \mathcal{B}(E) : \pi \circ \rho_E \circ \overline{a}(x' - \tilde{\eta}(y)) \leq e \ \text{for some} \ x \in X \}.
\]

Since \(x' \in \hat{X}\), we have

\[
\inf_{x' \in \hat{X}} \rho_E \circ \overline{a}(x' - \tilde{\eta}(x)) = 0.
\]

Hence, the set \(E'\) is dense in the band \(\mathcal{B}_e(E)\) generated by the band projection \(pr_e\). By the exhaustion principle, there exists a partition \((\sigma_\gamma)_{\gamma \in \Omega} \subseteq E'\) of \(pr_e\). In accordance with the definition of \(E'\), there is a family \((x'_\gamma)_{\gamma \in \Omega} \subseteq X\) for which

\[
\sigma_\gamma \circ \rho_E \circ \overline{a}(x' - \tilde{\eta}(x'_\gamma)) \leq e \ (\gamma \in \Omega). \tag{18}
\]

Take a \(\gamma_0 \notin \Omega\) and assign \(\Gamma := \Omega \cup \{\gamma_0\}\), \(\sigma_{\gamma_0} := pr_{\gamma_0}^e\), and \(x'_{\gamma_0} := 0\). For each \(\gamma \in \Gamma\), due to decomposability of \((X, a, E)\), there exists a unique band projection \(\tau_\gamma\) in
the space $X$ which satisfies $a \circ \tau_\gamma = \delta_\gamma \circ a$. Define a family $(x_\gamma)_{\gamma \in \Gamma} \subseteq X$ so that $x_\gamma := \tau_\gamma x'_\gamma$ for every $\gamma \in \Gamma$. From (18) it follows that

$$a(x_\gamma) = a(\tau_\gamma x'_\gamma) = \sigma_\gamma \circ a(x'_\gamma) = \sigma_\gamma \circ \rho_E \circ \bar{a}(\bar{\eta}(x'_\gamma))$$

$$\leq \sigma_\gamma \circ \rho_E \circ \bar{a}(x') + \sigma_\gamma \circ \rho_E \circ \bar{a}(x'_\gamma - \bar{\eta}(x'_\gamma)) \leq 2 \rho_E \circ \bar{a}(x') + e$$

for each $\gamma \in \Gamma$. Thus, the family $(x_\gamma)_{\gamma \in \Gamma}$ is bounded with respect to the norm $a$. By $(o)$-completeness of $(\tilde{X}, \rho_E \circ \bar{a}, E)$, the mixing $\text{mix}(\delta_\gamma, \bar{\eta}(x_\gamma))_{\gamma \in \Gamma} \in \tilde{X}$ exists. Using (18) again, we see that

$$\text{pr}_e \circ \rho_E \circ \bar{a}(x' - \text{mix}(\sigma_\gamma, \bar{\eta}(x_\gamma))) \leq e.$$ 

Since the choice of the elements $x'_\gamma \in \tilde{X}$ and $e \in E_+$ is arbitrary, property 4.0.12 (3) is established. The proof of the theorem is complete. 

4.11.7. Denote the space of all regular (respectively, order continuous) operators from $E$ into $F$ by $L_\tau(E, F) (L_n(E, F))$, and denote by $\mathcal{M}(\mathcal{X}, \mathcal{Y})$ the set of all internal linear operators from $\mathcal{X}$ into $\mathcal{Y}$ which admit some standard dominant $^*Q, Q \in L_\tau(E, F)$ (see 4.0.15). Next, let $\mathcal{M}_n(\mathcal{X}, \mathcal{Y})$ be the set of all operators in $\mathcal{M}(\mathcal{X}, \mathcal{Y})$ each of which admits a dominant of the form $^*S$ with $S \in L_n(E, F)$.

