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The moment function for the ratio of correlated generalized gamma variables

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ABSTRACT

The moment function for the ratio of correlated generalized gamma variables is expressed in terms of special functions. The expression presented generalizes the known moment expression for the integer valued moments to the real valued moments. Approximate formulas, in terms of elementary functions, are provided for low and high correlation regions and some application examples are given.

1. Introduction

The generalized gamma distribution is an extension of the standard gamma distribution which is found to be useful in many applications including radar signal processing and communications (Piboongungon et al., 2005; Candan and Koc, 2012; Zhang, 2000; Bithas et al., 2007; Cui et al., 2013). In communications, Nakagami and Weibull distributions, which are special cases of the generalized gamma distribution, are frequently used in the modeling of fading channels. In radar signal processing, the classical target fluctuation models, known as Swerling models, use the gamma distribution to model the distribution of return power from a target. In this work, we present the moment function for the *ratio* of generalized gamma distributed variables. Unlike the results available in the literature, the derivation presented here is valid for real valued moments; hence the moment expression given generalizes the moment function in Tubbs (1986) given for the integer valued moments. The expression presented is surprisingly elegant and can be utilized in many applications involving gamma variables.

2. The moment function

The joint probability density function for the correlated bivariate gamma variables p_1 and p_2 can be given as follows:

$$f_{p_1,p_2}(p_1,p_2) = \frac{m^{m+1}}{\Gamma(m)\Omega_1\Omega_2(1-\rho)}\left(\frac{p_1p_2}{\rho\Omega_1\Omega_2}\right)^{\frac{m-1}{2}} \exp\left\{-\frac{m}{1-\rho}\left(\frac{p_1}{\Omega_1}+\frac{p_2}{\Omega_2}\right)\right\} \times I_{m-1}\left(\frac{2m}{1-\rho}\sqrt{\rho\frac{p_1p_2}{\Omega_1\Omega_2}}\right)\mathrm{u}(p_1)\mathrm{u}(p_2). \quad (1)$$

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In the expression above, $\Gamma(\cdot)$ is the gamma function, $I_\alpha(\cdot)$ is the modified Bessel function of order α and $\mathrm{u}(\cdot)$ is the unit step function. Here Ω_k is the mean value of p_k, that is $\Omega_k = E\{p_k\}$ for $k = \{1, 2\}$ and $E\{\cdot\}$ is the statistical expectation operator. The correlation coefficient between p_1 and p_2 is shown with ρ $(0 \leq \rho \leq 1)$. The shape parameter is denoted by m $(m \geq 1/2)$. The shape parameter is also known as the fading parameter or the Nakagami-m parameter in the communications literature (e.g., Yacoub et al. (1999)).

The *generalized gamma variables*, x_k, are defined through the following power relation: $x_k = (p_k)^{1/(2\upsilon)}$ for $\upsilon > 0$. In this work, we are interested in the moments of $r = x_1/x_2$ where x_1 and x_2 are generalized gamma distributed random variables. The density of the ratio shown by r and the analytical expression for its integer valued moments are given in Lee et al. (1979) and Tubbs (1986), respectively. Our goal is to extend the known moment expression given in Tubbs (1986) to real valued moments. Applications of the relation presented are given at the end of the present section.

Theorem 1. *The moment function of $r = p_1/p_2$ where p_1 and p_2 is jointly gamma distributed with $\Omega_1 = \Omega_2$ is*[1]

$$\Phi_r(s) = E\{r^s\} = \frac{\Gamma(m+s)\Gamma(m-s)}{\Gamma(m)\Gamma(m)}\,{}_2F_1(-s, s; m; \rho), \quad s < m.$$

