Chapter 3

Formulation of FEM for Two-Dimensional Problems

3.1 Two-Dimensional FEM Formulation

Many details of 1D and 2D formulations are the same. To demonstrate how a 2D formulation works we'll use the following steady, AD equation

$$\vec{V} \cdot \nabla T - k\nabla^2 T = f \qquad \text{in } \ \Omega \tag{3.1}$$

where $\vec{V}$ is the known velocity field, k is the known and constant conductivity, f is the known force function and T is the scalar unknown. Weighted residual statement of this DE is

$$\int_\Omega \left(w\,\vec{V} \cdot \nabla T - wk\nabla^2 T - wf\right) d\Omega = 0 \tag{3.2}$$

We can apply the divergence theorem (similar to integration by parts we used in 1D) to the second term, which has the second derivative, as follows

$$\int_\Omega -wk\nabla^2 T \, d\Omega = \int_\Omega k\,\nabla w \cdot \nabla T \, d\Omega - \int_\Gamma wk\,(\vec{n} \cdot \nabla T)\, d\Gamma \tag{3.3}$$

Substituting this into the weighted residual statement we obtain the following weak form

$$\int_\Omega \left(w\,\vec{V} \cdot \nabla T + k\,\nabla w \cdot \nabla T\,\right) d\Omega = \int_\Omega wf \, d\Omega \; + \int_\Gamma wk\,(\vec{n} \cdot \nabla T)\, d\Gamma \tag{3.4}$$

By looking at the boundary term, *PV* of the problem is T and *SV* is $k\,\vec{n} \cdot \nabla T$, where $\vec{n}$ is the unit outward normal of the boundary of the problem domain.

$$SV = k\,(\vec{n} \cdot \nabla T) = k\left(n_x \frac{\partial T}{\partial x} + n_y \frac{\partial T}{\partial y}\right) \tag{3.5}$$

Now substituting the following approximate solution into the weak form

$$T_{app}(x,y) = \sum_{j=1}^{NN} T_j\, S_j(x,y) \tag{3.6}$$

and selecting the weight function to be the i^{th} shape function, i.e. $w = S_i(x,y)$ we get the following i^{th} linear algebraic equation for NN many nodal unknowns.

$$\sum_{j=1}^{NN} \left\{ \int_\Omega \left[S_i \left(V_x \frac{\partial S_j}{\partial x} + V_y \frac{\partial S_j}{\partial y} \right) + k \left(\frac{\partial S_i}{\partial x} \frac{\partial S_j}{\partial x} + \frac{\partial S_i}{\partial y} \frac{\partial S_j}{\partial y} \right) \right] d\Omega \right\} T_j = \int_\Omega S_i f \, d\Omega + \int_\Gamma S_i (SV) d\Gamma \qquad (3.7)$$

This equation is valid for $i = 1, 2, \ldots, NN$. All NN equations can be represented in the following compact form

$$[K]\{T\} = \{F\} + \{B\} \qquad (3.8)$$

where $\{T\}$ is the unknown vector with NN entries. $[K]$ is the stiffness matrix of size NNxNN with entries given below

$$K_{ij} = \int_\Omega \left[S_i \left(V_x \frac{\partial S_j}{\partial x} + V_y \frac{\partial S_j}{\partial y} \right) + k \left(\frac{\partial S_i}{\partial x} \frac{\partial S_j}{\partial x} + \frac{\partial S_i}{\partial y} \frac{\partial S_j}{\partial y} \right) \right] d\Omega \qquad (3.9)$$

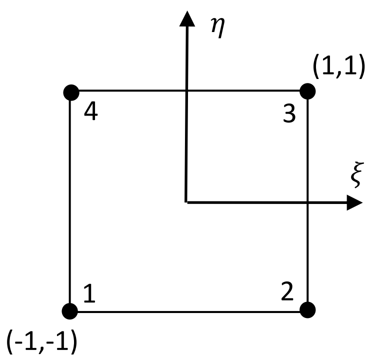
and $\{F\}$ and $\{B\}$ are the force vector and boundary integral vector, respectively.

Similar to the 1D case, in a computer code $[K]$, $\{F\}$ and $\{B\}$ are computed as a summation/assembly of NE integrals, each taken over a separate element.

In order to be able to take the integrals numerically using GQ integration we need to introduce 2D master elements and be able to work with master element coordinates.

3.2 Two Dimensional Master Elements and Shape Functions

In 2D, <u>triangular and quadrilateral elements</u> are the most commonly used ones. Figure 3.1 shows the bilinear (4 node) quadrilateral master element. Master element coordinates, ξ and η, vary between -1 and 1. Local node numbering starts from the lower left corner and goes CCW. Shape functions can be determined either by considering the general form and using the Kronecker-delta property or simply by combining proper linear, 1D shape functions.



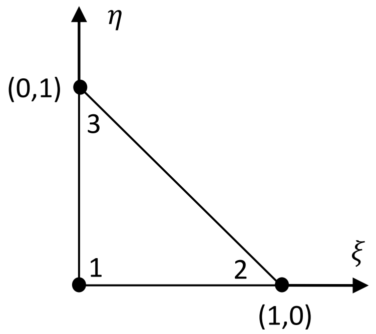
General form : $S = A + B\xi + C\eta + D\xi\eta$

$$S_1 = \frac{1}{4}(1-\xi)(1-\eta) \ , \qquad S_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$S_3 = \frac{1}{4}(1+\xi)(1+\eta) \ , \qquad S_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Figure 3.1 Bilinear (4 node) quadrilateral master element and shape functions

Figure 4.2 shows the bilinear (3 node) triangular master element. Note that master element coordinates do not vary between -1 and 1 for a triangular element and we need to be aware of this during GQ integration.



General form : $S = A + B\xi + C\eta$

$S_1 = 1 - \xi - \eta$

$S_2 = \xi$

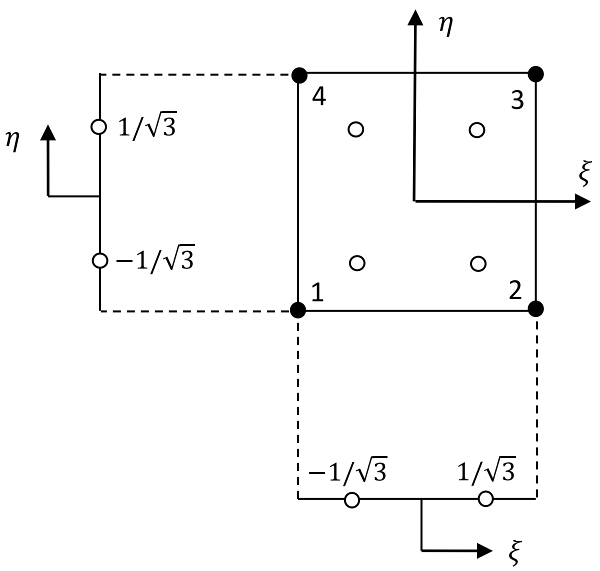
$S_3 = \eta$

Figure 3.2 Bilinear (3 node) triangular master element and shape functions

It is possible to construct higher order 2D elements such as 9 node quadrilateral or 6 node triangular elements, too.

3.3 Gauss Quadrature Integration in 2D

GQ points and weights for quadrilateral elements are directly related to the ones used for 1D GQ. We simply think about two integrals, one in ξ and the other in η direction and combine two 1D GQ integrations. Figure 3.3 shows how a sample 4 point GQ on a 2D quadrilateral element works. Table 3.1 provides GQ points and weights for NGP = 1, 4 and 9 to be used for quadrilateral elements.



GQ point 1: $\xi = -1/\sqrt{3}$, $\eta = -1/\sqrt{3}$, $W = 1$

GQ point 2: $\xi = 1/\sqrt{3}$, $\eta = -1/\sqrt{3}$, $W = 1$

GQ point 3: $\xi = -1/\sqrt{3}$, $\eta = 1/\sqrt{3}$, $W = 1$

GQ point 4: $\xi = 1/\sqrt{3}$, $\eta = 1/\sqrt{3}$, $W = 1$

Figure 3.3 GQ integration points and weights on 2D quadrilateral master element for NGP = 4. White circles are the GQ points.

For a triangular element, master element coordinates do not vary between -1 and 1, and we need to use a completely different GQ table, specifically designed for triangular elements. Table 3.2 can be used for GQ integration of triangular elements.

