Probability Random Variables and Stochastic Processes, 3rd Edition. Papoulis

PART II

STOCHASTIC PROCESSES

CHAPTER 10

GENERAL CONCEPTS

10-1 DEFINITIONS

As we recall, an RV $\mathbf{x}$ is a rule for assigning to every outcome ζ of an experiment $\mathscr{S}$ a *number* $\mathbf{x}(\zeta)$. A stochastic process $\mathbf{x}(t)$ is a rule for assigning to every ζ a *function* $\mathbf{x}(t,\zeta)$. Thus a stochastic process is a family of time functions depending on the parameter ζ or, equivalently, a function of t and ζ. The domain of ζ is the set of all experimental outcomes and the domain of t is a set R of real numbers.

If R is the real axis, then $\mathbf{x}(t)$ is a *continuous-time* process. If R is the set of integers, then $\mathbf{x}(t)$ is a *discrete-time* process. A discrete-time process is, thus, a sequence of random variables. Such a sequence will be denoted by $\mathbf{x}_n$ as in Sec. 8-4, or, to avoid double indices, by $\mathbf{x}[n]$.

We shall say that $\mathbf{x}(t)$ is a *discrete-state* process if its values are countable. Otherwise, it is a *continuous-state* process.

Most results in this investigation will be phrased in terms of continuous-time processes. Topics dealing with discrete-time processes will be introduced either as illustrations of the general theory, or when their discrete-time version is not self-evident.

We shall use the notation $\mathbf{x}(t)$ to represent a stochastic process omitting, as in the case of random variables, its dependence on ζ. Thus $\mathbf{x}(t)$ has the following interpretations:

1. It is a family (or an *ensemble*) of functions $\mathbf{x}(t,\zeta)$. In this interpretation, t and ζ are variables.

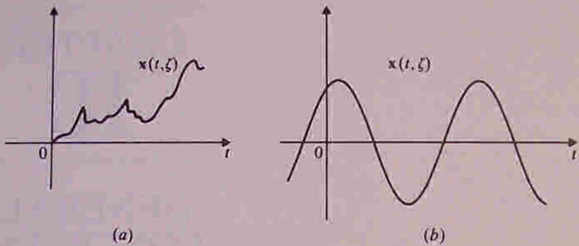

FIGURE 10-1

2. It is a single time function (or a *sample* of the given process). In this case, t is a variable and ζ is fixed.
3. If t is fixed and ζ is variable, then $\mathbf{x}(t)$ is a random variable equal to the *state* of the given process at time t.
4. If t and ζ are fixed, then $\mathbf{x}(t)$ is a *number*.

A physical example of a stochastic process is the motion of microscopic particles in collision with the molecules in a fluid (*brownian motion*). The resulting process $\mathbf{x}(t)$ consists of the motions of all particles (ensemble). A single realization $\mathbf{x}(t, \zeta_i)$ of this process (Fig. 10-1a) is the motion of a specific particle (sample). Another example is the voltage

$$\mathbf{x}(t) = \mathbf{r}\cos(\omega t + \boldsymbol{\varphi})$$

of an ac generator with random amplitude $\mathbf{r}$ and phase $\boldsymbol{\varphi}$. In this case, the process $\mathbf{x}(t)$ consists of a family of pure sine waves and a single sample is the function (Fig. 10-1b)

$$\mathbf{x}(t, \zeta_i) = \mathbf{r}(\zeta)\cos[\omega t + \boldsymbol{\varphi}(\zeta)]$$

According to our definition, both examples are stochastic processes. There is, however, a fundamental difference between them. The first example (regular) consists of a family of functions that cannot be described in terms of a finite number of parameters. Furthermore, the future of a sample $\mathbf{x}(t, \zeta)$ of $\mathbf{x}(t)$ cannot be determined in terms of its past. Finally, under certain conditions, the statistics† of a regular process $\mathbf{x}(t)$ can be determined in terms of a single sample (see Sec. 13-1). The second example (predictable) consists of a family of pure sine waves and it is completely specified in terms of the RVs $\mathbf{r}$ and $\boldsymbol{\varphi}$. Furthermore, if $\mathbf{x}(t, \zeta)$ is known for $t \le t_o$, then it is determined for $t > t_o$. Finally, a single sample $\mathbf{x}(t, \zeta)$ of $\mathbf{x}(t)$ does not specify the properties of the

†Recall that *statistics* hereafter will mean statistical properties.

entire process because it depends only on the particular values $\mathbf{r}(\zeta)$ and $\varphi(\zeta)$ of $\mathbf{r}$ and φ. A formal definition of regular and predictable processes is given in Sec. 12-3.

Equality. We shall say that two stochastic processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are equal (everywhere) if their respective samples $\mathbf{x}(t,\zeta)$ and $\mathbf{y}(t,\zeta)$ are identical for every ζ. Similarly, the equality $\mathbf{z}(t) = \mathbf{x}(t) + \mathbf{y}(t)$ means that $\mathbf{z}(t,\zeta) = \mathbf{x}(t,\zeta) + \mathbf{y}(t,\zeta)$ for every ζ. Derivatives, integrals, or any other operations involving stochastic processes are defined similarly in terms of the corresponding operations for each sample.

As in the case of limits, the above definitions can be relaxed. We give below the meaning of MS equality and in App. 10A we define MS derivatives and integrals. Two processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are equal in the MS sense iff

$$E\{|\mathbf{x}(t) - \mathbf{y}(t)|^2\} = 0 \tag{10-1}$$

for every t. Equality in the MS sense leads to the following conclusions: We denote by $\mathscr{A}_t$ the set of outcomes ζ such that $\mathbf{x}(t,\zeta) = \mathbf{y}(t,\zeta)$ for a *specific* t, and by $\mathscr{A}_\infty$ the set of outcomes ζ such that $\mathbf{x}(t,\zeta) = \mathbf{y}(t,\zeta)$ for *every* t. From (10-1) it follows that $\mathbf{x}(t,\zeta) - \mathbf{y}(t,\zeta) = 0$ with probability 1; hence $P(\mathscr{A}_t) = P(\mathscr{S}) = 1$. It does not follow, however, that $P(\mathscr{A}_\infty) = 1$. In fact, since $\mathscr{A}_\infty$ is the intersection of all sets $\mathscr{A}_t$ as t ranges over the entire axis, $P(\mathscr{A}_\infty)$ might even equal 0.

Statistics of Stochastic Processes

A stochastic process is a noncountable infinity of random variables, one for each t. For a specific t, $\mathbf{x}(t)$ is an RV with distribution

$$F(x,t) = P\{\mathbf{x}(t) \le x\} \tag{10-2}$$

This function depends on t, and it equals the probability of the event $\{\mathbf{x}(t) \le x\}$ consisting of all outcomes ζ such that, at the specific time t, the samples $\mathbf{x}(t,\zeta)$ of the given process do not exceed the number x. The function $F(x,t)$ will be called the *first-order distribution* of the process $\mathbf{x}(t)$. Its derivative with respect to x:

$$f(x,t) = \frac{\partial F(x,t)}{\partial x} \tag{10-3}$$

is the *first-order density* of $\mathbf{x}(t)$.

Frequency interpretation If the experiment is performed n times, then n functions $\mathbf{x}(t,\zeta_i)$ are observed, one for each trial (Fig. 10-2). Denoting by $n_t(x)$ the number of trials such that at time t the ordinates of the observed functions do not exceed x (solid lines), we conclude as in (4-3) that

$$F(x,t) \simeq \frac{n_t(x)}{n} \tag{10-4}$$

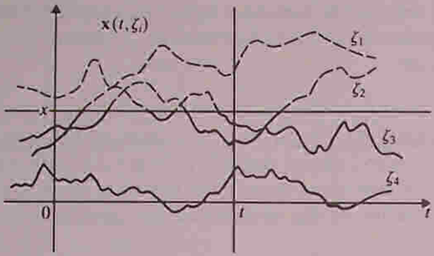

FIGURE 10-2

The *second-order distribution* of the process $\mathbf{x}(t)$ is the joint distribution

$$F(x_1, x_2; t_1, t_2) = P\{\mathbf{x}(t_1) \le x_1, \mathbf{x}(t_2) \le x_2\} \tag{10-5}$$

of the RVs $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$. The corresponding density equals

$$f(x_1, x_2; t_1, t_2) = \frac{\partial^2 F(x_1, x_2; t_1, t_2)}{\partial x_1 \, \partial x_2} \tag{10-6}$$

We note that (consistency conditions)

$$F(x_1; t_1) = F(x_1, \infty; t_1, t_2) \qquad f(x_1, t_1) = \int_{-\infty}^{\infty} f(x_1, x_2; t_1, t_2)\, dx_2$$

as in (6-9) and (6-10).

The *nth-order distribution* of $\mathbf{x}(t)$ is the joint distribution $F(x_1, \dots, x_n; t_1, \dots, t_n)$ of the RVs $\mathbf{x}(t_1), \dots, \mathbf{x}(t_n)$.

SECOND-ORDER PROPERTIES. For the determination of the statistical properties of a stochastic process, knowledge of the function $F(x_1, \dots, x_n; t_1, \dots, t_n)$ is required for every x_i, t_i, and n. However, for many applications, only certain averages are used, in particular, the expected value of $\mathbf{x}(t)$ and of $\mathbf{x}^2(t)$. These quantities can be expressed in terms of the second-order properties of $\mathbf{x}(t)$ defined as follows:

Mean The mean $\eta(t)$ of $\mathbf{x}(t)$ is the expected value of the RV $\mathbf{x}(t)$:

$$\eta(t) = E\{\mathbf{x}(t)\} = \int_{-\infty}^{\infty} x f(x, t)\, dx \tag{10-7}$$

Autocorrelation The autocorrelation $R(t_1, t_2)$ of $\mathbf{x}(t)$ is the expected value of the product $\mathbf{x}(t_1)\mathbf{x}(t_2)$:

$$R(t_1, t_2) = E\{\mathbf{x}(t_1)\mathbf{x}(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; t_1, t_2)\, dx_1\, dx_2 \tag{10-8}$$

The value of $R(t_1, t_2)$ on the diagonal $t_1 = t_2 = t$ is the *average power* of $\mathbf{x}(t)$:

$$E\{\mathbf{x}^2(t)\} = R(t, t)$$

The *autocovariance* $C(t_1, t_2)$ of $\mathbf{x}(t)$ is the covariance of the RVs $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$:

$$C(t_1, t_2) = R(t_1, t_2) - \eta(t_1)\eta(t_2) \tag{10-9}$$

and its value $C(t, t)$ on the diagonal $t_1 = t_2 = t$ equals the variance of $\mathbf{x}(t)$.

Note The following is an explanation of the reason for introducing the function $R(t_1, t_2)$ even in problems dealing only with average power: Suppose that $\mathbf{x}(t)$ is the input to a linear system and $\mathbf{y}(t)$ is the resulting output. In Sec. 10-2 we show that the mean of $\mathbf{y}(t)$ can be expressed in terms of the mean of $\mathbf{x}(t)$. However, the average power of $\mathbf{y}(t)$ cannot be found if only $E\{\mathbf{x}^2(t)\}$ is given. For the determination of $E\{\mathbf{y}^2(t)\}$, knowledge of the function $R(t_1, t_2)$ is required, not just on the diagonal $t_1 = t_2$, but for every t_1 and t_2. The following identity is a simple illustration

$$E\{[\mathbf{x}(t_1) + \mathbf{x}(t_2)]^2\} = R(t_1, t_1) + 2R(t_1, t_2) + R(t_2, t_2)$$

This follows from (10-8) if we expand the square and use the linearity of expected values.

Example 10-1. An extreme example of a stochastic process is a deterministic signal $\mathbf{x}(t) = f(t)$. In this case,

$$\eta(t) = E\{f(t)\} = f(t) \qquad R(t_1, t_2) = E\{f(t_1)f(t_2)\} = f(t_1)f(t_2)$$

Example 10-2. Suppose that $\mathbf{x}(t)$ is a process with

$$\eta(t) = 3 \qquad R(t_1, t_2) = 9 + 4e^{-0.2|t_1 - t_2|}$$

We shall determine the mean, the variance, and the covariance of the RVs $\mathbf{z} = \mathbf{x}(5)$ and $\mathbf{w} = \mathbf{x}(8)$.

Clearly, $E\{\mathbf{z}\} = \eta(5) = 3$ and $E\{\mathbf{w}\} = \eta(8) = 3$. Furthermore,

$$E\{\mathbf{z}^2\} = R(5, 5) = 13 \qquad E\{\mathbf{w}^2\} = R(8, 8) = 13$$

$$E\{\mathbf{zw}\} = R(5, 8) = 9 + 4e^{-0.6} = 11.195$$

Thus $\mathbf{z}$ and $\mathbf{w}$ have the same variance $\sigma^2 = 4$ and their covariance equals $C(5, 8) = 4e^{-0.6} = 2.195$.

Example 10-3. The integral

$$\mathbf{s} = \int_a^b \mathbf{x}(t)\, dt$$

of a stochastic process $\mathbf{x}(t)$ is an RV $\mathbf{s}$ and its value $\mathbf{s}(\zeta)$ for a specific outcome ζ is the area under the curve $\mathbf{x}(t, \zeta)$ in the interval (a, b) (see also App. 10A). Interpreting the above as a Riemann integral, we conclude from the linearity of expected values that

$$\eta_s = E\{\mathbf{s}\} = \int_a^b E\{\mathbf{x}(t)\}\, dt = \int_a^b \eta(t)\, dt \tag{10-10}$$

Similarly, since

$$\mathbf{s}^2 = \int_a^b \int_a^b \mathbf{x}(t_1)\mathbf{x}(t_2)\, dt_1\, dt_2$$

we conclude, using again the linearity of expected values, that

$$E\{\mathbf{s}^2\} = \int_a^b \int_a^b E\{\mathbf{x}(t_1)\mathbf{x}(t_2)\}\,dt_1\,dt_2 = \int_a^b \int_a^b R(t_1, t_2)\,dt_1\,dt_2 \qquad (10\text{-}11)$$

Example 10-4. We shall determine the autocorrelation $R(t_1, t_2)$ of the process

$$\mathbf{x}(t) = \mathbf{r}\cos(\omega t + \boldsymbol{\varphi})$$

where we assume that the RVs $\mathbf{r}$ and $\boldsymbol{\varphi}$ are independent and $\boldsymbol{\varphi}$ is uniform in the interval $(-\pi, \pi)$.

Using simple trigonometric identities, we find

$$E\{\mathbf{x}(t_1)\mathbf{x}(t_2)\} = \tfrac{1}{2}E\{\mathbf{r}^2\}E\{\cos\omega(t_1 - t_2) + \cos(\omega t_1 + \omega t_2 + 2\boldsymbol{\varphi})\}$$

and since

$$E\{\cos(\omega t_1 + \omega t_2 + 2\boldsymbol{\varphi})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t_1 + \omega t_2 + 2\varphi)\,d\varphi = 0$$

we conclude that

$$R(t_1, t_2) = \tfrac{1}{2}E\{\mathbf{r}^2\}\cos\omega(t_1 - t_2) \qquad (10\text{-}12)$$

Example 10-5 Poisson process. In Sec. 3-4 we introduced the concept of Poisson points and we showed that these points are specified by the following properties:

P_1: The number $\mathbf{n}(t_1, t_2)$ of the points $\mathbf{t}_i$ in an interval (t_1, t_2) of length $t = t_2 - t_1$ is a Poisson RV with parameter λt:

$$P\{\mathbf{n}(t_1, t_2) = k\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!} \qquad (10\text{-}13)$$

P_2: If the intervals (t_1, t_2) and (t_3, t_4) are nonoverlapping, then the RVs $\mathbf{n}(t_1, t_2)$ and $\mathbf{n}(t_3, t_4)$ are independent.