**Lemma.** For every internal linear operator $T : \mathcal{X} \to \mathcal{Y}$, the following hold:

1. $T \in \mathcal{M}(\mathcal{X}, \mathcal{Y}) \Rightarrow T(\text{fin}(\mathcal{X})) \subseteq \text{fin}(\mathcal{Y})$;
2. $T \in \mathcal{M}_n(\mathcal{X}, \mathcal{Y}) \Rightarrow T(\eta(\mathcal{X})) \subseteq \eta(\mathcal{Y})$.

\(< We verify only (1), since (2) is established similarly. According to the condition $T \in \mathcal{M}(\mathcal{X}, \mathcal{Y})$, there is an operator $Q \in L_\tau(E, F)$ for which

$$\beta(Tx) \leq ^*Q\alpha(x) \quad (x \in \mathcal{X}).$$ (19)

Take an arbitrary $x \in \text{fin}(\mathcal{X})$. Then $\alpha(x) \leq e$ for some $e \in E$. By (19), we have $\beta(Tx) \leq Q(e)$ and, consequently, $Tx \in \text{fin}(\mathcal{Y})$. 

Throughout the sequel, the vector lattice $L_n(E, F)$ is denoted by $L$.

4.11.8. Suppose that $T \in \mathcal{M}_n(\mathcal{X}, \mathcal{Y})$. According to Lemma 4.11.7, the mapping $\overline{T} : (o)-\mathcal{X} \to (o)-\mathcal{Y}$ acting as

$$\overline{T}(\langle x \rangle) := \langle Tx \rangle \quad (x \in \text{fin}(\mathcal{X}))$$ (20)

is soundly defined.
Theorem. The mapping $\overline{\mathcal{T}}$ is a dominated linear operator from the associated BKS $((o)\overline{\mathcal{X}}, \rho_E \circ \alpha, E)$ to the associated BKS $((o)\overline{\mathcal{Y}}, \rho_F \circ \beta, F)$. Furthermore, $\langle \langle \overline{\mathcal{T}} \rangle \rangle \leq \rho_L(\langle \langle \mathcal{T} \rangle \rangle)$.

Before proving, we make necessary clarifications. By $\langle \langle \mathcal{T} \rangle \rangle$ (by $\langle \langle \mathcal{T} \rangle \rangle$) we denote the least (least internal) dominant of $\overline{\mathcal{T}}$ (of $\mathcal{T}$). By $\rho_L$ we denote the $L$-valued norm in the LNS $((o)\overline{\mathcal{L}}, \rho_L, L)$.

It is sufficient to establish that the operator $\rho_L(\langle \langle \mathcal{T} \rangle \rangle) \in L_n(E, F)$ is a dominant for $\overline{\mathcal{T}}$. Take an arbitrary operator $S \in L$ that satisfies the condition $\ast S \geq \langle \langle \mathcal{T} \rangle \rangle$. Then, using relation (19), for every $\kappa \in \text{fin}(\mathcal{X})$, we obtain

$$\rho_F \circ \beta(\overline{\mathcal{T}}(\kappa)) = \inf_F \{ f \in F : f \geq \beta(\mathcal{T}(\kappa)) \}$$

$$\leq \inf_F \{ f \in F : f \geq \langle \langle \mathcal{T} \rangle \rangle \alpha(\kappa) \} = \inf_F \{ S \in E : e \geq \alpha(\kappa) \}$$

$$= S \inf_F \{ e \in E : e \geq \alpha(\kappa) \} = S \circ \rho_E \circ \alpha(\kappa).$$

Consequently, $S \geq \langle \langle \overline{\mathcal{T}} \rangle \rangle$. Using Lemma 4.11.1, we find

$$\rho_L(\langle \langle \mathcal{T} \rangle \rangle) = \inf_L \{ S \in L : \ast S \geq \langle \langle \mathcal{T} \rangle \rangle \} \geq \langle \langle \overline{\mathcal{T}} \rangle \rangle,$$

as required. $\triangleright$

4.11.9. Denote by $M_n(\mathcal{X}, \mathcal{Y})$ the set of all linear operators from $\mathcal{X}$ into $\mathcal{Y}$ which admit $(o)$-continuous dominants. It is clear that the conditions $T \in M_n(\mathcal{X}, \mathcal{Y})$ and $\ast T \in M_n(\mathcal{X}, \mathcal{Y})$ are equivalent. Take $T \in M_n(\mathcal{X}, \mathcal{Y})$. Then, according to (20), there exists a mapping $\overline{T} : (o)\overline{\mathcal{X}} \to (o)\overline{\mathcal{Y}}$ such that

$$\overline{T}(\kappa)) = \langle \langle \mathcal{T} \rangle \rangle \quad (\kappa \in \text{fin}(\mathcal{X})).$$