Table 3.1 Gauss Quadrature points and weights for 2D quadrilateral elements

NGP	ξ_k	η_k	W_k
1	0	0	2
4	$-\sqrt{1/3}$	$-\sqrt{1/3}$	1
	$\sqrt{1/3}$	$-\sqrt{1/3}$	1
	$-\sqrt{1/3}$	$\sqrt{1/3}$	1
	$\sqrt{1/3}$	$\sqrt{1/3}$	1
9	$-\sqrt{3/5}$	$-\sqrt{3/5}$	25/81
	0	$-\sqrt{3/5}$	40/81
	$\sqrt{3/5}$	$-\sqrt{3/5}$	25/81
	$-\sqrt{3/5}$	0	40/81
	0	0	64/81
	$\sqrt{3/5}$	0	40/81
	$-\sqrt{3/5}$	$\sqrt{3/5}$	25/81
	0	$\sqrt{3/5}$	40/81
	$\sqrt{3/5}$	$\sqrt{3/5}$	25/81

Table 3.2 Gauss Quadrature points and weights for 2D triangular elements

NGP	ξ_k	η_k	W_k
1	1/3	1/3	1/2
3	0.5	0	1/6
	0	0.5	1/6
	0.5	0.5	1/6
4	1/3	1/3	-27/96
	0.6	0.2	25/96
	0.2	0.6	25/96
	0.2	0.2	25/96

3.4 Coordinate Transformation and Jacobian Matrix in 2D

Remember that for 1D problems the relation between the global x coordinate and the master element coordinate ξ is

$$x = \frac{h^e}{2}\xi + \frac{x_1^e + x_2^e}{2} \tag{3.10}$$

which is used to obtain the following Jacobian formula

$$J = \frac{dx}{d\xi} = \frac{h^e}{2} \tag{3.11}$$

Similar relations are necessary in 2D so that the derivatives of shape functions with respect to x and y can be expressed as derivatives with respect to ξ and η.

In 2D (x, y) coordinates can be written in terms of (ξ, η) coordinates by using the previously defined 2D shape functions as follows

$$x(\xi,\eta) = \sum_{j=1}^{NEN} x_j^e \; S_j(\xi,\eta) \quad , \quad y(\xi,\eta) = \sum_{j=1}^{NEN} y_j^e \; S_j(\xi,\eta) \tag{3.12}$$

where NEN is the node number of an element (equal to 4 and 3 for bilinear quadrilateral and bilinear triangular elements, respectively) and x_j^e and y_j^e are the known nodal coordinates of the element. Actually these equations are not different than what we previously used in 1D, i.e. equation (3.10) is a direct outcome of using equation (3.12) in 1D. Using the same shape functions for both unknown approximation and coordinate transformation is known as iso-parametric formulation. It is possible to use different order shape functions for unknown approximation and coordinate transformation, known as sub-parametric or super-parametric formulation.

Using equation (3.12) we are now able to express ξ and η derivatives of the i^{th} shape function in terms of derivatives with respect to x and y as follows

$$\frac{\partial S_i}{\partial \xi} = \frac{\partial S_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial S_i}{\partial y}\frac{\partial y}{\partial \xi} \quad , \quad \frac{\partial S_i}{\partial \eta} = \frac{\partial S_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial S_i}{\partial y}\frac{\partial y}{\partial \eta} \tag{3.13}$$

In a more compact form these equations can be written as

$$\begin{Bmatrix} \dfrac{\partial S_i}{\partial \xi} \\ \dfrac{\partial S_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial S_i}{\partial x} \\ \dfrac{\partial S_i}{\partial y} \end{Bmatrix} \tag{3.14}$$

where the 2x2 matrix is known as the Jacobian matrix. Its entries can be calculated using equation (3.12) as shown below for a 4 node element (NEN=4)

$$[J^e] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial S_1}{\partial \xi} & \frac{\partial S_2}{\partial \xi} & \frac{\partial S_3}{\partial \xi} & \frac{\partial S_4}{\partial \xi} \\ \frac{\partial S_1}{\partial \eta} & \frac{\partial S_2}{\partial \eta} & \frac{\partial S_3}{\partial \eta} & \frac{\partial S_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \\ x_4^e & y_4^e \end{bmatrix} \tag{3.15}$$

where the last matrix has the x and y coordinates of the four corners of element e. In a computer code Jacobian is calculated in exactly this way. Note that in general each element of a FE mesh has a different Jacobian matrix and entries of Jacobian matrices are not constants but functions of (ξ, η).

Remember that in the integrals of elemental stiffness matrix and elemental force vector we have shape function derivatives with respect to x and y that need to be converted to derivatives wrt ξ and η. In other words we need the inverse of the Jacobian matrix as shown below

$$\begin{Bmatrix} \frac{\partial S_i}{\partial x} \\ \frac{\partial S_i}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \frac{\partial S_i}{\partial \xi} \\ \frac{\partial S_i}{\partial \eta} \end{Bmatrix} = [J^e]^{-1} \begin{Bmatrix} \frac{\partial S_i}{\partial \xi} \\ \frac{\partial S_i}{\partial \eta} \end{Bmatrix} \tag{3.16}$$

Jacobian matrix is 2x2 and its inverse can be evaluated simply as follows

$$J^e = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \rightarrow \quad J^{e^{-1}} = \frac{1}{|J^e|} \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} \quad , \quad \text{where} \quad |J^e| = AD - BC \tag{3.17}$$

Now we can write the elemental stiffness matrix integral given in equation (3.9) in terms of ξ and η as follows

$$\begin{aligned} K_{ij}^e = \int_{\Omega^e} \Bigg\{ & S_i \left[V_x \left({J^{e^{-1}}}_{11} \frac{\partial S_j}{\partial \xi} + {J^{e^{-1}}}_{12} \frac{\partial S_j}{\partial \eta} \right) + V_y \left({J^{e^{-1}}}_{21} \frac{\partial S_j}{\partial \xi} + {J^{e^{-1}}}_{22} \frac{\partial S_j}{\partial \eta} \right) \right] \\ & + k \left[\left({J^{e^{-1}}}_{11} \frac{\partial S_i}{\partial \xi} + {J^{e^{-1}}}_{12} \frac{\partial S_i}{\partial \eta} \right) \left({J^{e^{-1}}}_{11} \frac{\partial S_j}{\partial \xi} + {J^{e^{-1}}}_{12} \frac{\partial S_j}{\partial \eta} \right) \right. \\ & \left. + \left({J^{e^{-1}}}_{21} \frac{\partial S_i}{\partial \xi} + {J^{e^{-1}}}_{22} \frac{\partial S_i}{\partial \eta} \right) \left({J^{e^{-1}}}_{21} \frac{\partial S_j}{\partial \xi} + {J^{e^{-1}}}_{22} \frac{\partial S_j}{\partial \eta} \right) \right] \Bigg\} \; |J^e| \, d\xi d\eta \end{aligned} \tag{3.18}$$

This last integral is ready to be evaluated in a computer code using GQ integration. Determinant of the Jacobian that appears at the end of the integral is coming from the following relation

$$d\Omega \; = \; dxdy \; = \; |J^e| \, d\xi d\eta \tag{3.19}$$

3.5 Assembly in 2D

Assembly rule given in equation (2.27) can directly be used in 2D. Consider the 4 element mesh with 8 nodes shown in Figure 3.4. Elemental systems for the quadrilateral and triangular elements will be 4x4 and 3x3, respectively. Four elemental systems will be assembled into an 8x8 global system .

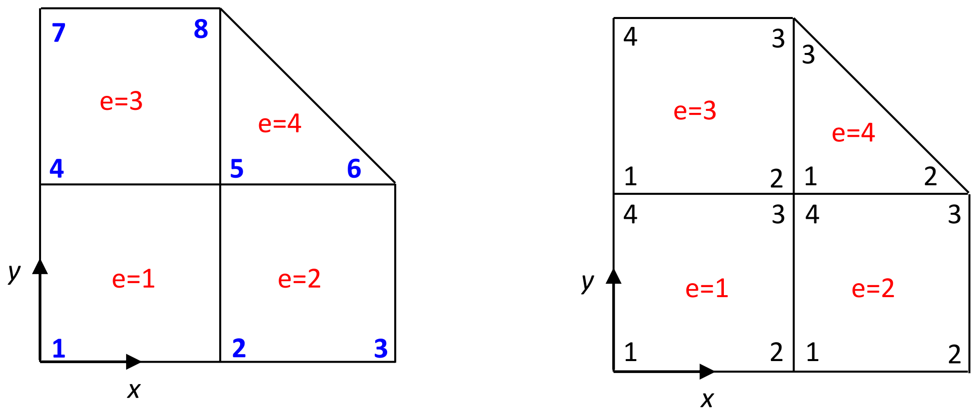


Figure 3.4 Element and global and local node numbering for a sample 2D mesh

To perform the assembly we need to write the local-to-global node mapping matrix. To do this first we need to select a global node numbering and then a local node numbering for each element, which can be as the ones shown in Figure 3.4. With the selected global and local node numberings local-to-global node mapping matrix can be written as follows

$$LtoG = \begin{bmatrix} 1 & 2 & 5 & 4 \\ 2 & 3 & 6 & 5 \\ 4 & 5 & 8 & 7 \\ 5 & 6 & 8 & \times \end{bmatrix} \tag{3.20}$$

where the entry of the last row does not exist since the third element has only three nodes. Using the assembly rule and this $LtoG$ matrix, the following global stiffness matrix