Using the points $\mathbf{t}_i$, we form the stochastic process

$$\mathbf{x}(t) = \mathbf{n}(0, t)$$

shown in Fig. 10-3*a*. This is a discrete-state process consisting of a family of increasing staircase functions with discontinuities at the points $\mathbf{t}_i$.

For a specific t, $\mathbf{x}(t)$ is a Poisson RV with parameter λt; hence

$$E\{\mathbf{x}(t)\} = \eta(t) = \lambda t$$

We shall show that its autocorrelation equals

$$R(t_1, t_2) = \begin{cases} \lambda t_2 + \lambda^2 t_1 t_2 & t_1 \ge t_2 \\ \lambda t_1 + \lambda^2 t_1 t_2 & t_1 \le t_2 \end{cases} \qquad (10\text{-}14)$$

or equivalently that

$$C(t_1, t_2) = \lambda\min(t_1, t_2) = \lambda t_1 U(t_2 - t_1) + \lambda t_2 U(t_1 - t_2)$$

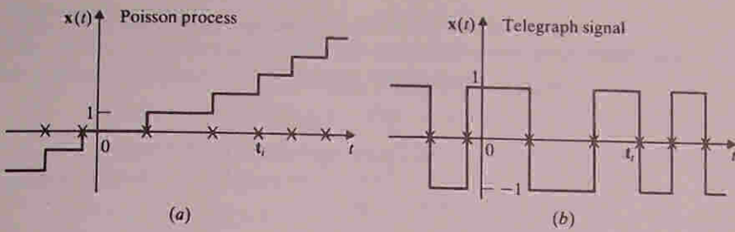

FIGURE 10-3

Proof. The above is true for $t_1 = t_2$ because [see (5-36)]

$$E\{\mathbf{x}^2(t)\} = \lambda t + \lambda^2 t^2 \tag{10-15}$$

Since $R(t_1, t_2) = R(t_2, t_1)$, it suffices to prove (10-14) for $t_1 < t_2$. The RVs $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2) - \mathbf{x}(t_1)$ are independent because the intervals $(0, t_1)$ and (t_1, t_2) are nonoverlapping. Furthermore, they are Poisson distributed with parameters λt_1 and $\lambda(t_2 - t_1)$ respectively. Hence

$$E\{\mathbf{x}(t_1)[\mathbf{x}(t_2) - \mathbf{x}(t_1)]\} = E\{\mathbf{x}(t_1)\}E\{\mathbf{x}(t_2) - \mathbf{x}(t_1)\} = \lambda t_1 \lambda(t_2 - t_1)$$

Using the identity

$$\mathbf{x}(t_1)\mathbf{x}(t_2) = \mathbf{x}(t_1)[\mathbf{x}(t_1) + \mathbf{x}(t_2) - \mathbf{x}(t_1)]$$

we conclude from the above and (10-15) that

$$R(t_1, t_2) = \lambda t_1 + \lambda^2 t_1^2 + \lambda t_1 \lambda(t_2 - t_1)$$

and (10-14) results.

Nonuniform case If the points $\mathbf{t}_i$ have a nonuniform density $\lambda(t)$ as in (3-54), then the preceding results still hold provided that the product $\lambda(t_2 - t_1)$ is replaced by the integral of $\lambda(t)$ from t_1 to t_2.

Thus

$$E\{\mathbf{x}(t)\} = \int_0^t \lambda(\alpha)\, d\alpha \tag{10-16}$$

and

$$R(t_1, t_2) = \int_0^{t_1} \lambda(t)\, dt \left[1 + \int_0^{t_2} \lambda(t)\, dt\right] \qquad t_1 \le t_2 \tag{10-17}$$

Example 10-6 Telegraph signal. Using the Poisson points $\mathbf{t}_i$, we form a process $\mathbf{x}(t)$ such that $\mathbf{x}(t) = 1$ if the number of points in the interval $(0, t)$ is even, and $\mathbf{x}(t) = -1$ if this number is odd (Fig. 10-3b).

Denoting by $p(k)$ the probability that the number of points in the interval $(0,t)$ equals k, we conclude that [see (10-13)]

$$P\{\mathbf{x}(t) = 1\} = p(0) + p(2) + \cdots$$
$$= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \cdots\right] = e^{-\lambda t}\cosh \lambda t$$
$$P\{\mathbf{x}(t) = -1\} = p(1) + p(3) + \cdots$$
$$= e^{-\lambda t}\left[\lambda t + \frac{(\lambda t)^3}{3!} + \cdots\right] = e^{-\lambda t}\sinh \lambda t$$

Hence

$$E\{\mathbf{x}(t)\} = e^{-\lambda t}(\cosh \lambda t - \sinh \lambda t) = e^{-2\lambda t} \tag{10-18}$$

To determine $R(t_1, t_2)$, we note that, if $\mathbf{x}(t_1) = 1$, then $\mathbf{x}(t_2) = 1$ if the number of points in the interval (t_1, t_2) is even. Hence

$$P\{\mathbf{x}(t_2) = 1 | \mathbf{x}(t_1) = 1\} = e^{-\lambda t}\cosh \lambda t \qquad t = |t_2 - t_1|$$

Multiplying by $P\{\mathbf{x}(t_1) = 1\}$, we obtain

$$P\{\mathbf{x}(t_1) = 1, \mathbf{x}(t_2) = 1\} = e^{-\lambda t}\cosh \lambda t e^{-\lambda t_2}\cosh \lambda t_2$$

Similarly,

$$P\{\mathbf{x}(t_1) = -1, \mathbf{x}(t_2) = -1\} = e^{-\lambda t}\cosh \lambda t e^{-\lambda t_2}\sinh \lambda t_2$$
$$P\{\mathbf{x}(t_1) = 1, \mathbf{x}(t_2) = -1\} = e^{-\lambda t}\sinh \lambda t e^{-\lambda t_2}\sinh \lambda t_2$$
$$P\{\mathbf{x}(t_1) = -1, \mathbf{x}(t_2) = 1\} = e^{-\lambda t}\sinh \lambda t e^{-\lambda t_2}\cosh \lambda t_2$$

Since the product $\mathbf{x}(t_1)\mathbf{x}(t_2)$ equals 1 or -1, we conclude omitting details that

$$R(t_1, t_2) = e^{-2\lambda|t_1 - t_2|} \tag{10-19}$$

The above process is called *semirandom* telegraph signal because its value $\mathbf{x}(0) = 1$ at $t = 0$ is not random. To remove this certainty, we form the product

$$\mathbf{y}(t) = \mathbf{a}\mathbf{x}(t)$$

where $\mathbf{a}$ is an RV taking the values $+1$ and -1 with equal probability and is independent of $\mathbf{x}(t)$. The process $\mathbf{y}(t)$ so formed is called *random* telegraph signal. Since $E\{\mathbf{a}\} = 0$ and $E\{\mathbf{a}^2\} = 1$, the mean of $\mathbf{y}(t)$ equals $E\{\mathbf{a}\}E\{\mathbf{x}(t)\} = 0$ and its autocorrelation is given by

$$E\{\mathbf{y}(t_1)\mathbf{y}(t_2)\} = E\{\mathbf{a}^2\}E\{\mathbf{x}(t_1)\mathbf{x}(t_2)\} = e^{-2\lambda|t_1 - t_2|}$$

We note that as $t \to \infty$ the processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ have asymptotically equal statistics.

General Properties

The statistical properties of a real stochastic process $\mathbf{x}(t)$ are completely determined† in terms of its nth-order distribution

$$F(x_1, \ldots, x_n; t_1, \ldots, t_n) = P\{\mathbf{x}(t_1) \le x_1, \ldots, \mathbf{x}(t_n) \le x_n\} \qquad (10\text{-}20)$$

The joint statistics of two real processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are determined in terms of the joint distribution of the RVs

$$\mathbf{x}(t_1), \ldots, \mathbf{x}(t_n), \mathbf{y}(t'_1), \ldots, \mathbf{y}(t'_m)$$

The *complex process* $\mathbf{z}(t) = \mathbf{x}(t) + j\mathbf{y}(t)$ is specified in terms of the joint statistics of the real processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$.

A *vector process* (n-dimensional process) is a family of n stochastic processes.

Correlation and covariance. The autocorrelation of a process $\mathbf{x}(t)$, real or complex, is by definition the mean of the product $\mathbf{x}(t_1)\mathbf{x}^*(t_2)$. This function, will be denoted by $R(t_1, t_2)$ or $R_x(t_1, t_2)$ or $R_{xx}(t_1, t_2)$. Thus

$$R_{xx}(t_1, t_2) = E\{\mathbf{x}(t_1)\mathbf{x}^*(t_2)\} \qquad (10\text{-}21)$$

where the conjugate term is associated with the second variable in $R_{xx}(t_1, t_2)$. From this it follows that

$$R(t_2, t_1) = E\{\mathbf{x}(t_2)\mathbf{x}^*(t_1)\} = R^*(t_1, t_2) \qquad (10\text{-}22)$$

We note, further, that

$$R(t, t) = E\{|\mathbf{x}(t)|^2\} \ge 0 \qquad (10\text{-}23)$$

The last two equations are special cases of the following: The autocorrelation $R(t_1, t_2)$ of a stochastic process $\mathbf{x}(t)$ is a *positive definite* (p.d.) function, that is, for any a_i and a_j:

$$\sum_{i,j} a_i a_j^* R(t_i, t_j) \ge 0 \qquad (10\text{-}24)$$

This is a consequence of the identity

$$0 \le E\left\{\left|\sum_i a_i \mathbf{x}(t_i)\right|^2\right\} = \sum_{i,j} a_i a_j^* E\{\mathbf{x}(t_i)\mathbf{x}^*(t_j)\}$$

We show later that the converse is also true: Given a p.d. function $R(t_1, t_2)$, we can find a process $\mathbf{x}(t)$ with autocorrelation $R(t_1, t_2)$.

†There are processes (nonseparable) for which this is not true. However, such processes are mainly of mathematical interest.

Example 10-7. (*a*) If $\mathbf{x}(t) = \mathbf{a}e^{j\omega t}$ then

$$R(t_1, t_2) = E\{\mathbf{a}e^{j\omega t_1}\mathbf{a}^*e^{-j\omega t_2}\} = E\{|\mathbf{a}|^2\}e^{j\omega(t_1-t_2)}$$

(*b*) Suppose that the RVs $\mathbf{a}_i$ are uncorrelated with zero mean and variance σ_i^2. If

$$\mathbf{x}(t) = \sum_i \mathbf{a}_i e^{j\omega_i t}$$

then (10-21) yields

$$R(t_1, t_2) = \sum_i \sigma_i^2 e^{j\omega_i(t_1-t_2)}$$

The *autocovariance* $C(t_1, t_2)$ of a process $\mathbf{x}(t)$ is the covariance of the RVs $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$:

$$C(t_1, t_2) = R(t_1, t_2) - \eta(t_1)\eta^*(t_2) \tag{10-25}$$

In the above, $\eta(t) = E\{\mathbf{x}(t)\}$ is the *mean* of $\mathbf{x}(t)$.

The ratio

$$r(t_1, t_2) = \frac{C(t_1, t_2)}{\sqrt{C(t_1, t_1)C(t_2, t_2)}} \tag{10-26}$$

is the *correlation coefficient*† of the process $\mathbf{x}(t)$.

Note The autocovariance $C(t_1, t_2)$ of a process $\mathbf{x}(t)$ is the autocorrelation of the *centered process*

$$\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \eta(t)$$

Hence it is p.d.

The correlation coefficient $r(t_1, t_2)$ of $\mathbf{x}(t)$ is the autocovariance of the *normalized process* $\mathbf{x}(t)/\sqrt{C(t,t)}$; hence it is also p.d. Furthermore [see (7-9)]

$$|r(t_1, t_2)| \le 1 \qquad r(t, t) = 1 \tag{10-27}$$

Example 10-8. If

$$\mathbf{s} = \int_a^b \mathbf{x}(t)\,dt \qquad \text{then} \quad \mathbf{s} - \eta_s = \int_a^b \tilde{\mathbf{x}}(t)\,dt$$

where $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \eta_x(t)$. Using (10-11), we conclude from the above note that

$$\sigma_s^2 = E\{|\mathbf{s} - \eta_s|^2\} = \int_a^b \int_a^b C_x(t_1, t_2)\,dt_1\,dt_2 \tag{10-28}$$

The *cross-correlation* of two processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ is the function

$$R_{xy}(t_1, t_2) = E\{\mathbf{x}(t_1)\mathbf{y}^*(t_2)\} = R_{yx}^*(t_2, t_1) \tag{10-29}$$

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†In optics, $C(t_1, t_2)$ is called the *coherence function* and $r(t_1, t_2)$ is called the *complex degree of coherence* (see Papoulis, 1968).

Similarly,

$$C_{xy}(t_1, t_2) = R_{xy}(t_1, t_2) - \eta_x(t_1)\eta_y^*(t_2) \tag{10-30}$$

is their *cross-covariance.*

Two processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are called (mutually) *orthogonal* if

$$R_{xy}(t_1, t_2) = 0 \qquad \text{for every} \quad t_1 \text{ and } t_2 \tag{10-31}$$

They are called *uncorrelated* if

$$C_{xy}(t_1, t_2) = 0 \qquad \text{for every} \quad t_1 \text{ and } t_2 \tag{10-32}$$

a-dependent processes In general, the values $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ of a stochastic process $\mathbf{x}(t)$ are statistically dependent for any t_1 and t_2. However, in most cases this dependence decreases as $|t_1 - t_2| \to \infty$. This leads to the following concept: A stochastic process $\mathbf{x}(t)$ is called *a-dependent* if all its values $\mathbf{x}(t)$ for $t < t_o$ and for $t > t_o + a$ are mutually *independent*. From this it follows that

$$C(t_1, t_2) = 0 \qquad \text{for} \quad |t_1 - t_2| > a \tag{10-33}$$

A process $\mathbf{x}(t)$ is called *correlation a-dependent* if its autocorrelation satisfies (10-33). Clearly, if $\mathbf{x}(t)$ is correlation a-dependent, then any linear combination of its values for $t < t_o$ is uncorrelated with any linear combination of its values for $t > t_o + a$.

White noise We shall say that a process $\boldsymbol{\nu}(t)$ is white noise if its values $\boldsymbol{\nu}(t_i)$ and $\boldsymbol{\nu}(t_j)$ are uncorrelated for every t_i and $t_j \neq t_i$:

$$C(t_i, t_j) = 0 \qquad t_i \neq t_j$$

As we explain later, the autocovariance of a nontrivial white-noise process must be of the form

$$C(t_1, t_2) = q(t_1)\delta(t_1 - t_2) \qquad q(t) \geq 0 \tag{10-34}$$

If the RVs $\boldsymbol{\nu}(t_i)$ and $\boldsymbol{\nu}(t_j)$ are not only uncorrelated but also independent, then $\boldsymbol{\nu}(t)$ will be called *strictly* white noise. Unless otherwise stated, it will be assumed that the mean of a white-noise process is identically 0.