Proof. The density of the ratio r can be written as $f_r(r) = \int_0^\infty z f_{p_1,p_2}(rz, z)dz$ (Papoulis, 1991, Eq. (6–43)). Here $f_{p_1,p_2}(p_1, p_2)$ is the joint density given in (1). The moment function can be written as follows:

$$\begin{aligned} E\{r^s\} &= \int_0^\infty r^s f_r(r)dr \\ &= \int_0^\infty z\left[\int_0^\infty r^s f_{p_1,p_2}(rz, z)dr\right]dz \\ &= \int_0^\infty K(z)\left[\int_0^\infty r^{\frac{m-1}{2}+s} e^{-\frac{mz}{1-\rho}r} I_{m-1}\left(\frac{2mz}{1-\rho}\sqrt{r\rho}\right)dr\right]dz. \end{aligned} \tag{2}$$

In the last expression, $K(z)$ contains terms irrelevant for the evaluation of the inner integral shown with the square brackets. The inner integral can be evaluated using (Gradshteyn and Ryzhik, 2007, 6.643 item 2) and can be simplified to the following expression using (Gradshteyn and Ryzhik, 2007, 9.220 item 2) and (Gradshteyn and Ryzhik, 2007, 9.212 item 1):

$$\frac{\Gamma(m+s)}{\Gamma(m)}\rho^{\frac{m-1}{2}} t^{-s-1} e^{t\rho}\,{}_1F_1(-s; m; -t\rho). \tag{3}$$

In the last relation, t is a space holder for $mz/(1-\rho)$. Substituting (3) for the contents of the square brackets in (2), we get

$$\begin{aligned} E\{r^s\} &= \frac{m^{m-s}(1-\rho)^s\Gamma(m+s)}{\Gamma(m)\Gamma(m)}\int_0^\infty z^{m-1-s}e^{-mz}\,{}_1F_1\left(-s; m; -\frac{\rho}{1-\rho}mz\right)dz \\ &= \frac{\Gamma(m+s)}{\Gamma(m)\Gamma(m)}(1-\rho)^s\int_0^\infty z^{m-1-s}e^{-z}\,{}_1F_1\left(-s; m; -\frac{\rho}{1-\rho}z\right)dz. \end{aligned}$$

The integral appearing in the last line can be evaluated using (Gradshteyn and Ryzhik, 2007, 7.522 item 9) as $\Gamma(m-s)$ ${}_2F_1(-s, m-s; m; -\rho/(1-\rho))$ for $s < m$. Using Pfaff's transformation formula given in Gradshteyn and Ryzhik (2007, 9.131 item 1) on this result finalizes the proof:

$$E\{r^s\} = \frac{\Gamma(m+s)\Gamma(m-s)}{\Gamma(m)\Gamma(m)}\,{}_2F_1(-s, s; m; \rho), \quad s < m. \quad \square$$

It should be noted that the moments for $s > m$ are unbounded.

Corollary 1. *The moment function of $r = p_1/p_2$ where p_1 and p_2 is jointly gamma distributed with $\Omega_1 \neq \Omega_2$ is*

$$E\{r^s\} = \left(\frac{\Omega_1}{\Omega_2}\right)^s \frac{\Gamma(m+s)\Gamma(m-s)}{\Gamma(m)\Gamma(m)}\,{}_2F_1(-s, s; m; \rho), \quad s < m.$$

Corollary 2. *The moment function of $r = x_1/x_2$ where x_1, x_2 are generalized gamma distributed, that is $x_1 = p_1^{1/2\upsilon}$, $x_2 = p_2^{1/2\upsilon}$ where p_1 and p_2 are gamma distributed, is*

$$E\{r^s\} = \left(\frac{\Omega_1}{\Omega_2}\right)^{\frac{s}{2\upsilon}} \frac{\Gamma\left(m+\frac{s}{2\upsilon}\right)\Gamma\left(m-\frac{s}{2\upsilon}\right)}{\Gamma(m)\Gamma(m)}\,{}_2F_1\left(\frac{s}{2\upsilon}, \frac{s}{2\upsilon}; m; \rho\right), \quad s < 2\upsilon m.$$

[1] ${}_2F_1(\cdot)$ is the generalized hypergeometric function with the definition and notation of Gradshteyn and Ryzhik (2007, 9.100).