$$K = \begin{bmatrix} K_{11}^1 & K_{12}^1 & & K_{14}^1 & K_{13}^1 & & & \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & K_{24}^1 & K_{23}^1 + K_{14}^2 & K_{13}^2 & & \\ & K_{21}^2 & K_{22}^2 & & K_{24}^2 & K_{23}^2 & & \\ K_{41}^1 & K_{42}^1 & & K_{44}^1 + K_{11}^3 & K_{43}^1 + K_{12}^3 & & K_{14}^3 & K_{13}^3 \\ K_{31}^1 & K_{32}^1 + K_{41}^2 & K_{42}^2 & K_{34}^1 + K_{21}^3 & K_{33}^1 + K_{44}^2 + K_{22}^3 + K_{11}^4 & K_{43}^2 + K_{12}^4 & K_{24}^3 & K_{23}^3 + K_{13}^4 \\ & K_{31}^2 & K_{32}^2 & & K_{34}^2 + K_{21}^4 & K_{33}^2 + K_{22}^4 & & K_{23}^4 \\ & & & K_{41}^3 & K_{42}^3 & & K_{44}^3 & K_{43}^3 \\ & & & K_{31}^3 & K_{32}^3 + K_{31}^4 & K_{32}^4 & K_{34}^3 & K_{33}^3 + K_{33}^4 \end{bmatrix} \tag{3.21}$$

and the following global force vector and boundary integral vector can be obtained

$$F = \begin{Bmatrix} F_1^1 \\ F_2^1 + F_1^2 \\ F_2^2 \\ F_4^1 + F_1^3 \\ F_3^1 + F_4^2 + F_2^3 + F_1^4 \\ F_3^2 \\ F_4^3 + F_2^4 \\ F_3^3 + F_3^4 \end{Bmatrix} \quad , \quad B = \begin{Bmatrix} B_1^1 \\ B_2^1 + B_1^2 \\ B_2^2 \\ B_4^1 + B_1^3 \\ B_3^1 + B_4^2 + B_2^3 + B_1^4 \\ B_3^2 \\ B_4^3 + B_2^4 \\ B_3^3 + B_3^4 \end{Bmatrix} \tag{3.22}$$

The assembled 8x8 global system can also be written as

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\ K_{21} & \ddots & & & & & & K_{28} \\ K_{31} & & \ddots & & & & & K_{38} \\ K_{41} & & & \ddots & & & & K_{48} \\ K_{51} & & & & \ddots & & & K_{58} \\ K_{61} & & & & & \ddots & & K_{68} \\ K_{71} & & & & & & \ddots & K_{78} \\ K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix} + \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \end{Bmatrix} \qquad (3.23)$$

3.6 Evaluation of Boundary Integrals in 2D

Application of EBCs in 2D is the same as 1D. We can again apply reduction to the global system and delete the necessary rows/columns of it and modify the right hand side of the remaining equations accordingly. However, NBCs and MBCs need more detailed calculations, because in 2D the problem boundary is not composed of just two nodes, but line segments, and the boundary integrals require the calculation of line integrals.

As seen in equation (3.7), for the i^{th} equation of the global system, entry of the boundary integral vector $\{B\}$ is given as

$$B_i = \int_\Gamma S_i\,(SV) d\Gamma \qquad (3.24)$$

This integral should be evaluated only on the boundaries of the problem domain. For the problem shown in Figure 3.6 problem boundary consists of 7 element faces.

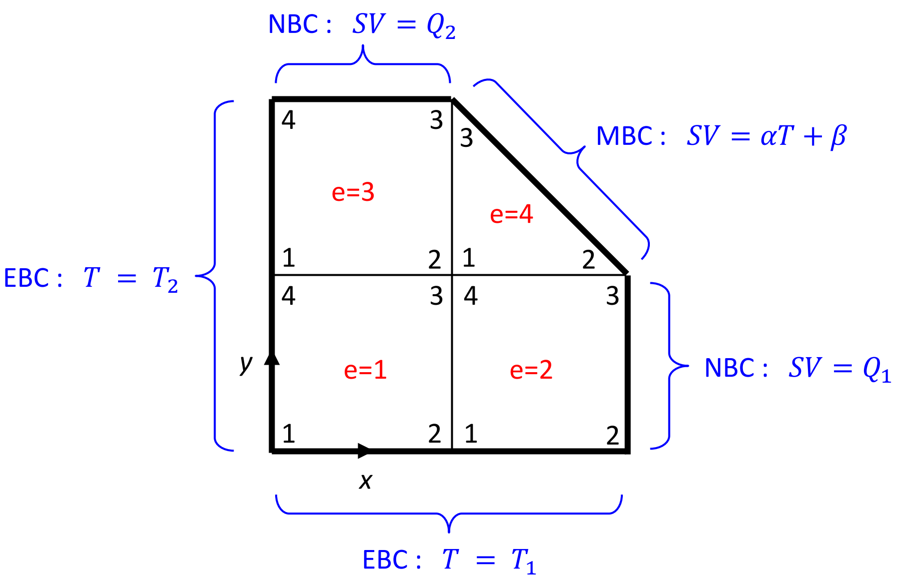


Figure 3.6 BCs for the sample 2D mesh. Boundary faces are shown as thick lines

Consider the BCs shown in Figure 3.6. There are 2 EBCs, 2 NBCs and 1 MBC. The following important observations can be made about the $\{B\}$ vector of equation (3.24)

- Similar to 1D, entry of the $\{B\}$ vector corresponding to an internal node (a node that is not on the problem boundary) is zero. For the sample 2D problem we are studying, 5th global node (global node numbers are given in Figure 3.4) is an internal node and $B_5 = 0$.
- If an EBC is specified at a boundary node, the corresponding boundary integral entry is not necessary for the solution. As seen in Figure 3.6, EBCs are specified at global nodes 1, 2, 3, 4 and 7. Therefore we do not need to calculate B_1, B_2, B_3, B_4 and B_7.
- In short, we only need to calculate B_6 and B_8.

Before calculating B_6 and B_8 let's first select a <u>face numbering notation</u>. As shown in Figure 3.7, the face between local nodes 1 and 2 of an element will be the 1st face and the other faces will be numbered consecutively in a counter clockwise order. According to this notation, boundary faces of the problem shown in Figure 3.6 are the 1st and 4th faces of element 1, 1st and 2nd faces of element 2, 3rd and 4th faces of element 3 and 2nd face of element 4.

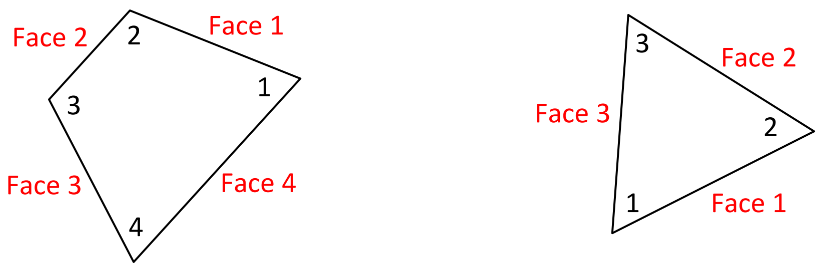


Figure 3.7 Face numbering notation for 2D elements

Now we are ready to calculate B_6 and B_8. Similar to $[K]$ and $\{F\}$ calculations, we can calculate the $\{B\}$ vector element by element, followed by an assembly. Since the 1st element of Figure 3.6 has no NBC or MBC faces, we do not need to perform any boundary integral calculation for it. Next comes the 2nd element, shown below.

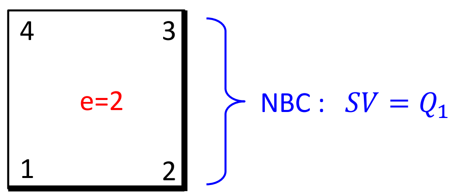


Figure 3.8 Second element of sample 2D mesh. Faces of this element located at the problem boundary are shown as thick lines. One of these faces has NBC.