Example 10-9. Suppose that $\boldsymbol{\nu}(t)$ is white noise and

$$\mathbf{x}(t) = \int_0^t \boldsymbol{\nu}(\alpha)\, d\alpha \tag{10-35}$$

Inserting (10-34) into (10-35), we obtain

$$E\{\mathbf{x}^2(t)\} = \int_0^t \int_0^t q(t_1)\delta(t_1 - t_2)\, dt_1\, dt_2 = \int_0^t q(t_1)\, dt_1 \tag{10-36}$$

because

$$\int_0^t \delta(t_1 - t_2)\, dt_2 = 1 \qquad \text{for} \quad 0 < t_1 < t$$

Uncorrelated and independent increments If the increments $\mathbf{x}(t_2) - \mathbf{x}(t_1)$ and $\mathbf{x}(t_4) - \mathbf{x}(t_3)$ of a process $\mathbf{x}(t)$ are uncorrelated (independent) for any

$t_1 < t_2 < t_3 < t_4$, then we say that $\mathbf{x}(t)$ is a process with uncorrelated (independent) increments. The Poisson process is a process with independent increments. The integral (10-35) of white noise is a process with uncorrelated increments.

Independent processes If two processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are such that the RVs $\mathbf{x}(t_1), \ldots, \mathbf{x}(t_n)$ and $\mathbf{y}(t'_1), \ldots, \mathbf{y}(t'_n)$ are mutually independent, then these processes are called independent.

Normal processes. A process $\mathbf{x}(t)$ is called normal, if the RVs $\mathbf{x}(t_1), \ldots, \mathbf{x}(t_n)$ are jointly normal for any n and $t_1, \ldots, t_n$.

The statistics of a normal process are completely determined in terms of its mean $\eta(t)$ and autocovariance $C(t_1, t_2)$. Indeed, since

$$E\{\mathbf{x}(t)\} = \eta(t) \qquad \sigma_x^2(t) = C(t, t)$$

we conclude that the first-order density $f(x, t)$ of $\mathbf{x}(t)$ is the normal density $N[\eta(t); \sqrt{C(t, t)}\,]$.

Similarly, since the function $r(t_1, t_2)$ in (10-26) is the correlation coefficient of the RVs $\mathbf{x}(t_1)$ and $\mathbf{x}(t_1)$, the second-order density $f(x_1, x_2; t_1, t_2)$ of $\mathbf{x}(t)$ is the jointly normal density

$$N\left[\eta(t_1), \eta(t_2); \sqrt{C(t_1, t_1)}\,, \sqrt{C(t_2, t_2)}\,; r(t_1, t_2)\right]$$

The nth-order characteristic function of the process $\mathbf{x}(t)$ is given by [see (8-60)]

$$\exp\left\{ j \sum_i \eta(t_i)\omega_i - \frac{1}{2} \sum_{i,k} C(t_i, t_k)\omega_i\omega_k \right\} \tag{10-37}$$

Its inverse $f(x_1, \ldots, x_n; t_1, \ldots, t_n)$ is the nth-order density of $\mathbf{x}(t)$.

Existence theorem. Given an arbitrary function $\eta(t)$ and a p.d. function $C(t_1, t_2)$, we can construct a normal process with mean $\eta(t)$ and autocovariance $C(t_1, t_2)$. This follows if we use in (10-37) the given functions $\eta(t)$ and $C(t_1, t_2)$. The inverse of the resulting characteristic function is a density because the function $C(t_1, t_2)$ is p.d. by assumption.

Example 10-10. Suppose that $\mathbf{x}(t)$ is a normal process with

$$\eta(t) = 3 \qquad C(t_1, t_2) = 4e^{-0.2|t_1 - t_2|}$$

(a) Find the probability that $\mathbf{x}(5) \le 2$.

Clearly, $\mathbf{x}(5)$ is a normal RV with mean $\eta(5) = 3$ and variance $C(5, 5) = 4$. Hence

$$P\{\mathbf{x}(5) \le 2\} = \mathbb{G}(-1/2) = 0.309$$

(b) Find the probability that $|\mathbf{x}(8) - \mathbf{x}(5)| \le 1$.

The difference $\mathbf{s} = \mathbf{x}(8) - \mathbf{x}(5)$ is a normal RV with mean $\eta(8) - \eta(5) = 0$ and variance

$$C(8, 8) + C(5, 5) - 2C(8, 5) = 8(1 - e^{-0.6}) = 3.608$$

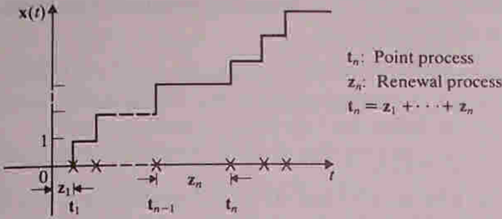

FIGURE 10-4

Hence

$$P\{|\mathbf{x}(8) - \mathbf{x}(5)| \le 1\} = 2\mathbb{G}(1/1.9) - 1 = 0.4$$

Point and renewal processes. A *point process* is a set of random points $\mathbf{t}_i$ on the time axis. To every point process we can associate a stochastic process $\mathbf{x}(t)$ equal to the number of points $\mathbf{t}_i$ in the interval $(0, t)$. An example is the Poisson process. To every point process $\mathbf{t}_i$ we can associate a sequence of RVs $\mathbf{z}_n$ such that

$$\mathbf{z}_1 = \mathbf{t}_1 \qquad \mathbf{z}_2 = \mathbf{t}_2 - \mathbf{t}_1 \cdots \mathbf{z}_n = \mathbf{t}_n - \mathbf{t}_{n-1}$$

where $\mathbf{t}_1$ is the first random point to the right of the origin. This sequence is called a *renewal process*. An example is the life history of light bulbs that are replaced as soon as they fail. In this case, $\mathbf{z}_i$ is the total time the ith bulb is in operation and $\mathbf{t}_i$ is the time of its failure.

We have thus established a correspondence between the following three concepts (Fig. 10-4): (a) a point process $\mathbf{t}_i$, (b) a discrete-state stochastic process $\mathbf{x}(t)$ increasing in unit steps at the points $\mathbf{t}_i$, (c) a renewal process consisting of the RVs $\mathbf{z}_i$ and such that $\mathbf{t}_n = \mathbf{z}_1 + \cdots + \mathbf{z}_n$. This correspondence is developed further in Sec. 16-1.

Stationary Processes

A stochastic process $\mathbf{x}(t)$ is called *strict-sense stationary* (abbreviated SSS) if its statistical properties are invariant to a shift of the origin. This means that the processes $\mathbf{x}(t)$ and $\mathbf{x}(t + c)$ have the same statistics for any c.

Two processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are called *jointly stationary* if the joint statistics of $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are the same as the joint statistics of $\mathbf{x}(t + c)$ and $\mathbf{y}(t + c)$ for any c.

A complex process $\mathbf{z}(t) = \mathbf{x}(t) + j\mathbf{y}(t)$ is stationary if the processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are jointly stationary.

From the definition it follows that the nth-order density of an SSS process must be such that

$$f(x_1, \ldots, x_n; t_1, \ldots, t_n) = f(x_1, \ldots, x_n; t_1 + c, \ldots, t_n + c) \quad (10\text{-}38)$$

for any c.

From the above it follows that $f(x; t) = f(x; t + c)$ for any c. Hence the first-order density of $\mathbf{x}(t)$ is independent of t:

$$f(x; t) = f(x) \tag{10-39}$$

Similarly, $f(x_1, x_2; t_1 + c, t_2 + c)$ is independent of c for any c. This leads to the conclusion that

$$f(x_1, x_2; t_1, t_2) = f(x_1, x_2; \tau) \qquad \tau = t_1 - t_2 \tag{10-40}$$

Thus the joint density of the RVs $\mathbf{x}(t + \tau)$ and $\mathbf{x}(t)$ is independent of t and it equals $f(x_1, x_2; \tau)$.

WIDE SENSE. A stochastic process $\mathbf{x}(t)$ is called *wide-sense stationary* (abbreviated WSS) if its mean is constant

$$E\{\mathbf{x}(t)\} = \eta \tag{10-41}$$

and its autocorrelation depends only on $\tau = t_1 - t_2$:

$$E\{\mathbf{x}(t + \tau)\mathbf{x}^*(t)\} = R(\tau) \tag{10-42}$$

Since τ is the distance from t to $t + \tau$, the function $R(\tau)$ can be written in the symmetrical form

$$R(\tau) = E\left\{\mathbf{x}\left(t + \frac{\tau}{2}\right)\mathbf{x}^*\left(t - \frac{\tau}{2}\right)\right\} \tag{10-43}$$

Note in particular that

$$E\{|\mathbf{x}(t)|^2\} = R(0)$$

Thus the average power of a stationary process is independent of t and it equals $R(0)$.

Example 10-11. Suppose that $\mathbf{x}(t)$ is a WSS process with autocorrelation

$$R(\tau) = Ae^{-\alpha|\tau|}$$

We shall determine the second moment of the RV $\mathbf{x}(8) - \mathbf{x}(5)$. Clearly,

$$E\{[\mathbf{x}(8) - \mathbf{x}(5)]^2\} = E\{\mathbf{x}^2(8)\} + E\{\mathbf{x}^2(5)\} - 2E\{\mathbf{x}(8)\mathbf{x}(5)\}$$
$$= R(0) + R(0) - 2R(3) = 2A - 2Ae^{-3\alpha}$$

Note As the above example suggests, the autocorrelation of a stationary process $\mathbf{x}(t)$ can be defined as average power. Assuming for simplicity that $\mathbf{x}(t)$ is real, we conclude from (10-42) that

$$E\{[\mathbf{x}(t + \tau) - \mathbf{x}(t)]^2\} = 2[R(0) - R(\tau)] \tag{10-44}$$

From (10-42) it follows that the autocovariance of a WSS process depends only on $\tau = t_1 - t_2$:

$$C(\tau) = R(\tau) - |\eta|^2 \tag{10-45}$$

and its correlation coefficient [see (10-26)] equals

$$r(\tau) = C(\tau)/C(0) \tag{10-46}$$

Thus $C(\tau)$ is the covariance, and $r(\tau)$ the correlation coefficient of the RVs $\mathbf{x}(t+\tau)$ and $\mathbf{x}(t)$.

Two processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are called jointly WSS if each is WSS and their cross-correlation depends only on $\tau = t_1 - t_2$:

$$R_{xy}(\tau) = E\{\mathbf{x}(t+\tau)\mathbf{y}^*(t)\} \qquad C_{xy}(\tau) = R_{xy}(\tau) - \eta_x\eta_y^* \tag{10-47}$$

If $\mathbf{x}(t)$ is WSS white noise, then [see (10-34)]

$$C(\tau) = q\delta(\tau) \tag{10-48}$$

If $\mathbf{x}(t)$ is an a-dependent process, then $C(\tau) = 0$ for $|\tau| > a$. In this case, the constant a is called the *correlation time* of $\mathbf{x}(t)$. This term is also used for arbitrary processes and it is defined as the ratio

$$\tau_c = \frac{1}{C(0)}\int_0^\infty C(\tau)\,d\tau \tag{10-49}$$

In general $C(\tau) \neq 0$ for every τ. However, for most regular processes

$$C(\tau) \xrightarrow[|\tau|\to\infty]{} 0 \qquad R(\tau) \xrightarrow[|\tau|\to\infty]{} |\eta|^2$$

Example 10-12. If $\mathbf{x}(t)$ is WSS and

$$\mathbf{s} = \int_{-T}^{T} \mathbf{x}(t)\,dt$$

then [see (10-28)]

$$\sigma_s^2 = \int_{-T}^{T}\int_{-T}^{T} C(t_1 - t_2)\,dt_1\,dt_2 = \int_{-2T}^{2T} (2T - |\tau|)C(\tau)\,d\tau \tag{10-50}$$

The last equality follows with $\tau = t_1 - t_2$ (see Fig. 10-5); the details, however, are omitted [see also (10-143)].

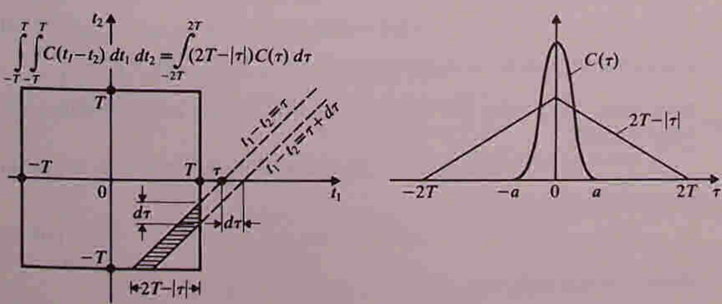

FIGURE 10-5

Special cases. (*a*) If $C(\tau) = q\delta(\tau)$, then

$$\sigma_s^2 = q\int_{-2T}^{2T}(2T - |\tau|)\delta(\tau)\,d\tau = 2Tq$$

(*b*) If the process $\mathbf{x}(t)$ is a-dependent and $a \ll T$, then (10-50) yields

$$\sigma_s^2 = \int_{-2T}^{2T}(2T - |\tau|)C(\tau)\,d\tau \simeq 2T\int_{-a}^{a}C(\tau)\,d\tau$$

This shows that, in the evaluation of the variance of $\mathbf{s}$, an a-dependent process with $a \ll T$ can be replaced by white noise as in (10-48) with

$$q = \int_{-a}^{a}C(\tau)\,d\tau$$

If a process is SSS, then it is also WSS. This follows readily from (10-39) and (10-40). The converse, however, is not in general true. As we show next, normal processes are an important exception.

Indeed, suppose that $\mathbf{x}(t)$ is a normal WSS process with mean η and autocovariance $C(\tau)$. As we see from (10-37), its nth-order characteristic function equals

$$\exp\left\{j\eta\sum_i\omega_i - \frac{1}{2}\sum_{i,k}C(t_i - t_k)\omega_i\omega_k\right\} \tag{10-51}$$

This function is invariant to a shift of the origin. And since it determines completely the statistics of $\mathbf{x}(t)$, we conclude that $\mathbf{x}(t)$ is SSS.

Example 10-13. We shall establish necessary and sufficient conditions for the stationarity of the process

$$\mathbf{x}(t) = \mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t \tag{10-52}$$

The mean of this process equals

$$E\{\mathbf{x}(t)\} = E\{\mathbf{a}\}\cos\omega t + E\{\mathbf{b}\}\sin\omega t$$

This function must be independent of t. Hence the condition

$$E\{\mathbf{a}\} = E\{\mathbf{b}\} = 0 \tag{10-53}$$

is necessary for both forms of stationarity. We shall assume that it holds.