Corollary 1 follows from Theorem 1 by defining $r = (\Omega_1/\Omega_2)\widehat{r}$ where $\widehat{r}$ is the ratio of gamma variables with identical mean values, as in Theorem 1. The moment generating function of r is then $(\Omega_1/\Omega_2)^s \Phi_{\widehat{r}}(s)$ and the corollary follows from Theorem 1. For Corollary 2, it can be noted that $E\{r^s\} = E\{(p_1/p_2)^{s/(2v)}\}$ and the result follows from Corollary 1.

Theorem 1 gives an elegant and compact characterization of the moment function in terms of special functions. It can be noted that the expression reduces to the known integer valued moments, given in Tubbs (1986, Eq. (2.6)), for $s = \{0, 1, 2, \ldots\}$.

In some applications, the gamma variables appearing in the numerator and denominator of the ratio can have extremely low correlation ($\rho \approx 0$) or high correlation ($\rho \approx 1$). Corollaries 3 and 4 present the Taylor series approximations of the moment function for such low and high correlation values. (Corollaries 3 and 4 are given under the conditions of Theorem 1 and can be easily generalized as in Corollaries 1 and 2.)

Corollary 3. *The moment function of $r = p_1/p_2$ where p_1 and p_2 is jointly gamma distributed with $\Omega_1 = \Omega_2$ can be approximated for small ρ ($\rho \approx 0$) as*

$$\begin{aligned}\Phi_r(s) &= \frac{\Gamma(m+s)\Gamma(m-s)}{\Gamma(m)\Gamma(m)} \sum_{k=0}^{\infty} \frac{(-s)_k (s)_k}{(m)_k} \frac{\rho^k}{k!} \\ &\approx \frac{\Gamma(m+s)\Gamma(m-s)}{\Gamma(m)\Gamma(m)} \left(1 - \frac{s^2}{m}\rho - \frac{s^2(1-s^2)}{m(m+1)} \frac{\rho^2}{2}\right), \quad s < m.\end{aligned}$$

Corollary 4. *The moment function of $r = p_1/p_2$ where p_1 and p_2 is jointly gamma distributed with $\Omega_1 = \Omega_2$ can be approximated for large ρ ($\rho \approx 1$) as*

$$\begin{aligned}\Phi_r(s) &= \sum_{k=0}^{\infty} \frac{(s)_k (s-k+1)_k}{(m-k)_k} \frac{(1-\rho)^k}{k!} \\ &\approx 1 + \frac{s^2}{(m-1)}(1-\rho) + \frac{s^2(s^2-1)}{(m-2)(m-1)} \frac{(1-\rho)^2}{2}, \quad s < m.\end{aligned}$$

Corollary 3 follows from the definition of the hypergeometric function given in Gradshteyn and Ryzhik (2007, 9.100). Here, the function $(\alpha)_m$ is the Pochhammer symbol which is defined as $(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1)$ for $m > 0$ and $(\alpha)_0 = 1$. The present definition for the Pochhammer symbol is also called the rising factorial.

The derivation of Corollary 4 is a bit more involved than the earlier one. When the moment function $\Phi_r(s)$ of Theorem 1 is rewritten by expressing the hypergeometric function in integral form using (Gradshteyn and Ryzhik, 2007, 9.111), we get the following alternative expression for $\Phi_r(s)$:

$$\Phi_r(s) = \frac{\Gamma(m+s)}{\Gamma(m)\Gamma(s)} \int_0^1 x^{s-1}(1-x)^{m-s-1}(1-\rho x)^s dx. \tag{4}$$