For this 2nd element the elemental boundary integral vector is

$$\{B^2\} = \begin{Bmatrix} B_1^2 \\ B_2^2 \\ B_3^2 \\ B_4^2 \end{Bmatrix} \tag{3.25}$$

We do not need to calculate B_1^2, B_2^2 or B_4^2, because they will be assembled into B_2, B_3 and B_5 locations of the global $\{B\}$ vector, respectively and we already know that these are not necessary in the FE solution. Therefore we only need to calculate B_3^2. Using the elemental version of equation (3.24) B_3^2 can be calculated as

$$B_3^2 = \int_{\Gamma_b^2} S_3\,(SV) d\Gamma = \int_{\Gamma_b^2} S_3\, Q_1\, d\Gamma \tag{3.26}$$

where S_3 is the third local shape function of the second element and Γ_b^2 is the part of the boundary of element 2 that is located at the problem boundary. Only 2 of the four faces are located on the problem's boundary and therefore it is possible to write the above integral as the summation of 2 line integrals as follows

$$B_3^2 = \underbrace{\int_{\text{Face 1}} S_3\, Q_1\, d\Gamma}_{Zero} + \int_{\text{Face 2}} S_3\, Q_1\, d\Gamma \tag{3.27}$$

Due to the Kronecker-delta property of the shape functions, S_3 is equal to zero on the first face (See Figure 3.1). Therefore B_3^2 is equal to

$$B_3^2 = \int_{\text{Face 2}} S_3\, Q_1\, d\Gamma \tag{3.28}$$

In order to evaluate this integral we need to know how Q_1 changes over the 2nd face of element 2. For simplicity let's consider Q_1 to be constant, which is actually the most common case. B_3^2 becomes

$$B_3^2 = Q_1 \int_{\text{Face 2}} S_3 d\Gamma \tag{3.29}$$

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S_3 is a 2D shape function that varies with both x and y (or ξ and η). On face 2 it varies linearly as shown below

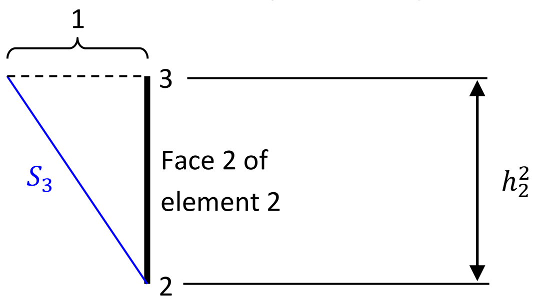


Figure 3.9 Variation of S_3 on the second face of element 2. Local node numbers are also shown

Using the shown linear variation of S_3 over face 2, the integral of equation (3.29) can be evaluated as $h_2^2/2$ and B_3^2 becomes

$$B_3^2 = \frac{Q_1 h_2^2}{2} \tag{3.30}$$

where h_2^2 is the length of the second face of the second element. Therefore the elemental boundary integral vector for element 2 is

$$\{B^2\} = \begin{Bmatrix} B_1^2 \\ B_2^2 \\ Q_1 h_2^2/2 \\ B_4^2 \end{Bmatrix} \tag{3.31}$$

Now we can do similar calculations for the third element to get $\{B^3\}$. Similar to the second element, third element also has a single NBC face. Again considering that the SV specified on that face is constant, i.e. Q_2 is constant, we do not need to repeat the calculations. The following $\{B^3\}$ can be written automatically

$$\{B^3\} = \begin{Bmatrix} B_1^3 \\ B_2^3 \\ Q_2 h_3^3/2 \\ B_4^3 \end{Bmatrix} \tag{3.32}$$

We did not calculate B_1^3, B_2^3 or B_4^3, because they will be assembled into B_4, B_5 and B_8 locations of the global $\{B\}$ vector, respectively and we know that these values are not necessary in the FE solution.

Finally $\{B^4\}$ needs to be calculated and for this the mixed BC given on face 2 of element 4, shown below, is to be considered.

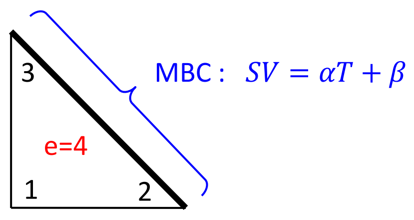


Figure 3.10 Fourth element of sample 2D mesh. Only face 2 of this element is located at the problem boundary and MBC is specified there.

For element 4 we do not need to evaluate B_1^4 because it will assemble into global B_5, which is not necessary for the FE solution. But we need to evaluate B_2^4 and B_3^4. Since only the second face of element 4 is located at the problem boundary these can be evaluated as

$$B_2^4 = \int_{\text{Face 2}} S_2 \, (SV) \, d\Gamma \;=\; \int_{\text{Face 2}} S_2 \, (\alpha T + \beta) \, d\Gamma$$
$$B_3^4 = \int_{\text{Face 2}} S_3 \, (SV) \, d\Gamma \;=\; \int_{\text{Face 2}} S_3 \, (\alpha T + \beta) \, d\Gamma \tag{3.33}$$

Let's consider B_2^4 first. To calculate it we need to know how S_2 varies on face 2. We discussed a similar case before and we know that it varies linearly on face 2, as shown below. We also need to know how T varies on face 2. For a 3-node triangular element with 2-node faces, T also varies linearly on the faces as shown below

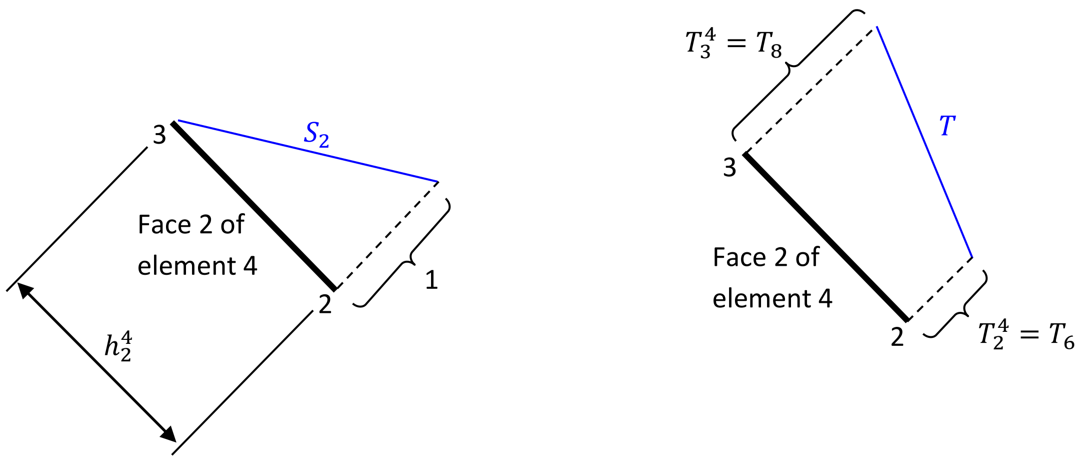


Figure 3.11 Variation of S_2 and T on the 2nd face of element 4. Local node numbers are also shown.

Using the shown linear variations of S_2 and T over the 2nd face of element 4, the integral for B_2^4 shown in equation (3.33) can be evaluated as

$$B_2^4 = \frac{h_2^4}{6}(3\beta + 2\alpha T_6 + \alpha T_8) \tag{3.34}$$

Similarly the integral for B_3^4 given in equation (3.33) can be evaluated as

$$B_3^4 = \frac{h_2^4}{6}(3\beta + \alpha T_6 + 2\alpha T_8) \tag{3.35}$$

And the elemental boundary integral vector of element 4 becomes

$$\{B^4\} = \begin{Bmatrix} B_1^4 \\ \frac{h_2^4}{6}(3\beta + 2\alpha T_6 + \alpha T_8) \\ \frac{h_2^4}{6}(3\beta + \alpha T_6 + 2\alpha T_8) \end{Bmatrix} \tag{3.36}$$

Now we can assemble all $\{B^e\}$'s to get the following global $\{B\}$ vector

$$\{B\} = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \end{Bmatrix} = \begin{Bmatrix} B_1^1 \\ B_2^1 + B_1^2 \\ B_2^2 \\ B_4^1 + B_1^3 \\ B_3^1 + B_4^2 + B_2^3 + B_1^4 \\ B_3^2 \\ B_4^3 + B_2^4 \\ B_3^3 + B_3^4 \end{Bmatrix} = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ 0 \\ \dfrac{Q_1 h_2^2}{2} + \dfrac{h_2^4}{6}(3\beta + 2\alpha T_6 + \alpha T_8) \\ B_7 \\ \dfrac{Q_2 h_3^3}{2} + \dfrac{h_2^4}{6}(3\beta + \alpha T_6 + 2\alpha T_8) \end{Bmatrix} \tag{3.37}$$

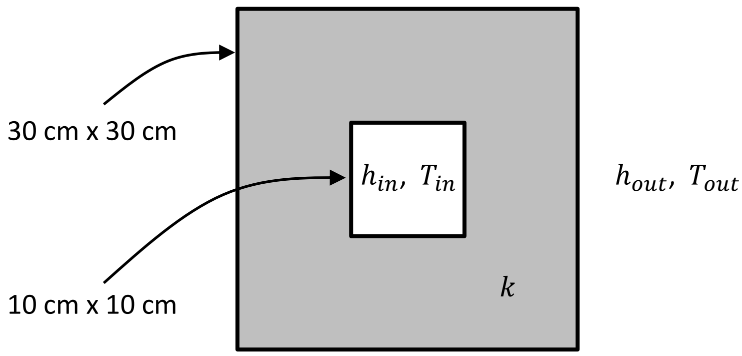
To summarize, natural and mixed BCs are treated as follows

- Boundary integrals should only be calculated for the faces where a NBC or MBC is specified, not on internal faces or faces where EBC is specified.
- On a 2-node face, if a NBC is specified as a constant value, we can directly use the result given by equation (3.30). We do not need to repeat the calculations.
- On a 2-node face, if a MBC is specified with constant α and β values, we can directly use the results given by equations (3.33) and (3.34). We do not need to repeat the calculations.
- If the specified SV as a NBC or α and β values specified as a MBC are not constant, then the boundary integrals need to be evaluated again, but we'll NOT consider these cases in this course.
- If boundary integrals are taken over a face with 3 or more nodes, then the results obtained in this section cannot be used, but the integrals need to be evaluated again.