Wide sense. The process $\mathbf{x}(t)$ is WSS iff the RVs $\mathbf{a}$ and $\mathbf{b}$ are uncorrelated with equal variance:

$$E\{\mathbf{ab}\} = 0 \qquad E\{\mathbf{a}^2\} = E\{\mathbf{b}^2\} = \sigma^2 \tag{10-54}$$

If this holds, then

$$R(\tau) = \sigma^2\cos\omega\tau \tag{10-55}$$

Proof. If $\mathbf{x}(t)$ is WSS, then

$$E\{\mathbf{x}^2(0)\} = E\{\mathbf{x}^2(\pi/2\omega)\} = R(0)$$

But $\mathbf{x}(0) = \mathbf{a}$ and $\mathbf{x}(\pi/2\omega) = \mathbf{b}$; hence $E\{\mathbf{a}^2\} = E\{\mathbf{b}^2\}$. Using the above, we obtain

$$E\{\mathbf{x}(t+\tau)\mathbf{x}(t)\} = E\{[\mathbf{a}\cos\omega(t+\tau) + \mathbf{b}\sin\omega(t+\tau)][\mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t]\}$$
$$= \sigma^2\cos\omega\tau + E\{\mathbf{ab}\}\sin\omega(2t+\tau) \qquad (10\text{-}56)$$

This is independent of t only if $E\{\mathbf{ab}\} = 0$ and (10-54) results.

Conversely, if (10-54) holds, then, as we see from (10-56), the autocorrelation of $\mathbf{x}(t)$ equals $\sigma^2\cos\omega\tau$; hence $\mathbf{x}(t)$ is WSS.

Strict sense. The process $\mathbf{x}(t)$ is SSS iff the joint density $f(a,b)$ of the RVs $\mathbf{a}$ and $\mathbf{b}$ has circular symmetry, that is, if

$$f(a,b) = f\left(\sqrt{a^2+b^2}\right) \qquad (10\text{-}57)$$

Proof. If $\mathbf{x}(t)$ is SSS, then the RVs

$$\mathbf{x}(0) = \mathbf{a} \qquad \mathbf{x}(\pi/2\omega) = \mathbf{b}$$

and

$$\mathbf{x}(t) = \mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t \qquad \mathbf{x}(t+\pi/2\omega) = \mathbf{b}\cos\omega t - \mathbf{a}\sin\omega t$$

have the same joint density for every t. Hence [see (6-70)], $f(a,b)$ must have circular symmetry.

We shall now show that, if $f(a,b)$ has circular symmetry, then $\mathbf{x}(t)$ is SSS. With τ a given number and

$$\mathbf{a}_1 = \mathbf{a}\cos\omega\tau + \mathbf{b}\sin\omega\tau \qquad \mathbf{b}_1 = \mathbf{b}\cos\omega\tau - \mathbf{a}\sin\omega\tau$$

we form the process

$$\mathbf{x}_1(t) = \mathbf{a}_1\cos\omega t + \mathbf{b}_1\sin\omega t = \mathbf{x}(t+\tau)$$

Clearly, the statistics of $\mathbf{x}(t)$ and $\mathbf{x}_1(t)$ are determined in terms of the joint densities $f(a,b)$ and $f(a_1,b_1)$ of the RVs $\mathbf{a},\mathbf{b}$ and $\mathbf{a}_1,\mathbf{b}_1$. But [see (6-67)] the RVs $\mathbf{a},\mathbf{b}$ and $\mathbf{a}_1,\mathbf{b}_1$ have the same joint density. Hence the processes $\mathbf{x}(t)$ and $\mathbf{x}(t+\tau)$ have the same statistics for every τ.

Corollary. If the process $\mathbf{x}(t)$ is SSS and the RVs $\mathbf{a}$ and $\mathbf{b}$ are independent, then they are normal.

Proof. It follows from (10-57) and (6-34).

Example 10-14. (*a*) Given an RV $\boldsymbol{\omega}$ with density $f(\omega)$ and an RV $\boldsymbol{\varphi}$ uniform in the interval $(-\pi,\pi)$ and independent of $\boldsymbol{\omega}$, we form the process

$$\mathbf{x}(t) = a\cos(\boldsymbol{\omega}t + \boldsymbol{\varphi}) \qquad (10\text{-}58)$$

We shall show that $\mathbf{x}(t)$ is WSS with zero mean and autocorrelation

$$R(\tau) = \frac{a^2}{2}E\{\cos\boldsymbol{\omega}\tau\} = \frac{a^2}{2}\operatorname{Re}\Phi_\omega(\tau) \qquad (10\text{-}59)$$

where

$$\Phi_\omega(\tau) = E\{e^{j\boldsymbol{\omega}\tau}\} = E\{\cos\boldsymbol{\omega}\tau\} + jE\{\sin\boldsymbol{\omega}\tau\} \qquad (10\text{-}60)$$

is the characteristic function of $\boldsymbol{\omega}$.

Proof. Clearly [see (7-59)]

$$E\{\cos(\omega t + \varphi)\} = E\{E\{\cos(\omega t + \varphi)|\omega\}\}$$

From the independence of ω and φ, it follows that

$$E\{\cos(\omega t + \varphi)|\omega\} = \cos\omega t\, E\{\cos\varphi\} - \sin\omega t\, E\{\sin\varphi\}$$

Hence $E\{\mathbf{x}(t)\} = 0$ because

$$E\{\cos\varphi\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\varphi\, d\varphi = 0 \qquad E\{\sin\varphi\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin\varphi\, d\varphi = 0$$

Reasoning similarly, we obtain $E\{\cos(2\omega t + \omega\tau + 2\varphi)\} = 0$. And since

$$2\cos[\omega(t+\tau)+\varphi]\cos(\omega t + \varphi) = \cos\omega\tau + \cos(2\omega t + \omega\tau + 2\varphi)$$

we conclude that

$$R(\tau) = a^2 E\{\cos[\omega(t+\tau)+\varphi]\cos(\omega t+\varphi)\} = \frac{a^2}{2}E\{\cos\omega\tau\}$$

(*b*) With ω and φ as above, the process

$$\mathbf{z}(t) = ae^{j(\omega t + \varphi)}$$

is WSS with zero mean and autocorrelation

$$E\{\mathbf{z}(t+\tau)\mathbf{z}^*(t)\} = a^2E\{e^{j\omega t}\} = a^2\Phi_\omega(\tau)$$

Centering. Given a process $\mathbf{x}(t)$ with mean $\eta(t)$ and autocovariance $C_x(t_1, t_2)$, we form difference

$$\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \eta(t) \tag{10-61}$$

This difference is called the *centered process* associated with the process $\mathbf{x}(t)$. Note that

$$E\{\tilde{\mathbf{x}}(t)\} = 0 \qquad R_{\tilde{x}}(t_1, t_2) = C_x(t_1, t_2)$$

From this it follows that if the process $\mathbf{x}(t)$ is covariance stationary, that is, if $C_x(t_1, t_2) = C_x(t_1 - t_2)$, then its centered process $\tilde{\mathbf{x}}(t)$ is WSS.

Other forms of stationarity. A process $\mathbf{x}(t)$ is *asymptotically stationary* if the statistics of the RVs $\mathbf{x}(t_1 + c), \ldots, \mathbf{x}(t_n + c)$ do not depend on c if c is large. More precisely, the function

$$f(x_1, \ldots, x_n; t_1 + c, \ldots, t_n + c)$$

tends to a limit (that does not depend on c) as $c \to \infty$. The semirandom telegraph signal is an example.

A process $\mathbf{x}(t)$ is *Nth-order stationary* if (10-38) holds not for every n, but only for $n \le N$.

A process $\mathbf{x}(t)$ is *stationary in an interval* if (10-38) holds for every t_i and $t_i + c$ in this interval.

We say that $\mathbf{x}(t)$ is a process with *stationary increments* if its increments $\mathbf{y}(t) = \mathbf{x}(t+h) - \mathbf{x}(t)$ form a stationary process for every h. The Poisson process is an example.

MEAN SQUARE PERIODICITY. A process $\mathbf{x}(t)$ is called MS periodic if

$$E\{|\mathbf{x}(t+T) - \mathbf{x}(t)|^2\} = 0 \tag{10-62}$$

for every t. From this it follows that, for a specific t,

$$\mathbf{x}(t+T) = \mathbf{x}(t) \tag{10-63}$$

with probability 1. It does not, however, follow that the set of outcomes ζ such that $\mathbf{x}(t+T, \zeta) = \mathbf{x}(t, \zeta)$ for all t has probability 1.

As we see from (10-63) the mean of an MS periodic process is periodic. We shall examine the properties of $R(t_1, t_2)$.

THEOREM. A process $\mathbf{x}(t)$ is MS periodic iff its autocorrelation is *doubly periodic*, that is, if

$$R(t_1 + mT, t_2 + nT) = R(t_1, t_2) \tag{10-64}$$

for every integer m and n.

Proof. As we know [see (7-12)]

$$E^2\{\mathbf{zw}\} \le E\{\mathbf{z}^2\}E\{\mathbf{w}^2\}$$

With $\mathbf{z} = \mathbf{x}(t_1)$ and $\mathbf{w} = \mathbf{x}(t_2 + T) - \mathbf{x}(t_2)$ the above yields

$$E^2\{\mathbf{x}(t_1)[\mathbf{x}(t_2+T) - \mathbf{x}(t_2)]\} \le E\{\mathbf{x}^2(t_1)\}E\{[\mathbf{x}(t_2+T) - \mathbf{x}(t_2)]^2\}$$

If $\mathbf{x}(t)$ is MS periodic, then the last term above is 0. Equating the left side to 0, we obtain

$$R(t_1, t_2 + T) - R(t_1, t_2) = 0$$

Repeated application of this yields (10-64).

Conversely, if (10-64) is true, then

$$R(t+T, t+T) = R(t+T, t) = R(t,t)$$

Hence

$$E\{[\mathbf{x}(t+T) - \mathbf{x}(t)]^2\} = R(t+T, t+T) + R(t,t) - 2R(t+T, t) = 0$$

therefore $\mathbf{x}(t)$ is MS periodic.

10-2 SYSTEMS WITH STOCHASTIC INPUTS

Given a stochastic process $\mathbf{x}(t)$, we assign according to some rule to each of its samples $\mathbf{x}(t, \zeta_i)$ a function $\mathbf{y}(t, \zeta_i)$. We have thus created another process

$$\mathbf{y}(t) = T[\mathbf{x}(t)]$$

whose samples are the functions $\mathbf{y}(t, \zeta_i)$. The process $\mathbf{y}(t)$ so formed can be considered as the output of a *system* (transformation) with input the process $\mathbf{x}(t)$. The system is completely specified in terms of the operator T, that is, the rule of correspondence between the samples of the input $\mathbf{x}(t)$ and the output $\mathbf{y}(t)$.

The system is *deterministic* if its operates only on the variable t treating ζ as a parameter. This means that if two samples $\mathbf{x}(t, \zeta_1)$ and $\mathbf{x}(t, \zeta_2)$ of the input are identical in t, then the corresponding samples $\mathbf{y}(t, \zeta_1)$ and $\mathbf{y}(t, \zeta_2)$ of the output are also identical in t. The system is called *stochastic* if T operates on both variables t and ζ. This means that there exist two outcomes ζ_1 and ζ_2 such that $\mathbf{x}(t, \zeta_1) = \mathbf{x}(t, \zeta_2)$ identically in t but $\mathbf{y}(t, \zeta_1) \neq \mathbf{y}(t, \zeta_2)$. These classifications are based on the terminal properties of the system. If the system is specified in terms of physical elements or by an equation, then it is deterministic (stochastic) if the elements or the coefficients of the defining equations are deterministic (stochastic). Throughout this book we shall consider only deterministic systems.

In principle, the statistics of the output of a system can be expressed in terms of the statistics of the input. However, in general this is a complicated problem. We consider next two important special cases.

Memoryless Systems

A system is called memoryless if its output is given by

$$\mathbf{y}(t) = g[\mathbf{x}(t)]$$

where $g(x)$ is a function of x. Thus, at a given time $t = t_1$, the output $\mathbf{y}(t_1)$ depends only on $\mathbf{x}(t_1)$ and not on any other past or future values of $\mathbf{x}(t)$.

From the above it follows that the first-order density $f_y(y; t)$ of $\mathbf{y}(t)$ can be expressed in terms of the corresponding density $f_x(x; t)$ of $\mathbf{x}(t)$ as in Sec. 5-2. Furthermore,

$$E\{\mathbf{y}(t)\} = \int_{-\infty}^{\infty} g(x) f_x(x; t)\, dx$$

Similarly, since $\mathbf{y}(t_1) = g[\mathbf{x}(t_1)]$ and $\mathbf{y}(t_2) = g[\mathbf{x}(t_2)]$, the second-order density $f_y(y_1, y_2; t_1, t_2)$ of $\mathbf{y}(t)$ can be determined in terms of the corresponding density $f_x(x_1, x_2; t_1, t_2)$ of $\mathbf{x}(t)$ as in Sec. 6-3. Furthermore,

$$E\{\mathbf{y}(t_1)\mathbf{y}(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1) g(x_2) f_x(x_1, x_2; t_1, t_2)\, dx_1\, dx_2$$

The nth-order density $f_y(y_1, \ldots, y_n; t_1, \ldots, t_n)$ of $\mathbf{y}(t)$ can be determined from the corresponding density of $\mathbf{x}(t)$ as in (8-8) where the underlying transformation is the system

$$\mathbf{y}(t_1) = g[\mathbf{x}(t_1)], \ldots, \mathbf{y}(t_n) = g[\mathbf{x}(t_n)] \qquad (10\text{-}65)$$

STATIONARITY. Suppose that the input to a memoryless system is an SSS process $\mathbf{x}(t)$. We shall show that the resulting output $\mathbf{y}(t)$ is also SSS.

Proof. To determine the nth-order density of $\mathbf{y}(t)$, we solve the system

$$g(x_1) = y_1, \ldots, g(x_n) = y_n \qquad (10\text{-}66)$$

If this system has a unique solution, then [see (8-8)]

$$f_y(y_1,\ldots,y_n;t_1,\ldots,t_n) = \frac{f_x(x_1,\ldots,x_n;t_1,\ldots,t_n)}{|g'(x_1)\cdots g'(x_n)|} \tag{10-67}$$

From the stationarity of $\mathbf{x}(t)$ it follows that the numerator in (10-67) is invariant to a shift of the time origin. And since the denominator does not depend on t, we conclude that the left side does not change if t_i is replaced by $t_i + c$. Hence $\mathbf{y}(t)$ is SSS. We can similarly show that this is true even if (10-66) has more than one solution.

Notes 1. If $\mathbf{x}(t)$ is stationary of order N, then $\mathbf{y}(t)$ is stationary of order N.
2. If $\mathbf{x}(t)$ is stationary in an interval, then $\mathbf{y}(t)$ is stationary in the same interval.
3. If $\mathbf{x}(t)$ is WSS stationary, then $\mathbf{y}(t)$ might not be stationary in any sense.

Square-law detector. A square-law detector is a memoryless system whose output equals

$$\mathbf{y}(t) = \mathbf{x}^2(t)$$

We shall determine its first- and second-order densities. If $y > 0$, then the system $y = x^2$ has the two solutions $\pm\sqrt{y}$. Furthermore, $y'(x) = \pm 2\sqrt{y}$; hence

$$f_y(y;t) = \frac{1}{2\sqrt{y}}\left[f_x(\sqrt{y};t) + f_x(-\sqrt{y};t)\right]$$

If $y_1 > 0$ and $y_2 > 0$, then the system

$$y_1 = x_1^2 \qquad y_2 = x_2^2$$

has the four solutions $(\pm\sqrt{y_1}, \pm\sqrt{y_2})$. Furthermore, its jacobian equals $\pm 4\sqrt{y_1 y_2}$; hence

$$f_y(y_1, y_2; t_1, t_2) = \frac{1}{4\sqrt{y_1 y_2}} \sum f_x\left(\pm\sqrt{y_1}, \pm\sqrt{y_2}; t_1, t_2\right)$$

where the summation has four terms.

