A Simple Proof of ${}_3F_2(1,1,1;2,2;\rho) = \frac{\text{dilog}(1-\rho)}{\rho}$

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\begin{aligned}
{}_3F_2(1,1,1;2,2;\rho) &\stackrel{(1)}{=} \sum_{k=0}^{\infty} \frac{(1)_k(1)_k(1)_k}{(2)_k(2)_k}\frac{\rho^k}{k!} && \text{(definition of hypergeometric func.)} \\
&\stackrel{(2)}{=} \sum_{k=0}^{\infty} \frac{k!k!k!}{(k+1)!(k+1)!}\frac{\rho^k}{k!} && \text{(definition of Pochhammer symbol)} \\
&\stackrel{(3)}{=} \sum_{k=0}^{\infty} \frac{\rho^k}{(k+1)^2} && \text{(clean up)} \\
&\stackrel{(4)}{=} \frac{1}{\rho}\sum_{k=0}^{\infty} \frac{\rho^{k+1}}{(k+1)^2} && \text{(multiply and divide by } \rho\text{)} \\
&\stackrel{(5)}{=} \frac{1}{\rho}\sum_{k=0}^{\infty} \frac{\int_0^\rho t^k dt}{k+1} && \text{(invent an integral!)} \\
&\stackrel{(6)}{=} \frac{1}{\rho}\int_0^\rho \left(\sum_{k=0}^{\infty} \frac{t^k}{k+1}\right) dt && \text{(do the exchange)} \\
&\stackrel{(7)}{=} \frac{-1}{\rho}\int_0^\rho \frac{\ln(1-t)}{t} dt && \text{(recognize the power series)} \\
&\stackrel{(8)}{=} \frac{1}{\rho}\int_1^{1-\rho} \frac{\ln(t')}{1-t'} dt' && (t' = 1-t) \\
&\stackrel{(9)}{=} \frac{\text{dilog}(1-\rho)}{\rho} && \text{(definition of dilog function)} \\
&\stackrel{(10)}{=} \frac{\text{Li}_2(\rho)}{\rho} && (\text{dilog}(1-x) = \text{Li}_2(x))
\end{aligned}
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Remember that $|\rho| \le 1$, which is the standard assumption for the hypergeometric series.

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