3.7 First 2D Solution

Hot combustion gases of a furnace are flowing through a chimney made of concrete ($k = 1.4\ \mathrm{W/(m°C)}$). The flow section of the chimney is $10\ \mathrm{cm} \times 10\ \mathrm{cm}$, and the thickness of the wall is $10\ \mathrm{cm}$. The average temperature of the hot gases in the chimney is $T_{in} = 300\ \mathrm{°C}$, and the average convection heat transfer coefficient inside the chimney is $h_{in} = 70\ \mathrm{W/(m^2°C)}$. The chimney is losing heat from its outer surface to the ambient air at $T_{out} = 20\ \mathrm{°C}$ by convection with a heat transfer coefficient of $h_{out} = 21\ \mathrm{W/m^2°C}$. Taking full advantage of symmetry, determine the temperature distribution inside the chimney and the rate of heat loss for a 1 m long section of the chimney.

This problem is taken from reference [1].



Heat transfer on the chimney is through conduction, which is governed by the following Laplace's equation

$$-\nabla \cdot (k \, \nabla \mathrm{T}) = 0$$

This equation is a simplified form of the AD equation studied at the beginning of this chapter. Although the thermal conductivity is constant, we do not cancel it out, but keep it inside the derivatives so that it will also appear in the boundary integral, resulting in a more physical secondary variable. Multiplying the residual of the above equation with a weight function, integrating it over the problem boundary and applying the divergence theorem we get the following weak form

$$\int_{\Omega} \nabla w \cdot (k \nabla T) \, d\Omega = \int_{\Gamma} w \, \underbrace{\vec{n} \cdot (k \nabla T)}_{SV} \, d\Gamma$$

where the secondary variable is

$$SV = \vec{n} \cdot (k \nabla T) = k \frac{\partial T}{\partial x} n_x + k \frac{\partial T}{\partial y} n_y$$

which is the heat flux passing through the boundary. It is positive if heat is coming into the problem domain and negative if heat is going out of the domain. Weak form takes the following form for the 2D problem we are studying in the xy plane

$$\int_{\Omega} \left(\frac{\partial w}{\partial x} k \frac{\partial T}{\partial x} + \frac{\partial w}{\partial y} k \frac{\partial T}{\partial y} \right) d\Omega = \int_{\Gamma} w \, (SV) \, d\Gamma$$

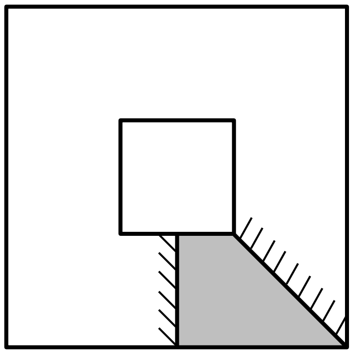
which yields the following elemental stiffness matrix, which is a simplified version of equation (3.9) and force vector

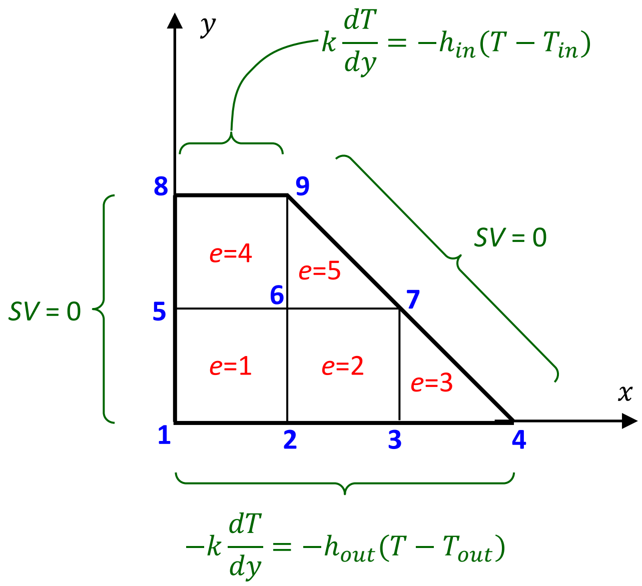
$$K_{ij}^e = \int_{\Omega^e} k \left(\frac{\partial S_i}{\partial x} \frac{\partial S_j}{\partial x} + \frac{\partial S_i}{\partial y} \frac{\partial S_j}{\partial y} \right) dxdy \quad , \qquad F_i^e = 0 \qquad (*)$$

K_{ij}^e integral can be written in terms of the master element coordinates ξ and η as follows, which is a simplified version of equation (3.18)

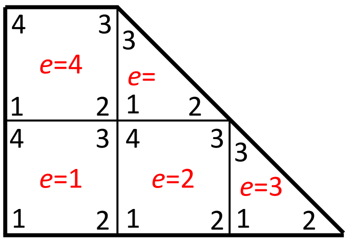
$$\begin{aligned} K_{ij}^e = \int_{\Omega^e} k \, \Bigg[& \left({J^{-1}}_{11} \frac{\partial S_i}{\partial \xi} + {J^{-1}}_{12} \frac{\partial S_i}{\partial \eta} \right) \left({J^{-1}}_{11} \frac{\partial S_j}{\partial \xi} + {J^{-1}}_{12} \frac{\partial S_j}{\partial \eta} \right) \\ & + \left({J^{-1}}_{21} \frac{\partial S_i}{\partial \xi} + {J^{-1}}_{22} \frac{\partial S_i}{\partial \eta} \right) \left({J^{-1}}_{21} \frac{\partial S_j}{\partial \xi} + {J^{-1}}_{22} \frac{\partial S_j}{\partial \eta} \right) \Bigg] \; |J^e| \, d\xi d\eta \end{aligned}$$

Now we can discretize the problem domain into elements. By considering existing symmetry planes, it is possible to study only a small portion of the problem domain as shown below. The symmetry planes act as insulated boundaries. The FE mesh shown below has 3 bilinear quadrilateral and 2 bilinear triangular elements. There are a total of 9 nodes. BCs are also shown below. There are 2 NBCs, 2 MBCs and no EBC. Note that NBCs are special in the sense that the SV is specified as zero. These are <u>"do nothing type" BCs</u>, because they do not modify the global system at all.





To evaluate $[K^e]$ matrices we first need to evaluate Jacobian matrices using equation (3.15). And for this we need to select local node numbering for each element. Our selection is seen below. <u>As a rule all local node numberings are CCW</u>, but the selection of the first node on an element is arbitrary.



Now the Jacobians and $[K^e]$'s can be calculated.

<u>Element 1:</u>

This is a quadrilateral element with shape functions given in Figure 3.1. Using the derivatives of these shape functions and the coordinates of four corners in equation (3.15) we get the following Jacobian matrix

$$[J^1] = \begin{bmatrix} -\frac{1}{4}(1-\eta) & \frac{1}{4}(1-\eta) & \frac{1}{4}(1+\eta) & -\frac{1}{4}(1+\eta) \\ -\frac{1}{4}(1-\xi) & -\frac{1}{4}(1+\xi) & \frac{1}{4}(1+\xi) & \frac{1}{4}(1-\xi) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0.05 & 0 \\ 0.05 & 0.05 \\ 0 & 0.05 \end{bmatrix} = \begin{bmatrix} 0.025 & 0 \\ 0 & 0.025 \end{bmatrix}$$

which is not a function of ξ or η and has two zero entries. This is due to the shape of the element, which is a square, being the same as that of the master element. Also the local numbering we used affected this result. Inverse of the Jacobian and its determinant are

$$[J^1]^{-1} = \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix} \quad , \quad |J^1| = 6.25 \times 10^{-4}$$

It's worth to note that the determinant of the Jacobian is equal to the ratio of the area of the actual element (0.0025 for *e*=1) to that of the master element (4 for a quadrilateral element). This is the case for all 2D elements. Actually in 1D Jacobian was half of the element's actual length, which is nothing but the ratio of the length of the actual element to that of the master element. Similarly, in 3D, determinant of the Jacobian corresponds to the volume ratio of an actual element and the 3D master element.