Note that, if $\mathbf{x}(t)$ is SSS, then $f_x(x;t) = f_x(x)$ is independent of t and $f_x(x_1, x_2; t_1, t_2) = f_x(x_1, x_2; \tau)$ depends only on $\tau = t_1 - t_2$. Hence $f_y(y)$ is independent of t and $f_y(y_1, y_2; \tau)$ depends only on $\tau = t_1 - t_2$.

Example 10-15. Suppose that $\mathbf{x}(t)$ is a normal stationary process with zero mean and autocorrelation $R_x(\tau)$. In this case, $f_x(x)$ is normal with variance $R_x(0)$.

If $\mathbf{y}(t) = \mathbf{x}^2(t)$ (Fig. 10-6), then $E\{\mathbf{y}(t)\} = R_x(0)$ and [see (5-8)]

$$f_y(y) = \frac{1}{\sqrt{2\pi R_x(0) y}} e^{-y/2R_x(0)} U(y)$$

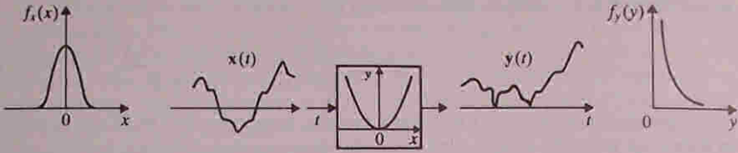

FIGURE 10-6

We shall show that

$$R_y(\tau) = R_x^2(0) + 2R_x^2(\tau) \tag{10-68}$$

Proof. The RVs $\mathbf{x}(t+\tau)$ and $\mathbf{x}(t)$ are jointly normal with zero mean. Hence [see (7-36)]

$$E\{\mathbf{x}^2(t+\tau)\mathbf{x}^2(t)\} = E\{\mathbf{x}^2(t+\tau)\}E\{\mathbf{x}^2(t)\} + 2E^2\{\mathbf{x}(t+\tau)\mathbf{x}(t)\}$$

and (10-68) results.

Note in particular that

$$E\{\mathbf{y}^2(t)\} = R_y(0) = 3R_x^2(0) \qquad \sigma_y^2 = 2R_x^2(0)$$

Hard limiter. Consider a memoryless system with

$$g(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \tag{10-69}$$

(Fig. 10-7). Its output $\mathbf{y}(t)$ takes the values ± 1 and

$$P\{\mathbf{y}(t) = 1\} = P\{\mathbf{x}(t) > 0\} = 1 - F_x(0)$$

$$P\{\mathbf{y}(t) = -1\} = P\{\mathbf{x}(t) < 0\} = F_x(0)$$

Hence

$$E\{\mathbf{y}(t)\} = 1 \times P\{\mathbf{y}(t) = 1\} - 1 \times P\{\mathbf{y}(t) = -1\} = 1 - 2F_x(0)$$

The product $\mathbf{y}(t+\tau)\mathbf{y}(t)$ equals 1 if $\mathbf{x}(t+\tau)\mathbf{x}(t) > 0$ and its equals -1 otherwise. Hence

$$R_y(\tau) = P\{\mathbf{x}(t+\tau)\mathbf{x}(t) > 0\} - P\{\mathbf{x}(t+\tau)\mathbf{x}(t) < 0\} \tag{10-70}$$

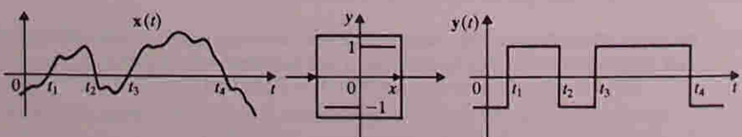

FIGURE 10-7

Thus, in the probability plane of the RVs $\mathbf{x}(t+\tau)$ and $\mathbf{x}(t)$, $R_y(\tau)$ equals the masses in the first and third quadrants minus the masses in the second and fourth quadrants.

Example 10-16. We shall show that if $\mathbf{x}(t)$ is a normal stationary process, then the autocorrelation of the output of a hard limiter equals

$$R_y(\tau) = \frac{2}{\pi} \arcsin \frac{R_x(\tau)}{R_x(0)} \tag{10-71}$$

This result is known as the *arcsine law*.†

PROOF. The RVs $\mathbf{x}(t+\tau)$ and $\mathbf{x}(t)$ are jointly normal with zero mean, variance $R_x(0)$, and correlation coefficient $R_x(\tau)/R_x(0)$. Hence [see (6-47)].

$$P\{\mathbf{x}(t+\tau)\mathbf{x}(t) > 0\} = \frac{1}{2} + \frac{\alpha}{\pi} \qquad \sin\alpha = \frac{R_x(\tau)}{R_x(0)}$$
$$P\{\mathbf{x}(t+\tau)\mathbf{x}(t) < 0\} = \frac{1}{2} - \frac{\alpha}{\pi}$$

Inserting in (10-70), we obtain

$$R_y(\tau) = \frac{1}{2} + \frac{\alpha}{\pi} - \left(\frac{1}{2} - \frac{\alpha}{\pi}\right) = \frac{2\alpha}{\pi}$$

and (10-71) follows.

Example 10-17 Bussgang's theorem. Using Price's theorem, we shall show that if the input to a memoryless system $y = g(x)$ is a zero-mean normal process $\mathbf{x}(t)$, the cross-correlation of $\mathbf{x}(t)$ with the resulting output $\mathbf{y}(t) = g[\mathbf{x}(t)]$ is proportional to $R_{xx}(\tau)$:

$$R_{xy}(\tau) = KR_{xx}(\tau) \qquad \text{where} \quad K = E\{g'[\mathbf{x}(t)]\} \tag{10-72}$$

Proof. For a specific τ, the RVs $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{z} = \mathbf{x}(t+\tau)$ are jointly normal with zero mean and covariance $\mu = E\{\mathbf{xz}\} = R_{xx}(\tau)$. With

$$I = E\{\mathbf{z}g(\mathbf{x})\} = E\{\mathbf{x}(t+\tau)\mathbf{y}(t)\} = R_{xy}(\tau)$$

it follows from (7-37) that

$$\frac{\partial I}{\partial \mu} = E\left\{\frac{\partial^2[\mathbf{z}g(\mathbf{x})]}{\partial x\,\partial z}\right\} = E\{g'[\mathbf{x}(t)]\} = K \tag{10-73}$$

If $\mu = 0$, the RVs $\mathbf{x}(t+\tau)$ and $\mathbf{x}(t)$ are independent; hence $I = 0$. Integrating (10-73) with respect to μ, we obtain $I = K\mu$ and (10-72) results.

†J. L. Lawson and G. E. Uhlenbeck: *Threshold Signals*, McGraw-Hill Book Company, New York, 1950.

***Special cases.*†** (*a*) (Hard limiter) Suppose that $g(x) = \operatorname{sgn} x$ as in (10-69). In this case, $g'(x) = 2\delta(x)$; hence

$$K = E\{2\delta(\mathbf{x})\} = 2\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = 2f(0)$$

where

$$f(x) = \frac{1}{\sqrt{2\pi R_{xx}(0)}} \exp\left\{-\frac{x^2}{2R_{xx}(0)}\right\}$$

is the first-order density of $\mathbf{x}(t)$. Inserting into (10-72), we obtain

$$R_{xy}(\tau) = R_{xx}(\tau)\sqrt{\frac{2}{\pi R_{xx}(0)}} \qquad \mathbf{y}(t) = \operatorname{sgn} \mathbf{x}(t) \tag{10-74}$$

(*b*) (Limiter) Suppose next that $\mathbf{y}(t)$ is the output of a limiter

$$g(x) = \begin{cases} x & |x| < c \\ c & |x| > c \end{cases} \qquad g'(x) = \begin{cases} 1 & |x| < c \\ 0 & |x| > 0 \end{cases}$$

In this case,

$$K = \int_{-c}^{c} f(x)\, dx = 2\mathbb{G}\left(\frac{c}{\sqrt{R_{xx}(0)}}\right) - 1 \tag{10-75}$$

Linear Systems

The notation

$$\mathbf{y}(t) = L[\mathbf{x}(t)] \tag{10-76}$$

will indicate that $\mathbf{y}(t)$ is the output of a *linear* system with input $\mathbf{x}(t)$. This means that

$$L[\mathbf{a}_1\mathbf{x}_1(t) + \mathbf{a}_2\mathbf{x}_2(t)] = \mathbf{a}_1 L[\mathbf{x}_1(t)] + \mathbf{a}_2 L[\mathbf{x}_2(t)] \tag{10-77}$$

for any $\mathbf{a}_1, \mathbf{a}_2, \mathbf{x}_1(t), \mathbf{x}_2(t)$.

The above is the familiar definition of linearity and it also holds if the coefficients $\mathbf{a}_1$ and $\mathbf{a}_2$ are random variables because, as we have assumed, the system is deterministic, that is, it operates only on the variable t.

Note If a system is specified by its internal structure or by a differential equation, then (10-77) holds only if $\mathbf{y}(t)$ is the *zero-state* response. The response due to the initial conditions (zero-input response) will not be considered.

A system is called *time-invariant* if its response to $\mathbf{x}(t + c)$ equals $\mathbf{y}(t + c)$. We shall assume throughout that all linear systems under consideration are time-invariant.

†H. E. Rowe, "Memoryless Nonlinearities with Gaussian Inputs," *BSTJ*, vol. 67, no. 7, September 1982.

It is well known that the output of a linear system is a convolution

$$\mathbf{y}(t) = \mathbf{x}(t) * h(t) = \int_{-\infty}^{\infty} \mathbf{x}(t-\alpha)h(\alpha)\,d\alpha \tag{10-78}$$

where

$$h(t) = L[\delta(t)]$$

in its impulse response. In the following, most systems will be specified by (10-78). However, we start our investigation using the operational notation (10-76) to stress the fact that various results based on the next theorem also hold for arbitrary linear operators involving one or more variables.

The following observations are immediate consequences of the linearity and time invariance of the system.

If $\mathbf{x}(t)$ is a normal process, then $\mathbf{y}(t)$ is also a normal process. This is an extension of the familiar property of linear transformations of normal RVs and can be justified if we approximate the integral in (10-78) by a sum:

$$\mathbf{y}(t_i) \simeq \sum_k \mathbf{x}(t_i - \alpha_k)\Delta(\alpha)$$

If $\mathbf{x}(t)$ is SSS, then $\mathbf{y}(t)$ is also SSS. Indeed, since $\mathbf{y}(t+c) = L[\mathbf{x}(t+c)]$ for every c, we conclude that if the processes $\mathbf{x}(t)$ and $\mathbf{x}(t+c)$ have the same statistical properties, so do the processes $\mathbf{y}(t)$ and $\mathbf{y}(t+c)$. We show later [see (10-133)] that if $\mathbf{x}(t)$ is WSS, the processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are jointly WSS.

Fundamental theorem. For any linear system

$$E\{L[\mathbf{x}(t)]\} = L[E\{\mathbf{x}(t)\}] \tag{10-79}$$

In other words, the mean $\eta_y(t)$ of the output $\mathbf{y}(t)$ equals the response of the system to the mean $\eta_x(t)$ of the input (Fig. 10-8*a*)

$$\eta_y(t) = L[\eta_x(t)] \tag{10-80}$$

The above is a simple extension of the linearity of expected values to arbitrary linear operators. In the context of (10-78) it can be deduced if we write the integral as a limit of a sum. This yields

$$E\{\mathbf{y}(t)\} = \int_{-\infty}^{\infty} E\{\mathbf{x}(t-\alpha)\}h(\alpha)\,d\alpha = \eta_x(t) * h(t) \tag{10-81}$$

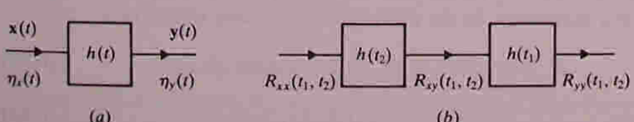

FIGURE 10-8

Frequency interpretation At the ith trial the input to our system is a function $\mathbf{x}(t, \zeta_i)$ yielding as output the function $\mathbf{y}(t, \zeta_i) = L[\mathbf{x}(t, \zeta_i)]$. For large n,

$$E\{\mathbf{y}(t)\} \simeq \frac{\mathbf{y}(t, \zeta_1) + \cdots + \mathbf{y}(t, \zeta_n)}{n} = \frac{L[\mathbf{x}(t, \zeta_1)] + \cdots + L[\mathbf{x}(t, \zeta_n)]}{n}$$

From the linearity of the system it follows that the last term above equals

$$L\left[\frac{\mathbf{x}(t, \zeta_1) + \cdots + \mathbf{x}(t, \zeta_n)}{n}\right]$$

This agrees with (10-79) because the fraction is nearly equal to $E\{\mathbf{x}(t)\}$.

Notes 1. From (10-80) it follows that if

$$\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \eta_x(t) \qquad \tilde{\mathbf{y}}(t) = \mathbf{y}(t) - \eta_y(t)$$

then

$$L[\tilde{\mathbf{x}}(t)] = L[\mathbf{x}(t)] - L[\eta_x(t)] = \tilde{\mathbf{y}}(t) \tag{10-82}$$

Thus the response of a linear system to the centered input $\tilde{\mathbf{x}}(t)$ equals the centered output $\tilde{\mathbf{y}}(t)$.

2. Suppose that

$$\mathbf{x}(t) = f(t) + \boldsymbol{\nu}(t) \qquad E\{\boldsymbol{\nu}(t)\} = 0$$

In this case, $E\{\mathbf{x}(t)\} = f(t)$; hence

$$\eta_y(t) = f(t) * h(t)$$

Thus, if $\mathbf{x}(t)$ is the sum of a deterministic signal $f(t)$ and a random component $\boldsymbol{\nu}(t)$, then for the determination of the mean of the output we can ignore $\boldsymbol{\nu}(t)$ provided that the system is linear and $E\{\boldsymbol{\nu}(t)\} = 0$.

Theorem (10-79) can be used to express the joint moments of any order of the output $\mathbf{y}(t)$ of a linear system in terms of the corresponding moments of the input. The following special cases are of fundamental importance in the study of linear systems with stochastic inputs.

OUTPUT AUTOCORRELATION. We wish to express the autocorrelation $R_{yy}(t_1, t_2)$ of the output $\mathbf{y}(t)$ of a linear system in terms of the autocorrelation $R_{xx}(t_1, t_2)$ of the input $\mathbf{x}(t)$. As we shall presently see, it is easier to find first the cross-correlation $R_{xy}(t_1, t_2)$ between $\mathbf{x}(t)$ and $\mathbf{y}(t)$.