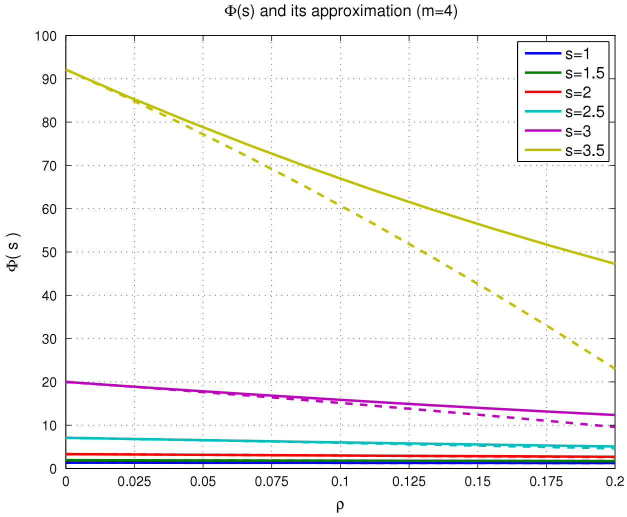
This expression allows us to calculate the Taylor series expansion of $\Phi_r(s)$ around $\rho = 1$. The resultant Taylor series expansion is given in Corollary 4.

Fig. 1 compares the low and high correlation approximations, which are calculated through the quadratic expressions for ρ given in the Corollaries 3 and 4, with the exact expression involving special functions. In this figure, m is set to 4 and the exact moment function (solid lines) and its approximations (dashed lines) are shown for different ρ and s values.

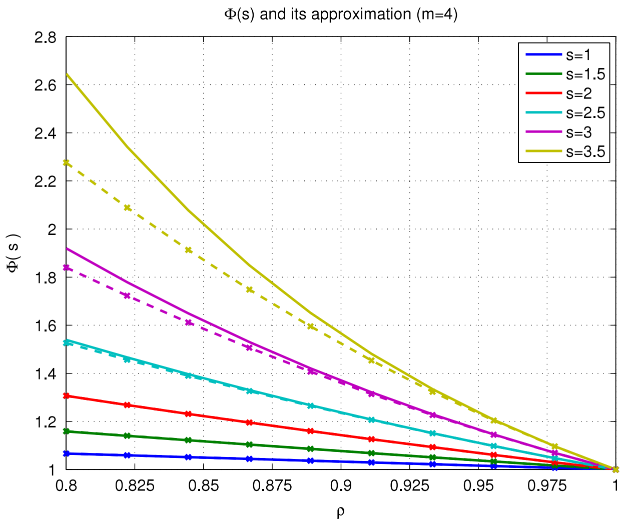
Application suggestions: The real valued moment expression given in this letter can be used to derive other expressions for the ratio of the gamma variables. In Candan and Koc (2012), the moments of $z = \ln(r)$, where r is the ratio of gamma variables, is needed to characterize the performance of a direction finding system. Unfortunately, the moments of z are difficult to calculate using standard techniques; but it can be noted that $E\{z^k\} = \frac{d^k}{ds^k}\Phi_r(s)\Big|_{s=0}$ and therefore, by taking the kth derivative of $\Phi_r(s)$ and evaluating the result at $s = 0$, we can immediately get the kth moment of $\ln(r)$. We believe that the expression for the real valued moments can be further utilized in other applications. Similarly, the calculation of the channel capacity for the zero-outage scheme is also enabled through the usage of the results presented in this work; see Candan (2013).

3. Conclusions

This work presents the moment expression for the ratio of generalized gamma variables. The compact relation, in terms of the generalized hypergeometric function, reduces to the known expressions for the integer valued moments and generalizes them to arbitrary non-negative real numbers. The relation presented could be useful in the analysis of communication and radar systems where ratios of gamma variables are frequently utilized (Piboongungon et al., 2005; Candan and Koc, 2012).



(a) Low correlation region.



(b) High correlation region.

Fig. 1. Moment function $\Phi_r(s)$ (solid line) and its quadratic approximation for low and high correlation values (dashed line) for $m = 4$.

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