Now we can evaluate the entries of the elemental stiffness matrix by using different values for i and j indices in the K_{ij}^e equation.

$$K_{11}^1 = \int_{-1}^{1}\int_{-1}^{1} 1.4 \left\{\left[40\left(-\frac{1}{4}(1-\eta)\right)+0\right]\left[40\left(-\frac{1}{4}(1-\eta)\right)+0\right] + \left[0+40\left(-\frac{1}{4}(1-\xi)\right)\right]\left[0+40\left(-\frac{1}{4}(1-\xi)\right)\right]\right\} 6.25\times 10^{-4}\ d\xi d\eta \ = \ 0.9333$$

$$K_{12}^1 = \int_{-1}^{1}\int_{-1}^{1} 1.4 \left\{\left[40\left(-\frac{1}{4}(1-\eta)\right)+0\right]\left[40\left(\frac{1}{4}(1-\eta)\right)+0\right] + \left[0+40\left(-\frac{1}{4}(1-\xi)\right)\right]\left[0+40\left(-\frac{1}{4}(1+\xi)\right)\right]\right\} 6.25\times 10^{-4}\ d\xi d\eta \ = \ -0.2333$$

$$K_{13}^1 = \int_{-1}^{1}\int_{-1}^{1} 1.4 \left\{\left[40\left(-\frac{1}{4}(1-\eta)\right)+0\right]\left[40\left(\frac{1}{4}(1+\eta)\right)+0\right] + \left[0+40\left(-\frac{1}{4}(1-\xi)\right)\right]\left[0+40\left(\frac{1}{4}(1+\xi)\right)\right]\right\} 6.25\times 10^{-4}\ d\xi d\eta \ = \ -0.4667$$

$$K_{14}^1 = \int_{-1}^{1}\int_{-1}^{1} 1.4 \left\{\left[40\left(-\frac{1}{4}(1-\eta)\right)+0\right]\left[40\left(-\frac{1}{4}(1+\eta)\right)+0\right] + \left[0+40\left(-\frac{1}{4}(1-\xi)\right)\right]\left[0+40\left(\frac{1}{4}(1-\xi)\right)\right]\right\} 6.25\times 10^{-4}\ d\xi d\eta \ = \ -0.2333$$

Skipping the details for the rest of the entries we get

$$[K^1] = \begin{bmatrix} 0.9333 & -0.2333 & -0.4667 & -0.2333 \\ -0.2333 & 0.9333 & -0.2333 & -0.4667 \\ -0.4667 & -0.2333 & 0.9333 & -0.2333 \\ -0.2333 & -0.4667 & -0.2333 & 0.9333 \end{bmatrix}$$

which is a <u>symmetric matrix</u>. This is because Laplace's equation with constant k yields symmetric stiffness matrices, irrespective of the details of the element. To see this better, have a look at equation ($*$) of page 3-15 and notice that nothing changes when we interchange i and j, i.e. $K_{ij}^e = K_{ji}^e$.

<u>Element 2 and 4:</u>

Shape, size, orientation and local node numbering of the 2nd and 4th elements are the same those of the 1st element. Therefore Jacobian matrix of them should be the same. Also in K^e integral k is constant and nothing depends on the position of the element. In short $[K^2] = [K^4] = [K^1]$.

<u>Element 3:</u>

3rd element is a triangular one. Its Jacobian matrix is

$$[J^3] = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ 0.15 & 0 \\ 0.1 & 0.05 \end{bmatrix} = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix} \quad , \quad [J^3]^{-1} = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} \quad , \quad |J^3| = 0.0025$$

With this information first entry of $[K^3]$ can be calculated as follows

$$K_{11}^3 = \int_{\xi=0}^{1} \int_{\eta=0}^{1-\xi} k \, \{[2(-1)+0][2(-1)+0] + [0+2(-1)][0+2(-1)]\} \frac{1}{4} \, d\eta d\xi = 1.4$$

Here it is important to use correct integral limits if we are not using numerical integration. As seen in Figure 3.2, on a master triangular element ξ changes between 0 and 1 and η changes between 0 and $1-\xi$.

Repeating similar calculations for the other entries, $[K^3]$ turns out to be

$$[K^3] = \begin{bmatrix} 1.4 & -0.7 & -0.7 \\ -0.7 & 0.7 & 0 \\ -0.7 & 0 & 0.7 \end{bmatrix}$$

<u>Element 5:</u>

The geometrical similarity between 1st , 2nd and 4th elements mentioned above also exists between 3rd and 5th elements. Therefore $[K^5] = [K^3]$.

Now we have five $[K^e]$'s that can be assembled using the following $LtoG$ mapping matrix

$$LtoG = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 2 & 3 & 7 & 6 \\ 3 & 4 & 7 & \times \\ 5 & 6 & 9 & 8 \\ 6 & 7 & 9 & \times \end{bmatrix}$$

Using the assembly rule and this $LtoG$ matrix the following global stiffness matrix can be obtained

$$K = \begin{bmatrix} 0.9333 & -0.2333 & 0 & 0 & -0.2333 & -0.4667 & 0 & 0 & 0 \\ -0.2333 & 1.8667 & -0.2333 & 0 & -0.4667 & -0.4667 & -0.4667 & 0 & 0 \\ 0 & -0.2333 & 2.3333 & -0.7 & 0 & -0.4667 & -0.9333 & 0 & 0 \\ 0 & 0 & -0.7 & 0.7 & 0 & 0 & 0 & 0 & 0 \\ -0.2333 & -0.4667 & 0 & 0 & 1.8667 & -0.4667 & 0 & -0.2333 & -0.4667 \\ -0.4667 & -0.4667 & -0.4667 & 0 & -0.4667 & 4.2 & -0.9333 & -0.4667 & -0.9333 \\ 0 & -0.4667 & -0.9333 & 0 & 0 & -0.9333 & 2.3333 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.2333 & -0.4667 & 0 & 0.9333 & -0.2333 \\ 0 & 0 & 0 & 0 & -0.4667 & -0.9333 & 0 & -0.2333 & 1.6333 \end{bmatrix}$$

Global force vector $\{F\}$ is zero for this problem.

Now the BCs should be considered. For the NBCs with $SV = 0$ we do not need to do anything. For the bottom and top surfaces of the domain, *SV*'s and provided MBCs in "α, β form" are as follows

Bottom surface $\left(n_x = 0, n_y = -1\right)$: $\quad SV = -k\dfrac{\partial T}{\partial y} = -h_{out}(T - T_{out}) = \underbrace{-h_{out}}_{\alpha} T + \underbrace{h_{out}T_{out}}_{\beta}$

Top surface $\left(n_x = 0, n_y = 1\right)$: $\quad SV = k\dfrac{\partial T}{\partial y} = -h_{in}(T - T_{in}) = \underbrace{-h_{in}}_{\alpha} T + \underbrace{h_{in}T_{in}}_{\beta}$

The first 4 elements have faces with MBC. Equations (3.34) and (3.35) can be used to construct $\{B^1\}$, $\{B^2\}$, $\{B^3\}$ and $\{B^4\}$ as follows

$$\{B^1\} = \begin{Bmatrix} \frac{0.05}{6}(3h_{out}T_{out} - 2h_{out}T_1 - h_{out}T_2) \\ \frac{0.05}{6}(3h_{out}T_{out} - 2h_{out}T_2 - h_{out}T_1) \\ B_3^1 \\ B_4^1 \end{Bmatrix} , \qquad \{B^2\} = \begin{Bmatrix} \frac{0.05}{6}(3h_{out}T_{out} - 2h_{out}T_2 - h_{out}T_3) \\ \frac{0.05}{6}(3h_{out}T_{out} - 2h_{out}T_3 - h_{out}T_2) \\ B_3^2 \\ B_4^2 \end{Bmatrix}$$

$$\{B^3\} = \begin{Bmatrix} \frac{0.05}{6}(3h_{out}T_{out} - 2h_{out}T_3 - h_{out}T_4) \\ \frac{0.05}{6}(3h_{out}T_{out} - 2h_{out}T_4 - h_{out}T_3) \\ B_3^3 \end{Bmatrix} , \qquad \{B^4\} = \begin{Bmatrix} B_1^4 \\ B_2^4 \\ \frac{0.05}{6}(3h_{in}T_{in} - 2h_{in}T_9 - h_{in}T_8) \\ \frac{0.05}{6}(3h_{in}T_{in} - 2h_{in}T_8 - h_{in}T_9) \end{Bmatrix}$$

These four $\{B^e\}$'s can be assembled into the following global $\{B\}$ vector

$$\{B\} = \begin{Bmatrix} \frac{0.05}{6}(3h_{out}T_{out} - 2h_{out}T_1 - h_{out}T_2) \\ \frac{0.05}{6}(3h_{out}T_{out} - 2h_{out}T_2 - h_{out}T_1) + \frac{0.05}{6}(3h_{out}T_{out} - 2h_{out}T_2 - h_{out}T_3) \\ \frac{0.05}{6}(3h_{out}T_{out} - 2h_{out}T_3 - h_{out}T_2) + \frac{0.05}{6}(3h_{out}T_{out} - 2h_{out}T_3 - h_{out}T_4) \\ \frac{0.05}{6}(3h_{out}T_{out} - 2h_{out}T_4 - h_{out}T_3) \\ 0 \\ 0 \\ 0 \\ \frac{0.05}{6}(3h_{in}T_{in} - 2h_{in}T_8 - h_{in}T_9) \\ \frac{0.05}{6}(3h_{in}T_{in} - 2h_{in}T_9 - h_{in}T_8) \end{Bmatrix}$$