THEOREM

(a)

$$R_{xy}(t_1, t_2) = L_2[R_{xx}(t_1, t_2)] \tag{10-83}$$

In the above notation, L_2 means that the system operates on the variable t_2, treating t_1 as a parameter. In the context of (10-78) this means that

$$R_{xy}(t_1, t_2) = \int_{-\infty}^{\infty} R_{xx}(t_1, t_2 - \alpha) h(\alpha)\, d\alpha \tag{10-84}$$

(b)

$$R_{yy}(t_1, t_2) = L_1[R_{xy}(t_1, t_2)] \tag{10-85}$$

In this case, the system operates on t_1:

$$R_{yy}(t_1, t_2) = \int_{-\infty}^{\infty} R_{xy}(t_1 - \alpha, t_2) h(\alpha)\, d\alpha \tag{10-86}$$

Proof. Multiplying (10-76) by $\mathbf{x}(t_1)$ and using (10-77), we obtain

$$\mathbf{x}(t_1)\mathbf{y}(t) = L_t[\mathbf{x}(t_1)\mathbf{x}(t)]$$

where L_t means that the system operates on t. Hence [see (10-79)]

$$E\{\mathbf{x}(t_1)\mathbf{y}(t)\} = L_t[E\{\mathbf{x}(t_1)\mathbf{x}(t)\}]$$

and (10-83) follows with $t = t_2$. The proof of (10-85) is similar: We multiply (10-76) by $\mathbf{y}(t_2)$ and use (10-79). This yields

$$E\{\mathbf{y}(t)\mathbf{y}(t_2)\} = L_t[E\{\mathbf{x}(t)\mathbf{y}(t_2)\}]$$

and (10-85) follows with $t = t_1$.

The preceding theorem is illustrated in Fig. 10-8*b*: If $R_{xx}(t_1, t_2)$ is the input to the given system and the system operates on t_2, the output equals $R_{xy}(t_1, t_2)$. If $R_{xy}(t_1, t_2)$ is the input and the system operates on t_1, the output equals $R_{yy}(t_1, t_2)$.

Inserting (10-84) into (10-86), we obtain

$$R_{yy}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t_1 - \alpha, t_2 - \beta) h(\alpha) h(\beta)\, d\alpha\, d\beta$$

This expresses $R_{yy}(t_1, t_2)$ directly in terms of $R_{xx}(t_1, t_2)$. However, conceptually and operationally, it is preferable to find first $R_{xy}(t_1, t_2)$.

Example 10-18. A stationary process $\boldsymbol{\nu}(t)$ with autocorrelation $R_{\nu\nu}(\tau) = q\delta(\tau)$ (white noise) is applied at $t = 0$ to a linear system with

$$h(t) = e^{-ct}U(t)$$

We shall show that the autocorrelation of the resulting output $\mathbf{y}(t)$ equals

$$R_{yy}(t_1, t_2) = \frac{q}{2c}(1 - e^{-2ct_1})e^{-c|t_2 - t_1|} \tag{10-87}$$

for $0 < t_1 < t_2$.

Proof. We can use the preceding results if we assume that the input to the system is the process

$$\mathbf{x}(t) = \boldsymbol{\nu}(t)U(t)$$

With this assumption, all correlations are 0 if $t_1 < 0$ or $t_2 < 0$. For $t_1 > 0$ and $t_2 > 0$,

$$R_{xx}(t_1, t_2) = E\{\boldsymbol{\nu}(t_1)\boldsymbol{\nu}(t_2)\} = q\delta(t_1 - t_2)$$

As we see from (10-83), $R_{xy}(t_1, t_2)$ equals the response of the system to $q\delta(t_1 - t_2)$ considered as a function of t_2. Since $\delta(t_1 - t_2) = \delta(t_2 - t_1)$ and $L[\delta(t_2 - t_1)] =$

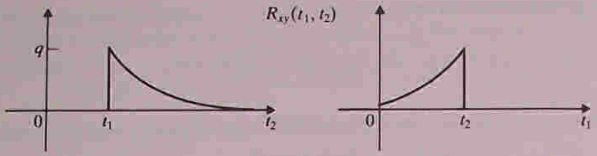

FIGURE 10-9

$h(t_2 - t_1)$ (time invariance), we conclude that

$$R_{xy}(t_1, t_2) = qh(t_2 - t_1) = qe^{-c(t_2 - t_1)}U(t_2 - t_1)$$

In Fig. 10-9, we show $R_{xy}(t_1, t_2)$ as a function of t_1 and t_2. Inserting into (10-86), we obtain

$$R_{yy}(t_1, t_2) = q\int_0^{t_1} e^{c(t_1 - \alpha - t_2)} e^{-c\alpha}\, d\alpha \qquad t_1 < t_2$$

and (10-87) results.

Note that

$$E\{\mathbf{y}^2(t)\} = R_{yy}(t) = \frac{q}{2c}(1 - e^{-2ct}) = q\int_0^t h^2(\alpha)\, d\alpha$$

COROLLARY. The autocovariance $C_{yy}(t_1, t_2)$ of $\mathbf{y}(t)$ is the autocorrelation of the process $\tilde{\mathbf{y}}(t) = \mathbf{y}(t) - \eta_y(t)$ and, as we see from (10-82), $\tilde{\mathbf{y}}(t)$ equals $L[\tilde{\mathbf{x}}(t)]$. Applying (10-84) and (10-86) to the centered processes $\tilde{\mathbf{x}}(t)$ and $\tilde{\mathbf{y}}(t)$, we obtain

$$\begin{aligned} C_{xy}(t_1, t_2) &= C_{xx}(t_1, t_2) * h(t_2) \\ C_{yy}(t_1, t_2) &= C_{xy}(t_1, t_2) * h(t_1) \end{aligned} \qquad (10\text{-}88)$$

where the convolutions are in t_1 and t_2 respectively.

Complex processes The preceding results can be readily extended to complex processes and to systems with complex-valued $h(t)$. Reasoning as in the real case, we obtain

$$\begin{aligned} R_{xy}(t_1, t_2) &= R_{xx}(t_1, t_2) * h^*(t_2) \\ R_{yy}(t_1, t_2) &= R_{xy}(t_1, t_2) * h(t_1) \end{aligned} \qquad (10\text{-}89)$$

Response to white noise. We shall determine the average intensity $E\{|\mathbf{y}(t)|^2\}$ of the output of a system driven by white noise. This is a special case of (10-89), however, because of its importance it is stated as a theorem.

THEOREM. If the input to a linear system is white noise with autocorrelation

$$R_{xx}(t_1, t_2) = q(t_1)\delta(t_1 - t_2)$$

then

$$E\{|\mathbf{y}(t)|^2\} = q(t) * |h(t)|^2 = \int_{-\infty}^{\infty} q(t - \alpha)|h(\alpha)|^2\, d\alpha \qquad (10\text{-}90)$$

Proof. From (10-89) it follows that

$$R_{xy}(t_1, t_2) = q(t_1)\delta(t_2 - t_1) * h^*(t_2) = q(t_1)h^*(t_2 - t_1)$$

$$R_{yy}(t_1, t_2) = \int_{-\infty}^{\infty} q(t_1 - \alpha)h^*[t_2 - (t_1 - \alpha)]h(\alpha)\,d\alpha$$

and with $t_1 = t_2 = t$, results.

Special cases (*a*) If $\mathbf{x}(t)$ is stationary white noise, then $q(t) = q$ and (10-90) yields

$$E\{\mathbf{y}^2(t)\} = qE \qquad \text{where} \quad E = \int_{-\infty}^{\infty} |h(t)|^2\,dt$$

is the energy of $h(t)$.

(*b*) If $h(t)$ is of short duration relative to the variations of $q(t)$, then

$$E\{\mathbf{y}^2(t)\} \simeq q(t)\int_{-\infty}^{\infty} |h(\alpha)|^2\,d\alpha = Eq(t) \tag{10-91}$$

This relationship justifies the term *average intensity* used to describe the function $q(t)$.

(*c*) If $R_{\nu\nu}(\tau) = q\delta(\tau)$ and $\boldsymbol{\nu}(t)$ is applied to the system at $t = 0$, then $q(t) = qU(t)$ and (10-90) yields

$$E\{\mathbf{y}^2(t)\} = q\int_{-\infty}^{t} |h(\alpha)|^2\,d\alpha$$

Example 10-19. The integral

$$\mathbf{y} = \int_0^t \boldsymbol{\nu}(\alpha)\,d\alpha$$

can be considered as the output of a linear system with input $\mathbf{x}(t) = \boldsymbol{\nu}(t)U(t)$ and impulse response $h(t) = U(t)$. If, therefore, $\boldsymbol{\nu}(t)$ is white noise with average intensity $q(t)$, then $\mathbf{x}(t)$ is white noise with average intensity $q(t)U(t)$ and (10-90) yields

$$E\{\mathbf{y}^2(t)\} = q(t)U(t) * U(t) = \int_0^t q(\alpha)\,d\alpha$$

Differentiators. A differentiator is a linear system whose output is the derivative of the input

$$L[\mathbf{x}(t)] = \mathbf{x}'(t)$$

We can, therefore, use the preceding results to find the mean and the autocorrelation of $\mathbf{x}'(t)$.

From (10-80) it follows that

$$\eta_{x'}(t) = L[\eta_x(t)] = \eta_x'(t) \tag{10-92}$$

Similarly [see (10-83)]

$$R_{xx'}(t_1, t_2) = L_2[R_{xx}(t_1, t_2)] = \frac{\partial R_{xx}(t_1, t_2)}{\partial t_2} \tag{10-93}$$

because, in this case, L_2 means differentiation with respect to t_2. Finally,

$$R_{x'x'}(t_1, t_2) = L_1[R_{xx'}(t_1, t_2)] = \frac{\partial R_{xx'}(t_1, t_2)}{\partial t_1} \tag{10-94}$$

Combining, we obtain

$$R_{x'x'}(t_1, t_2) = \frac{\partial^2 R_{xx}(t_1, t_2)}{\partial t_1 \, \partial t_2} \tag{10-95}$$

Stationary processes If $\mathbf{x}(t)$ is WSS, then $\eta_x(t)$ is constant; hence

$$E\{\mathbf{x}'(t)\} = 0 \tag{10-96}$$

Furthermore, since $R_{xx}(t_1, t_2) = R_{xx}(\tau)$, we conclude with $\tau = t_1 - t_2$ that

$$\frac{\partial R_{xx}(t_1 - t_2)}{\partial t_2} = -\frac{dR_{xx}(\tau)}{d\tau} \qquad \frac{\partial^2 R_{xx}(t_1 - t_2)}{\partial t_1 \, \partial t_2} = -\frac{d^2 R_{xx}(\tau)}{d\tau^2}$$

Hence

$$R_{xx'}(\tau) = -R_{xx'}(\tau) \qquad R_{x'x'}(\tau) = -R''_{xx}(\tau) \tag{10-97}$$

Poisson impulses. If the input $\mathbf{x}(t)$ to a differentiator is a Poisson process, the resulting output $\mathbf{z}(t)$ is a train of impulses (Fig. 10-10)

$$\mathbf{z}(t) = \sum_i \delta(t - \mathbf{t}_i) \tag{10-98}$$

We maintain that $\mathbf{z}(t)$ is a stationary process with mean

$$\eta_z = \lambda \tag{10-99}$$

and autocorrelation

$$R_{zz}(\tau) = \lambda^2 + \lambda\delta(\tau) \tag{10-100}$$

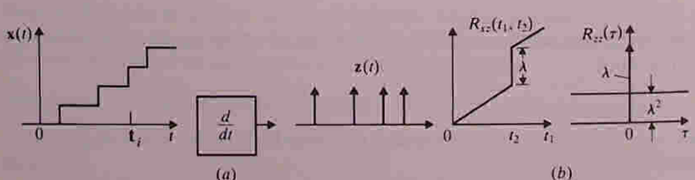

FIGURE 10-10

Proof. The first equation follows from (10-91) because $\eta_x(t) = \lambda t$. To prove the second, we observe that [see (10-14)]

$$R_{xx}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2) \tag{10-101}$$

And since $\mathbf{z}(t) = \mathbf{x}'(t)$, (10-93) yields

$$R_{xz}(t_1, t_2) = \frac{\partial R_{xx}(t_1, t_2)}{\partial t_2} = \lambda^2 t_1 + \lambda U(t_1 - t_2)$$

This function is plotted in Fig. 10-10*b* where the independent variable is t_1. As we see, it is discontinuous for $t_1 = t_2$ and its derivative with respect to t_1 contains the impulse $\lambda\delta(t_1 - t_2)$. This yields [see (10-94)]

$$R_{zz}(t_1, t_2) = \frac{\partial R_{xz}(t_1, t_2)}{\partial t_1} = \lambda^2 + \lambda\delta(t_1 - t_2)$$

DIFFERENTIAL EQUATIONS. A deterministic differential equation with random excitation is an equation of the form

$$a_n \mathbf{y}^{(m)}(t) + \cdots + a_0 \mathbf{y}(t) = \mathbf{x}(t) \tag{10-102}$$

where the coefficients a_k are given numbers and the driver $\mathbf{x}(t)$ is a stochastic process. We shall consider its solution $\mathbf{y}(t)$ under the assumption that the initial conditions are 0. With this assumption, $\mathbf{y}(t)$ is unique (zero-state response) and it satisfies the linearity condition (10-77). We can, therefore, interpret $\mathbf{y}(t)$ as the output of a linear system specified by (10-102).

In general, the determination of the complete statistics of $\mathbf{y}(t)$ is complicated. In the following, we evaluate only its second-order moments using the preceding results. The above system is an operator L specified as follows: Its output $\mathbf{y}(t)$ is a process with zero initial conditions satisfying (10-102).

Mean. As we know [see (10-80)] the mean $\eta_y(t)$ of $\mathbf{y}(t)$ is the output of L with input $\eta_x(t)$. Hence it satisfies the equation

$$a_n \eta_y^{(n)}(t) + \cdots + a_0 \eta_y(t) = \eta_x(t) \tag{10-103}$$

and the initial conditions

$$\eta_y(0) = \cdots = \eta_y^{(n-1)}(0) = 0 \tag{10-104}$$

This result can be established directly: Clearly,

$$E\{\mathbf{y}^{(k)}(t)\} = \eta_y^{(k)}(t) \tag{10-105}$$

Taking expected values of both sides of (10-102) and using the above, we obtain (10-103). Equation (10-104) follows from (10-105) because $\mathbf{y}^{(k)}(0) = 0$ by assumption.