Theorem. For every $T \in M_n(\mathcal{X}, \mathcal{Y})$, the mapping $\overline{T}$ is a dominated linear operator from the BKS $((o)\overline{\mathcal{X}}, \overline{a}, E)$ to the BKS $((o)\overline{\mathcal{Y}}, \overline{b}, F)$. Furthermore,

1. $\overline{T}(\overline{\mathcal{X}}(x)) = \overline{\mathcal{Y}}(Tx) \quad (x \in \mathcal{X});$

2. $\langle \langle \overline{T} \rangle \rangle = \langle \langle \mathcal{T} \rangle \rangle,$

where $\overline{\mathcal{X}} : \mathcal{X} \to (o)\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}} : \mathcal{Y} \to (o)\overline{\mathcal{Y}}$ are the canonical embeddings defined by (16).

$\langle \langle \overline{T} \rangle \rangle$ The mapping $\overline{T}$ is linear by construction. Equality (1) ensues immediately from the definitions of $\overline{\mathcal{X}}, \overline{\mathcal{Y}}$, and $\overline{T}$. The fact that the operator $\overline{T}$ is dominated, as well as the inequality $\langle \langle \overline{T} \rangle \rangle \leq \langle \langle \mathcal{T} \rangle \rangle$, is established in Theorem 5.2. It remains to verify the reverse inequality.
Let $x \in \mathcal{X}$. Taking property (1) into account, as well as the fact that $\hat{\eta}_{\mathcal{X}}$ and $\hat{\eta}_{\mathcal{Y}}$ are isometrically isomorphic embeddings, we obtain

$$b(T(x)) = \rho_F \circ \tilde{b}(\hat{\eta}_{\mathcal{Y}}(Tx)) = \rho_F \circ \tilde{b}(\hat{\eta}_{\mathcal{X}}(x))$$

$$\leq \langle \langle T \rangle \rangle (\rho_E \circ \tilde{a}(\hat{\eta}_{\mathcal{X}}(x))) = \langle \langle T \rangle \rangle a(x).$$

Since the element $x \in \mathcal{X}$ is chosen arbitrary, it follows that $\langle \langle T \rangle \rangle \leq \langle \langle T \rangle \rangle$. \(\triangleright\)

Let $\mathcal{X}$ and $\mathcal{Y}$ be the $(\sigma)$-completions of $\mathcal{X}$ and $\mathcal{Y}$, constructed in Theorem 4.11.6. From the previous theorem we obtain the following assertion (see [19, 14, Theorem 2.3.3]).

**Corollary** (A. G. Kusraev; V. Z. Strizhevskii). For every $T \in M_n(\mathcal{X}, \mathcal{Y})$, there exists a unique operator $\hat{T} \in M_n(\mathcal{X}, \mathcal{Y})$ extending $T$ in the sense that $\hat{T}(\hat{\eta}_{\mathcal{X}}(x)) = \hat{\eta}_{F}(Tx)$ for all $x \in \mathcal{X}$. Furthermore, $\langle \langle \hat{T} \rangle \rangle = \langle \langle T \rangle \rangle$.

$\triangleright$ It is sufficient to take as $\hat{T}$ the restriction of the operator $T$ onto $\mathcal{X}$. Uniqueness of extension ensues from the requirement $\hat{T} \in M_n(\mathcal{X}, \mathcal{Y})$ and the construction of $\mathcal{X}$. \(\triangleright\)

**References**


27. Wolff M. P. H., "An introduction to nonstandard functional analysis. Non-