The 9x9 global system that needs to be solved is

$$
\begin{bmatrix}
0.9333 & -0.2333 & 0 & 0 & -0.2333 & -0.4667 & 0 & 0 & 0 \\
-0.2333 & 1.8667 & -0.2333 & 0 & -0.4667 & -0.4667 & -0.4667 & 0 & 0 \\
0 & -0.2333 & 2.3333 & -0.7 & 0 & -0.4667 & -0.9333 & 0 & 0 \\
0 & 0 & -0.7 & 0.7 & 0 & 0 & 0 & 0 & 0 \\
-0.2333 & -0.4667 & 0 & 0 & 1.8667 & -0.4667 & 0 & -0.2333 & -0.4667 \\
-0.4667 & -0.4667 & -0.4667 & 0 & -0.4667 & 4.2 & -0.9333 & -0.4667 & -0.9333 \\
0 & -0.4667 & -0.9333 & 0 & 0 & -0.9333 & 2.3333 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.2333 & -0.4667 & 0 & 0.9333 & -0.2333 \\
0 & 0 & 0 & 0 & -0.4667 & -0.9333 & 0 & -0.2333 & 1.6333
\end{bmatrix}
\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix}
$$

$$
= \begin{Bmatrix}
\frac{0.05}{6}\left(3h_{out}T_{out} \boxed{-2h_{out}T_1 - h_{out}T_2}\right) \\
\frac{0.05}{6}\left(3h_{out}T_{out} \boxed{-2h_{out}T_2 - h_{out}T_1}\right) + \frac{0.05}{6}\left(3h_{out}T_{out} \boxed{-2h_{out}T_2 - h_{out}T_3}\right) \\
\frac{0.05}{6}\left(3h_{out}T_{out} \boxed{-2h_{out}T_3 - h_{out}T_2}\right) + \frac{0.05}{6}\left(3h_{out}T_{out} \boxed{-2h_{out}T_3 - h_{out}T_4}\right) \\
\frac{0.05}{6}\left(3h_{out}T_{out} \boxed{-2h_{out}T_4 - h_{out}T_3}\right) \\
0 \\
0 \\
0 \\
\frac{0.05}{6}\left(3h_{in}T_{in} \boxed{-2h_{in}T_8 - h_{in}T_9}\right) \\
\frac{0.05}{6}\left(3h_{in}T_{in} \boxed{-2h_{in}T_9 - h_{in}T_8}\right)
\end{Bmatrix}
$$

The boxed terms of the $\{B\}$ vector has unknown temperatures and they should be transferred to the $[K]$ matrix to get the following system (we also used $h_{in} = 70,\ T_{in} = 300,\ h_{out} = 21,\ T_{out} = 20$)

$$
\begin{bmatrix}
1.2833 & -0.0583 & 0 & 0 & -0.2333 & -0.4667 & 0 & 0 & 0 \\
-0.0583 & 2.5667 & -0.0583 & 0 & -0.4667 & -0.4667 & -0.4667 & 0 & 0 \\
0 & -0.0583 & 3.0333 & -0.525 & 0 & -0.4667 & -0.9333 & 0 & 0 \\
0 & 0 & -0.525 & 1.05 & 0 & 0 & 0 & 0 & 0 \\
-0.2333 & -0.4667 & 0 & 0 & 1.8667 & -0.4667 & 0 & -0.2333 & -0.4667 \\
-0.4667 & -0.4667 & -0.4667 & 0 & -0.4667 & 4.2 & -0.9333 & -0.4667 & -0.9333 \\
0 & -0.4667 & -0.9333 & 0 & 0 & -0.9333 & 2.3333 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.2333 & -0.4667 & 0 & 2.1 & 0.35 \\
0 & 0 & 0 & 0 & -0.4667 & -0.9333 & 0 & 0.35 & 2.8
\end{bmatrix}
\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix}
= \begin{Bmatrix} 10.5 \\ 21 \\ 21 \\ 10.5 \\ 0 \\ 0 \\ 0 \\ 525 \\ 525 \end{Bmatrix}
$$

where the blue entries of $[K]$ are the ones modified due to MBCs. This global system can be solved to get the following nodal temperatures

$$
\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix}
= \begin{Bmatrix} 97.1 \\ 88.9 \\ 73.2 \\ 46.6 \\ 163.3 \\ 151.8 \\ 107.8 \\ 263.1 \\ 232.4 \end{Bmatrix} \ ℃
$$

To find the rate of heat transfer through a 1 m section of the chimney, we must evaluate the heat passing through the 3rd face of the 4th element, which is given by

$$Q = \int_{x=0}^{0.05} k \left. \frac{\partial T^4}{\partial y} \right|_{\text{face 3}} (1) dx$$

where the temperature distribution over the 3rd face of 4th element is

$$T^4|_{\text{face 3}} = \sum_{j=1}^{4} T_j^4 S_j = \underbrace{T_1^4 S_1}_{=0 \text{ on face 3}} + \underbrace{T_2^4 S_2}_{=0 \text{ on face 3}} + T_3^4 S_3 + T_4^4 S_4$$

and its y-derivative is

$$\left. \frac{\partial T^4}{\partial y} \right|_{\text{face 3}} = T_3^4 \frac{\partial S_3}{\partial y} + T_4^4 \frac{\partial S_4}{\partial y}$$

where the y-derivatives of the shape functions can be written using equation (3.16) as follows

$$\frac{\partial S_3}{\partial y} = \frac{\partial \xi}{\partial y} \frac{\partial S_3}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial S_3}{\partial \eta}$$

$$\frac{\partial S_3}{\partial y} = J_{21}^{-1} \frac{\partial S_3}{\partial \xi} + J_{22}^{-1} \frac{\partial S_3}{\partial \eta}$$

$$\frac{\partial S_3}{\partial y} = (0) \frac{\partial S_3}{\partial \xi} + (40) \frac{1}{4} (1 + \xi) = 10 + 10\xi$$

Similarly

$$\frac{\partial S_4}{\partial y} = (0) \frac{\partial S_4}{\partial \xi} + (40) \frac{1}{4} (1 - \xi) = 10 - 10\xi$$

Using these in the above $\left. \frac{\partial T^4}{\partial y} \right|_{\text{face 3}}$ equation

$$\left. \frac{\partial T^4}{\partial y} \right|_{\text{face 3}} = T_3^4 (10 + 10\xi) + T_4^4 (10 - 10\xi)$$

$$\left. \frac{\partial T^4}{\partial y} \right|_{\text{face 3}} = (232.4)(10 + 10\xi) + (263.1)(10 - 10\xi)$$

$$\left. \frac{\partial T^4}{\partial y} \right|_{\text{face 3}} = 4955 - 307\xi$$

$$Q = \int_{x=0}^{0.05} (1.4)\,(4955 - 307\xi) dx$$

We need to do a conversion from x to ξ on face 3 of element 4. This is like the conversion of an actual element to the master element in 1D, with $dx = \frac{0.05}{2} d\xi$. The integral becomes

$$Q = \int_{\xi=-1}^{1} (1.4)\,(4955 - 307\xi)\, \frac{0.05}{2}\, d\xi$$

$$Q = 346.85 \text{ W}$$

Note that we modeled only 1/8 of the chimney. Therefore the total heat that passes through 1 m section of the chimney is 8 times of the above value

$$Q_{chimney} = 2775 \text{ W}$$

Same heat transfer can also be obtained using the following Newton's law of cooling with an average temperature on the inner wall of the chimney

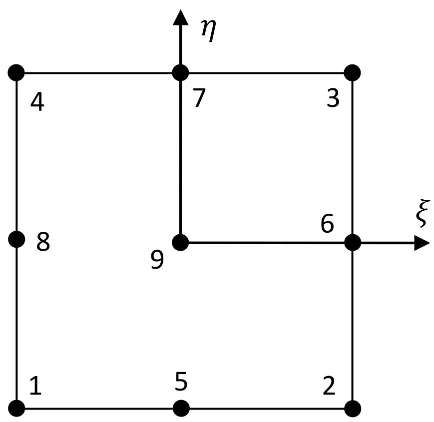
$$Q_{chimney} = -8\, h_{in} \left(\frac{T_8 + T_9}{2} - T_{in}\right)(0.05)(1)$$

$$Q_{chimney} = -8\,(70)\left(\frac{263.1 + 232.4}{2} - 300\right)(0.05)(1) = 1463 \text{ W}$$

As seen the results are not close, but they should approach to each other as the mesh is refined. The latter one is expected to be more accurate because the differentiation involved in the first one amplifies the errors already existing in the approximate solution. Note that the second result is nothing but equal to $B_3^4 + B_4^4$.