Correlation. To determine $R_{xy}(t_1, t_2)$, we use (10-83)

$$R_{xy}(t_1, t_2) = L_2[R_{xx}(t_1, t_2)]$$

In this case, L_2 means that $R_{xy}(t_1, t_2)$ satisfies the differential equation

$$a_n \frac{\partial^n R_{xy}(t_1, t_2)}{\partial t_2^n} + \cdots + a_0 R_{xy}(t_1, t_2) = R_{xx}(t_1, t_2) \qquad (10\text{-}106)$$

with the initial conditions

$$R_{xy}(t_1, 0) = \cdots = \frac{\partial^{n-1} R_{xy}(t_1, 0)}{\partial t_2^{n-1}} = 0 \qquad (10\text{-}107)$$

Similarly, since [see (10-85)]

$$R_{yy}(t_1, t_2) = L_1[R_{xy}(t_1, t_2)]$$

we conclude as above that

$$a_n \frac{\partial^n R_{yy}(t_1, t_2)}{\partial t_1^n} + \cdots + a_0 R_{yy}(t_1, t_2) = R_{xy}(t_1, t_2) \qquad (10\text{-}108)$$

$$R_{yy}(0, t_2) = \cdots = \frac{\partial^{n-1} R_{yy}(0, t_2)}{\partial t_1^{n-1}} = 0 \qquad (10\text{-}109)$$

The preceding results can be established directly: From (10-102) it follows that

$$\mathbf{x}(t_1)\left[a_n \mathbf{y}^{(n)}(t_2) + \cdots + a_0 \mathbf{y}(t_2)\right] = \mathbf{x}(t_1)\mathbf{x}(t_2)$$

This yields (10-106) because [see (10-119)]

$$E\{\mathbf{x}(t_1)\mathbf{y}^{(k)}(t_2)\} = \partial^k R_{xy}(t_1, t_2)/\partial t_2^k$$

Similarly, (10-108) is a consequence of the identity

$$\left[a_n \mathbf{y}^{(n)}(t_1) + \cdots + a_0 \mathbf{y}(t_1)\right]\mathbf{y}(t_2) = \mathbf{x}(t_1)\mathbf{y}(t_2)$$

because

$$E\{\mathbf{y}^{(k)}(t_1)\mathbf{y}(t_2)\} = \partial^k R_{yy}(t_1, t_2)/\partial t_1^k$$

Finally, the expected values of

$$\mathbf{x}(t_1)\mathbf{y}^{(k)}(0) = 0 \qquad \mathbf{y}^{(k)}(0)\mathbf{y}(t_2) = 0$$

yield (10-107) and (10-109).

General moments. The moments of any order of the output $\mathbf{y}(t)$ of a linear system can be expressed in terms of the corresponding moments of the input $\mathbf{x}(t)$. As an illustration, we shall determine the third-order moment

$$R_{yyy}(t_1, t_2, t_3) = E\{\mathbf{y}_1(t)\mathbf{y}_2(t)\mathbf{y}_3(t)\}$$

of $\mathbf{y}(t)$ in terms of the third-order moment $R_{xxx}(t_1, t_2, t_3)$ of $\mathbf{x}(t)$. Proceeding as in (10-83), we obtain

$$E\{\mathbf{x}(t_1)\mathbf{x}(t_2)\mathbf{y}(t_3)\} = L_3[E\{\mathbf{x}(t_1)\mathbf{x}(t_2)\mathbf{x}(t_3)\}]$$
$$= \int_{-\infty}^{\infty} R_{xxx}(t_1, t_2, t_3 - \gamma)h(\gamma)\,d\gamma \quad (10\text{-}110a)$$
$$E\{\mathbf{x}(t_1)\mathbf{y}(t_2)\mathbf{y}(t_3)\} = L_2[E\{\mathbf{x}(t_1)\mathbf{x}(t_2)\mathbf{y}(t_3)\}]$$
$$= \int_{-\infty}^{\infty} R_{xxy}(t_1, t_2, -\beta, t_3)h(\beta)\,d\beta \quad (10\text{-}110b)$$
$$E\{\mathbf{y}(t_1)\mathbf{y}(t_2)\mathbf{y}(t_3)\} = L_1[E\{\mathbf{x}(t_1)\mathbf{y}(t_2)\mathbf{y}(t_3)\}]$$
$$= \int_{-\infty}^{\infty} R_{xyy}(t_1 - \alpha, t_2, t_3)h(\alpha)\,d\alpha \quad (10\text{-}110c)$$

Note that for the evaluation of $R_{yyy}(t_1, t_2, t_3)$ for specific times t_1, t_2, t_3, the function $R_{xxx}(t_1, t_2, t_3)$ must be known for every t_1, t_2, t_3.

Vector Processes and Multiterminal Systems

We consider now systems with n inputs $\mathbf{x}_i(t)$ and r outputs $\mathbf{y}_j(t)$. As a preparation, we introduce the notion of autocorrelation and cross-correlation for vector processes starting with a review of the standard matrix notation.

The expression $A = [a_{ij}]$ will mean a matrix with elements a_{ij}. The notation

$$A^t = [a_{ji}] \qquad A^* = [a_{ij}^*] \qquad A^\dagger = [a_{ji}^*]$$

will mean the transpose, the conjugate, and the conjugate transpose of A.

A column vector will be identified by $A = [a_i]$. Whether A is a vector or a general matrix will be understood from the context. If $A = [a_i]$ and $B = [b_j]$ are two vectors with m elements each, the product $A^tB = a_1b_1 + \cdots + a_mb_m$ is a number, and the product $AB^t = [a_ib_j]$ is an $m \times m$ matrix with elements a_ib_j.

A vector process $\mathbf{X}(t) = [\mathbf{x}_i(t)]$ is a vector, the components of which are stochastic processes. The mean $\eta(t) = E\{\mathbf{X}(t)\} = [\eta_i(t)]$ of $\mathbf{X}(t)$ is a vector with components $\eta_i(t) = E\{\mathbf{x}_i(t)\}$. The autocorrelation $R(t_1, t_2)$ or $R_{xx}(t_1, t_2)$ of a vector process $\mathbf{X}(t)$ is an $m \times m$ matrix

$$R(t_1, t_2) = E\{\mathbf{X}(t_1)\mathbf{X}^\dagger(t_2)\} \quad (10\text{-}111)$$

with elements $E\{\mathbf{x}_i(t_1)\mathbf{x}_j^*(t_2)\}$. We define similarly the cross-correlation matrix

$$R_{xy}(t_1, t_2) = E\{\mathbf{X}(t_1)\mathbf{Y}^\dagger(t_2)\} \quad (10\text{-}112)$$

of the vector processes

$$\mathbf{X}(t) = [\mathbf{x}_i(t)] \quad i = 1, \ldots, m \qquad \mathbf{Y}(t) = [\mathbf{y}_j(t)] \quad j = 1, \ldots, r \quad (10\text{-}113)$$

A multiterminal system with m inputs $\mathbf{x}_i(t)$ and r outputs $\mathbf{y}_j(t)$ is a rule for assigning to an m vector $\mathbf{X}(t)$ an r vector $\mathbf{Y}(t)$. If the system is linear and time-invariant, it is specified in terms of its impulse response matrix. This is an $r \times m$ matrix

$$H(t) = [h_{ji}(t)] \qquad i = 1, \ldots, m \qquad j = 1, \ldots, r \tag{10-114}$$

defined as follows: Its component $h_{ji}(t)$ is the response of the jth output when the ith input equals $\delta(t)$ and all other inputs equal 0. From this and the linearity of the system, it follows that the response $\mathbf{y}_j(t)$ of the jth output to an arbitrary input $\mathbf{X}(t) = [\mathbf{x}_i(t)]$ equals

$$\mathbf{y}_j(t) = \int_{-\infty}^{\infty} h_{j1}(\alpha)\mathbf{x}_1(t-\alpha)\,d\alpha + \cdots + \int_{-\infty}^{\infty} h_{jm}(\alpha)\mathbf{x}_m(t-\alpha)\,d\alpha$$

Hence

$$\mathbf{Y}(t) = \int_{-\infty}^{\infty} H(\alpha)\mathbf{X}(t-\alpha)\,d\alpha \tag{10-115}$$

In the above, $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are column vectors and $H(t)$ is an $r \times m$ matrix. We shall use this relationship to determine the autocorrelation $R_{yy}(t_1, t_2)$ of $\mathbf{Y}(t)$. Premultiplying the conjugate transpose of (10-115) by $\mathbf{X}(t_1)$ and setting $t = t_2$, we obtain

$$\mathbf{X}(t_1)\mathbf{Y}^\dagger(t_2) = \int_{-\infty}^{\infty} \mathbf{X}(t_1)\mathbf{X}^\dagger(t_2-\alpha)H^\dagger(\alpha)\,d\alpha$$

Hence

$$R_{xy}(t_1, t_2) = \int_{-\infty}^{\infty} R_{xx}(t_1, t_2-\alpha)H^\dagger(\alpha)\,d\alpha \tag{10-116a}$$

Postmultiplying (10-115) by $\mathbf{Y}^\dagger(t_2)$ and setting $t = t_1$, we obtain

$$R_{yy}(t_1, t_2) = \int_{-\infty}^{\infty} H(\alpha)R_{xy}(t_1-\alpha, t_2)\,d\alpha \tag{10-116b}$$

as in (10-89). These results can be used to express the cross-correlation of the outputs of several scalar systems in terms of the cross-correlation of their inputs. The next example is an illustration.

Example 10-20. In Fig. 10-11 we show two systems with inputs $\mathbf{x}_1(t), \mathbf{x}_2(t)$ and outputs

$$\mathbf{y}_1(t) = \int_{-\infty}^{\infty} h_1(\alpha)\mathbf{x}_1(t-\alpha)\,d\alpha \qquad \mathbf{y}_2(t) = \int_{-\infty}^{\infty} h_2(\alpha)\mathbf{x}_2(t-\alpha)\,d\alpha \tag{10-117}$$

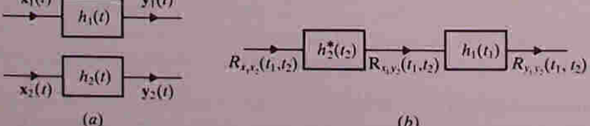

FIGURE 10-11

These signals can be considered as the components of the output vector $\mathbf{Y}'(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)]$ of a 2×2 system with input vector $\mathbf{X}'(t) = [\mathbf{x}_1(t), \mathbf{x}_2(t)]$ and impulse response matrix

$$H(t) = \begin{bmatrix} h_1(t) & 0 \\ 0 & h_2(t) \end{bmatrix}$$

Inserting into (10-116), we obtain

$$\begin{aligned} R_{x_1y_2}(t_1, t_2) &= \int_{-\infty}^{\infty} R_{x_1x_2}(t_1, t_2 - \alpha) h_2^*(\alpha)\, d\alpha \\ R_{y_1y_2}(t_1, t_2) &= \int_{-\infty}^{\infty} h_1(\alpha) R_{x_1y_2}(t_1 - \alpha, t_2)\, d\alpha \end{aligned} \qquad (10\text{-}118)$$

Thus, to find $R_{x_1y_2}(t_1, t_2)$, we use $R_{x_1x_2}(t_1, t_2)$ as the input to the conjugate $h_2^*(t)$ of $h_2(t)$, operating on the variable t_2. To find $R_{y_1y_2}(t_1, t_2)$, we use $R_{x_1y_2}(t_1, t_2)$ as the input to $h_1(t)$ operating on the variable t_1 (Fig. 10-11).

Example 10-21. The derivatives $\mathbf{y}_1(t) = \mathbf{z}^{(m)}(t)$ and $\mathbf{y}_2(t) = \mathbf{w}^{(n)}(t)$ of two processes $\mathbf{z}(t)$ and $\mathbf{w}(t)$ can be considered as the responses of two differentiators with inputs $\mathbf{x}_1(t) = \mathbf{z}(t)$ and $\mathbf{x}_2(t) = \mathbf{w}(t)$. Applying (10-118) suitably interpreted, we conclude that

$$E\{\mathbf{z}^{(m)}(t_1)\mathbf{w}^{(n)}(t_2)\} = \frac{\partial^{m+n} R_{zw}(t_1, t_2)}{\partial t_1^m\, \partial t_2^n} \qquad (10\text{-}119)$$

10-3 THE POWER SPECTRUM

In signal theory, spectra are associated with Fourier transforms. For deterministic signals, they are used to represent a function as a superposition of exponentials. For random signals, the notion of a spectrum has two interpretations. The first involves transforms of averages; it is thus essentially deterministic. The second leads to the representation of the process under consideration as superposition of exponentials with random coefficients. In this section, we introduce the first interpretation. The second is treated in Sec. 12-4. We shall consider only stationary processes. For nonstationary processes the notion of a spectrum is of limited interest.

DEFINITIONS. The *power spectrum* (or *spectral density*) of a WSS process $\mathbf{x}(t)$, real or complex, is the Fourier transform $S(\omega)$ of its autocorrelation $R(\tau) = E\{\mathbf{x}(t+\tau)\mathbf{x}^*(t)\}$:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau}\, d\tau \qquad (10\text{-}120)$$

Since $R(-\tau) = R^*(\tau)$ it follows that $S(\omega)$ is a real function of ω.

From the Fourier inversion formula, it follows that

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega\tau}\, d\omega \qquad (10\text{-}121)$$

TABLE 10-1

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}\,d\omega \leftrightarrow S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}\,d\tau$$

$$\delta(\tau) \leftrightarrow 1 \qquad 1 \leftrightarrow 2\pi\delta(\omega)$$

$$e^{j\beta\tau} \leftrightarrow 2\pi\delta(\omega - \beta) \qquad \cos\beta\tau \leftrightarrow \pi\delta(\omega - \beta) + \pi\delta(\omega + \beta)$$

$$e^{-\alpha|\tau|} \leftrightarrow \frac{2\alpha}{\alpha^2 + \omega^2} \qquad e^{-\alpha\tau^2} \leftrightarrow \sqrt{\frac{\pi}{\alpha}}\,e^{-\omega^2/4\alpha}$$

$$e^{-\alpha|\tau|}\cos\beta\tau \leftrightarrow \frac{\alpha}{\alpha^2 + (\omega - \beta)^2} + \frac{\alpha}{\alpha^2 + (\omega + \beta)^2}$$

$$2e^{-\alpha\tau^2}\cos\beta\tau \leftrightarrow \sqrt{\frac{\pi}{\alpha}}\,[e^{-(\omega-\beta)^2/4\alpha} + e^{-(\omega+\beta)^2/4\alpha}]$$

$$\begin{cases} 1 - \dfrac{|\tau|}{T} & |\tau| < T \\ 0 & |\tau| > T \end{cases} \leftrightarrow \frac{4\sin^2(\omega T/2)}{T\omega^2}$$

$$\frac{\sin\sigma\tau}{\pi\tau} \leftrightarrow \begin{cases} 1 & |\omega| < \sigma \\ 0 & |\omega| > \sigma \end{cases}$$

If $\mathbf{x}(t)$ is a real process, then $R(\tau)$ is real and even; hence $S(\omega)$ is also real and even. In this case,

$$\begin{aligned} S(\omega) &= \int_{-\infty}^{\infty} R(\tau)\cos\omega\tau\,d\tau = 2\int_{0}^{\infty} R(\tau)\cos\omega\tau\,d\tau \\ R(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\cos\omega\tau\,d\omega = \frac{1}{\pi}\int_{0}^{\infty} S(\omega)\cos\omega\tau\,d\omega \end{aligned} \qquad (10\text{-}122)$$

The *cross-power spectrum* of two processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ is the Fourier transform $S_{xy}(\omega)$ of their cross-correlation $R_{xy}(\tau) = E\{\mathbf{x}(t + \tau)\mathbf{y}^*(t)\}$:

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)e^{-j\omega\tau}\,d\tau \qquad R_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)e^{j\omega\tau}\,d\omega \qquad (10\text{-}123)$$

The function $S_{xy}(\omega)$ is, in general, complex even when both processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are real. In all cases,

$$S_{xy}(\omega) = S^*_{yx}(\omega) \qquad (10\text{-}124)$$

because $R_{xy}(-\tau) = E\{\mathbf{x}(t - \tau)\mathbf{y}^*(t)\} = R^*_{yx}(\tau)$.