3.8 Higher Order Elements in 2D

In Section 3.2 we introduced 4-node quadrilateral and 3-node triangular elements, which are also known as bilinear elements. Similar to 1D, it is possible to use higher order elements with more nodes, as shown below.



$$general\,form \quad S = A + B\xi + C\eta + D\xi\eta + E\xi^2 + F\eta^2 + G\xi^2\eta + H\xi\eta^2 + I\xi^2\eta^2$$

$$S_1 = \frac{1}{4}(\xi^2 - \xi)(\eta^2 - \eta)\ , \qquad S_2 = \frac{1}{4}(\xi^2 + \xi)(\eta^2 - \eta)$$

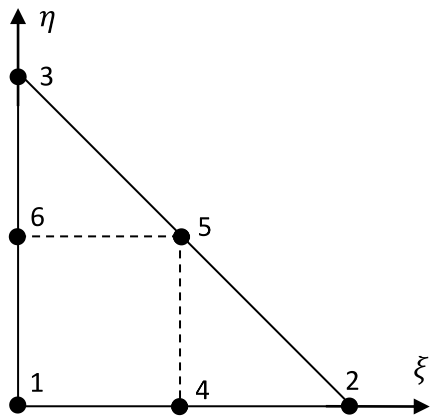
$$S_3 = \frac{1}{4}(\xi^2 + \xi)(\eta^2 + \eta)\ , \qquad S_4 = \frac{1}{4}(\xi^2 - \xi)(\eta^2 + \eta)$$

$$S_5 = \frac{1}{2}(1 - \xi^2)(\eta^2 - \eta)\ , \qquad S_6 = \frac{1}{2}(\xi^2 + \xi)(1 - \eta^2)$$

$$S_7 = \frac{1}{2}(1 - \xi^2)(\eta^2 + \eta)\ , \qquad S_8 = \frac{1}{2}(\xi^2 - \xi)(1 - \eta^2)$$

$$S_9 = (1 - \xi^2)(1 - \eta^2)$$

Figure 3.12 9-node quadrilateral master element and shape functions



$$general\ form\quad S = A + B\xi + C\eta + D\xi\eta + E\xi^2\eta + F\xi\eta^2$$

$$S_1 = 2(1-\xi-\eta)\left(\frac{1}{2}-\xi-\eta\right)$$

$$S_2 = 2\xi\left(\xi-\frac{1}{2}\right)$$

$$S_3 = 2\eta\left(\eta-\frac{1}{2}\right)$$

$$S_4 = 4(1-\xi-\eta)\xi$$

$$S_5 = 4\xi\eta$$

$$S_6 = 4(1-\xi-\eta)\eta$$

Figure 3.13 6-node triangular master element and shape functions

Note that these higher order elements may have curved faces, such as the ones given below. Geometric calculations on such elements generally use transfinite interpolation (TFI).

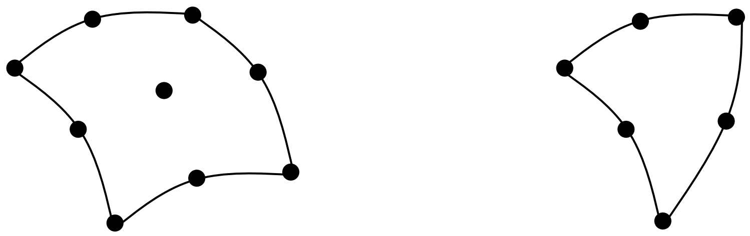
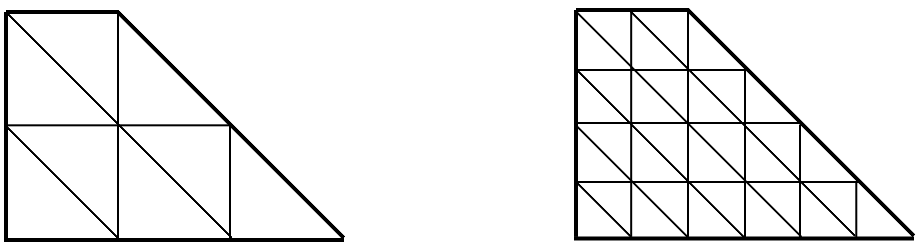


Figure 3.14 Higher order 2D elements with curved faces

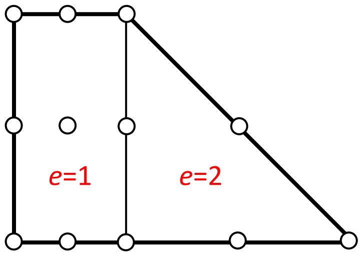
3.9 Exercises

E-3.1. Solve the chimney problem of Section 3.7 again using the following meshes and compare the results.



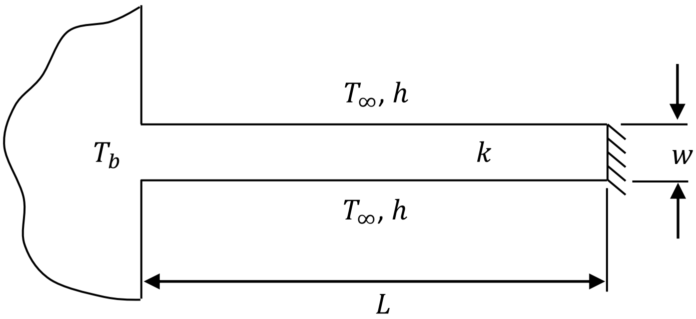
NE=8, NN=9 NE=32, NN=25

E-3.2. Solve the chimney problem of Section 3.7 using the following 2 element mesh, but the elements are not bilinear. Equations (3.34) and (3.35) are no longer valid and new equations need to be derived for MBCs specified at 3-node faces.



NE=2, NN=12

E-3.3. A straight fin of uniform cross section is fabricated from a material of thermal conductivity $50\ \mathrm{W/(mK)}$, thickness $w = 6\ \mathrm{mm}$, and length $L = 48\ \mathrm{mm}$, and is very long in the direction normal to the page. The convection heat transfer coefficient is $500\ \mathrm{W/(m^2\ K)}$ with an ambient temperature of $T_\infty = 30\ °\mathrm{C}$. The base of the fin is maintained at $T_b = 100\ °\mathrm{C}$, while the tip is well insulated.

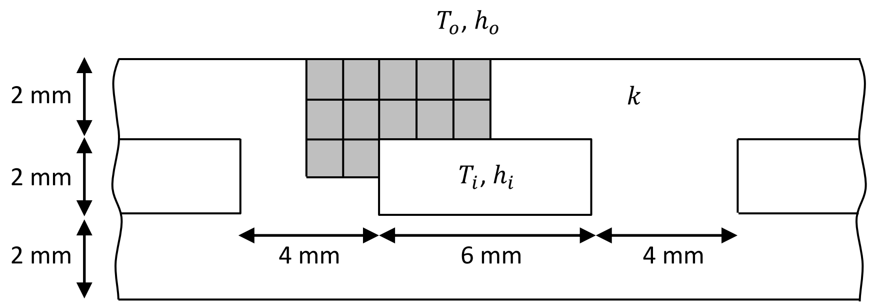


a) Using a mesh of eight 6 mm-length, linear 1D elements, obtain the temperature distribution over the fin and find the fin heat transfer rate per unit length normal to the page. Compare your results with the known analytical solution.
b) Using a mesh of sixteen (6 mm x 3 mm) bilinear quadrilateral elements, obtain the 2D temperature distribution over the fin. Determine the fin heat transfer rate and compare it with the one found in part (a).

This problem is taken from reference [2].

E-3.4. A major objective in advancing gas turbine engine technologies is to increase the temperature limit associated with operation of the gas turbine blades. This limit determines the permissible turbine gas inlet temperature, which, in turn, strongly influences overall system performance. In addition to fabricating turbine blades from special, high-temperature, high-strength superalloys, it is common to use internal cooling by machining flow channels within the blades and routing air through the channels. We wish to assess the effect of such a scheme by approximating the blade as a rectangular solid in which rectangular channels are machined. The blade, which has a thermal conductivity of $k = 25\ \mathrm{W/(mK)}$, is 6 mm thick, and each channel has a 2 mm x 6 mm rectangular cross section, with a 4 mm spacing between adjoining channels. Under operating conditions for which $h_o = 1000\ \mathrm{W/(m^2K)}$, $T_o = 1700\ \mathrm{K}$, $h_i = 200\ \mathrm{W/(m^2K)}$, and $T_i = 400\ \mathrm{K}$, determine the temperature field in the turbine blade and rate of heat transfer per unit length to the channel. At what location is the temperature a maximum?

Consider the symmetry planes and study only the shaded part of the problem domain using 12 quadrilateral elements of size 1cm x 1cm. This problem is taken from reference [2].



E-3.5. For the chimney problem of Section 3.7, consider that the chimney is losing heat not only to the ambient air by convection, but also to the sky by radiation. The emissivity of the outer surface of the wall is $\varepsilon = 0.9$, and the effective sky temperature is $260\ \mathrm{K}$. Radiative heat transfer can be modeled as

$$q''_{rad} = \varepsilon\, \sigma \left(T^4 - T_{sky}^4\right)$$

where $\sigma = 5.67 \times 10^{-8}\ \mathrm{W/\left(m^2K^4\right)}$ is the Stefan-Boltzmann constant. This BC is nonlinear in terms of the outer wall temperature of the chimney. What kind of a difficulty arises in the FE solution due to this BC nonlinearity? Determine the nonlinear global system and solve it. Note that you need to work with the Kelvin temperature scale because of the radiative heat transfer.

References

[1] Y. Cengel, Heat and Mass Transfer – A Practical Approach, Mc Graw-Hill, 2006.

[2] F. P. Incropera, D. P Dewitt, T. L. Bergman, A. S. Lavine, Fundamentals of Heat and Mass Transfer, 6th ed., John Wiley and Sons, 2007.