In Table 10-1 we list a number of frequently used autocorrelations and the corresponding spectra. Note that in all cases, $S(\omega)$ is positive. As we shall soon show, this is true for every spectrum.

Example 10-22. A random telegraph signal is a process $\mathbf{x}(t)$ taking the values +1 and −1 as in Example 10-6:

$$\mathbf{x}(t) = \begin{cases} 1 & \mathbf{t}_{2i} < t < \mathbf{t}_{2i+1} \\ -1 & \mathbf{t}_{2i-1} < t < \mathbf{t}_{2i} \end{cases}$$

where $\mathbf{t}_i$ is a set of Poisson points with average density λ. As we have shown in (10-19), its autocorrelation equals $e^{-2\lambda|\tau|}$. Hence

$$S(\omega) = \frac{4\lambda}{4\lambda^2 + \omega^2}$$

For most processes $R(\tau) \to \eta^2$ where $\eta = E\{\mathbf{x}(t)\}$ (see Sec. 12-4). If, therefore, $\eta \neq 0$, then $S(\omega)$ contains an impulse at $\omega = 0$. To avoid this, it is often convenient to express the spectral properties of $\mathbf{x}(t)$ in terms of the Fourier transform $S^c(\omega)$ of its autocovariance $C(\tau)$. Since $R(\tau) = C(\tau) + \eta^2$, it follows that

$$S(\omega) = S^c(\omega) + 2\pi\eta^2\delta(\omega) \tag{10-125}$$

The function $S^c(\omega)$ is called the *covariance spectrum* of $\mathbf{x}(t)$.

Example 10-23. We have shown in (10-100) that the autocorrelation of the Poisson impulses

$$\mathbf{z}(t) = \frac{d}{dt}\sum_i U(t - \mathbf{t}_i) = \sum_i \delta(t - \mathbf{t}_i)$$

equals $R_z(\tau) = \lambda^2 + \lambda\delta(\tau)$. From this it follows that

$$S_z(\omega) = \lambda + 2\pi\lambda^2\delta(\omega) \qquad S_z^c(\omega) = \lambda$$

We shall show that given an arbitrary positive function $S(\omega)$, we can find a process $\mathbf{x}(t)$ with power spectrum $S(\omega)$.

(*a*) Consider the process

$$\mathbf{x}(t) = ae^{j(\boldsymbol{\omega} t - \boldsymbol{\varphi})} \tag{10-126}$$

where a is a real constant, $\boldsymbol{\omega}$ is an RV with density $f_\omega(\omega)$, and $\boldsymbol{\varphi}$ is an RV independent of $\boldsymbol{\omega}$ and uniform in the interval $(0, 2\pi)$. As we know, this process is WSS with zero mean and autocorrelation

$$R_x(\tau) = a^2 E\{e^{j\boldsymbol{\omega}\tau}\} = a^2\int_{-\infty}^{\infty} f_\omega(\omega)e^{j\omega\tau}\,d\omega$$

From this and the uniqueness property of Fourier transforms, it follows that [see (10-121)] the power spectrum of $\mathbf{x}(t)$ equals

$$S_x(\omega) = 2\pi a^2 f_\omega(\omega) \tag{10-127}$$

If, therefore,

$$f_\omega(\omega) = \frac{S(\omega)}{2\pi a^2} \qquad a^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,d\omega = R(0)$$

then $f_\omega(\omega)$ is a density and $S_x(\omega) = S(\omega)$. To complete the specification of $\mathbf{x}(t)$, it suffices to construct an RV $\boldsymbol{\omega}$ with density $S(\omega)/2\pi a^2$ and insert it into (10-126).

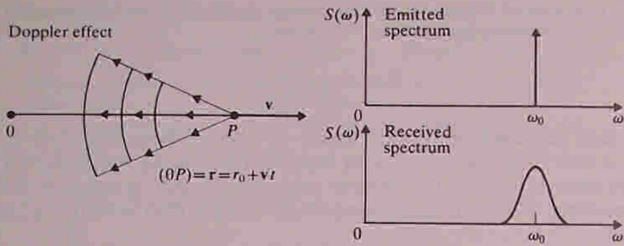

FIGURE 10-12

(*b*) We show next that if $S(-\omega) = S(\omega)$, we can find a real process with power spectrum $S(\omega)$. To do so, we form the process

$$\mathbf{y}(t) = a\cos(\boldsymbol{\omega} t + \boldsymbol{\varphi}) \tag{10-128}$$

In this case (see Example 10-14)

$$R_y(\tau) = \frac{a^2}{2}E\{\cos\boldsymbol{\omega}\tau\} = \frac{a^2}{2}\int_{-\infty}^{\infty} f(\omega)\cos\omega\tau\,d\omega$$

From this it follows that if $f_\omega(\omega) = S(\omega)/\pi a^2$, then $S_y(\omega) = S(\omega)$.

Example 10-24 Doppler effect. A harmonic oscillator located at point P of the x axis (Fig. 10-12) moves in the x direction with velocity $\mathbf{v}$. The emitted signal equals $e^{j\omega_0 t}$ and the signal received by an observer located at point O equals

$$\mathbf{s}(t) = ae^{j\omega_0(t - \mathbf{r}/c)}$$

where c is the velocity of propagation and $\mathbf{r} = r_0 + \mathbf{v}t$. We assume that $\mathbf{v}$ is an RV with density $f_v(v)$. Clearly,

$$\mathbf{s}(t) = ae^{j(\boldsymbol{\omega} t - \varphi)} \qquad \boldsymbol{\omega} = \omega_0\left(1 - \frac{\mathbf{v}}{c}\right) \qquad \varphi = \frac{r_0\omega_0}{c}$$

hence the spectrum of the received signal is given by (10-127)

$$S(\omega) = 2\pi a^2 f_\omega(\omega) = \frac{2\pi a^2 c}{\omega_0} f_v\left[\left(1 - \frac{\omega}{\omega_0}\right)c\right] \tag{10-129}$$

Note that if $\mathbf{v} = 0$, then

$$\mathbf{s}(t) = ae^{j(\omega_0 t - \varphi)} \qquad R(\tau) = a^2 e^{j\omega_0\tau} \qquad S(\omega) = 2\pi a^2\delta(\omega - \omega_0)$$

This is the spectrum of the emitted signal. Thus the motion causes broadening of the spectrum.

The above holds also if the motion forms an angle with the x axis provided that $\mathbf{v}$ is replaced by its projection $\mathbf{v}_x$ on OP. The following case is of special interest. Suppose that the emitter is a particle in a gas of temperature T. In this case, the x component of its velocity is a normal RV with zero mean and variance

kT/m (see Prob. 8-5). Inserting into (10-129), we conclude that

$$S(\omega) = \frac{2\pi a^2 c}{\omega_0\sqrt{2\pi kT/m}} \exp\left\{-\frac{mc^2}{2kT}\left(1-\frac{\omega}{\omega_0}\right)^2\right\}$$

$$R(\tau) = a^2 \exp\left\{-\frac{kT\omega_0^2\tau^2}{2mc^2}\right\}e^{j\omega_0\tau}$$

Line spectra. (*a*) We have shown in Example 10-7 that the process

$$\mathbf{x}(t) = \sum_i \mathbf{c}_i e^{j\omega_i t}$$

is WSS if the RVs $\mathbf{c}_i$ are uncorrelated with zero mean. From this and Table 10-1 it follows that

$$R(\tau) = \sum_i \sigma_i^2 e^{j\omega_i\tau} \qquad S(\omega) = 2\pi\sum \sigma_i^2\delta(\omega - \omega_i) \qquad (10\text{-}130)$$

where $\sigma_i^2 = E\{\mathbf{c}_i^2\}$. Thus $S(\omega)$ consists of lines. In Sec. 14-2 we show that such a process is predictable, that is, its present value is uniquely determined in terms of its past.

(*b*) Similarly, the process

$$\mathbf{y}(t) = \sum_i (\mathbf{a}_i \cos\omega_i t + \mathbf{b}_i \sin\omega_i t)$$

is WSS iff the RVs $\mathbf{a}_i$ and $\mathbf{b}_i$ are uncorrelated with zero mean and $E\{\mathbf{a}_i^2\} = E\{\mathbf{b}_i^2\} = \sigma_i^2$. In this case,

$$R(\tau) = \sum_i \sigma_i^2 \cos\omega_i\tau \qquad S(\omega) = \pi\sum_i \sigma_i^2[\delta(\omega-\omega_i) + \delta(\omega+\omega_i)] \qquad (10\text{-}131)$$

Linear systems. We shall express the autocorrelation $R_{yy}(\tau)$ and power spectrum $S_{yy}(\omega)$ of the response

$$\mathbf{y}(t) = \int_{-\infty}^{\infty} \mathbf{x}(t-\alpha)h(\alpha)\,d\alpha \qquad (10\text{-}132)$$

of a linear system in terms of the autocorrelation $R_{xx}(\tau)$ and power spectrum $S_{xx}(\omega)$ of the input $\mathbf{x}(t)$.

THEOREM

$$R_{xy}(\tau) = R_{xx}(\tau) * h^*(-\tau) \qquad R_{yy}(\tau) = R_{xy}(\tau) * h(\tau) \qquad (10\text{-}133)$$

$$S_{xy} = S_{xx}(\omega)H^*(\omega) \qquad S_{yy}(\omega) = S_{xy}(\omega)H(\omega) \qquad (10\text{-}134)$$

Proof. The two equations in (10-133) are special cases of (10-184) and (10-185). However, because of their importance they will be proved directly. Multiplying the conjugate of (10-132) by $\mathbf{x}(t+\tau)$ and taking expected values, we obtain

$$E\{\mathbf{x}(t+\tau)\mathbf{y}^*(t)\} = \int_{-\infty}^{\infty} E\{\mathbf{x}(t+\tau)\mathbf{x}^*(t-\alpha)\}h^*(\alpha)\,d\alpha$$

Since $E\{\mathbf{x}(t+\tau)\mathbf{x}^*(t-\alpha)\} = R_{xx}(\tau+\alpha)$, this yields

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} R_{xx}(\tau+\alpha)h^*(\alpha)\,d\alpha = \int_{-\infty}^{\infty} R_{xx}(\tau-\beta)h^*(-\beta)\,d\beta$$

Proceeding similarly, we obtain

$$E\{\mathbf{y}(t)\mathbf{y}^*(t-\tau)\} = \int_{-\infty}^{\infty} E\{\mathbf{x}(t-\alpha)\mathbf{y}^*(t-\tau)\}h(\alpha)\,d\alpha$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau-\alpha)h(\alpha)\,d\alpha$$

Equation (10-134) follows from (10-133) and the convolution theorem.

COROLLARY. Combining the two equations in (10-133) and (10-134), we obtain

$$R_{yy}(\tau) = R_{xx}(\tau) * h(\tau) * h^*(-\tau) = R_{xx}(\tau) * \rho(\tau) \qquad (10\text{-}135)$$

$$S_{yy}(\omega) = S_{xx}(\omega)H(\omega)H^*(\omega) = S_{xx}(\omega)|H(\omega)|^2 \qquad (10\text{-}136)$$

where

$$\rho(\tau) = h(\tau) * h^*(-\tau) = \int_{-\infty}^{\infty} h(t+\tau)h^*(t)\,dt \leftrightarrow |H(\omega)|^2 \quad (10\text{-}137)$$

Note, in particular, that if $\mathbf{x}(t)$ is white noise with average power q, then

$$\begin{aligned} R_{xx}(\tau) &= q\delta(\tau) \qquad & S_{xx}(\omega) &= q \\ S_{yy}(\omega) &= q|H(\omega)|^2 \qquad & R_{yy}(\tau) &= q\rho(\tau) \end{aligned} \qquad (10\text{-}138)$$

From (10-136) and the inversion formula (10-121), it follows that

$$E\{|\mathbf{y}(t)|^2\} = R_{yy}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)|H(\omega)|^2\,d\omega \geq 0 \quad (10\text{-}139)$$

This equation describes the filtering properties of a system when the input is a random process. It shows, for example, that if $H(\omega) = 0$ for $|\omega| > \omega_0$ and $S_{xx}(\omega) = 0$ for $|\omega| < \omega_0$, then $E\{\mathbf{y}^2(t)\} = 0$.

Note The preceding results hold if all correlations are replaced by the corresponding covariances and all spectra by the corresponding covariance spectra. This follows from the fact that the response to $\mathbf{x}(t) - \eta_x$ equals $\mathbf{y}(t) - \eta_y$. For example, (10-136) and (10-142) yield

$$S^c_{yy}(\omega) = S^c_{xx}(\omega)|H(\omega)|^2 \qquad (10\text{-}140)$$

$$\operatorname{Var}\mathbf{y}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S^c_{xx}(\omega)|H(\omega)|^2\,d\omega \qquad (10\text{-}141)$$

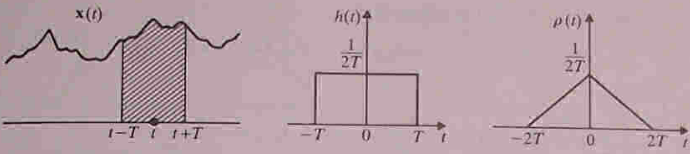

FIGURE 10-13

Example 10-25. (*a*) (*Moving average*) The integral

$$\mathbf{y}(t) = \frac{1}{2T}\int_{t-T}^{t+T}\mathbf{x}(\alpha)\,d\alpha$$

is the average of the process $\mathbf{x}(t)$ in the interval $(t - T, t + T)$. Clearly, $\mathbf{y}(t)$ is the output of a system with input $\mathbf{x}(t)$ and impulse response a rectangular pulse as in Fig. 10-13. The corresponding $\rho(\tau)$ is a triangle. In this case,

$$H(\omega) = \frac{1}{2T}\int_{-T}^{T} e^{-j\omega\tau}\,d\tau = \frac{\sin T\omega}{T\omega} \qquad S_{yy}(\omega) = S_{xx}(\omega)\frac{\sin^2 T\omega}{T^2\omega^2}$$

Thus $H(\omega)$ takes significant values only in an interval of the order of $1/T$ centered at the origin. Hence the moving average suppresses the high-frequency components of the input. It the thus a simple low-pass filter.

Since $\rho(\tau)$ is a triangle, it follows from (10-135) that

$$R_{yy}(\tau) = \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|\alpha|}{2T}\right)R_{xx}(\tau - \alpha)\,d\alpha \tag{10-142}$$

We shall use this result to determine the variance of the integral

$$\boldsymbol{\eta}_T = \frac{1}{2T}\int_{-T}^{T}\mathbf{x}(t)\,dt$$

Clearly, $\boldsymbol{\eta}_T = \mathbf{y}(0)$; hence

$$\operatorname{Var}\boldsymbol{\eta}_T = C_{yy}(0) = \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|\alpha|}{2T}\right)C_{xx}(\alpha)\,d\alpha \tag{10-143}$$

(*b*) (*High-pass filter*) The process $\mathbf{z}(t) = \mathbf{x}(t) - \mathbf{y}(t)$ is the output of a system with input $\mathbf{x}(t)$ and system function

$$H(\omega) = 1 - \frac{\sin T\omega}{T\omega}$$

This function is nearly 0 in an interval of the order of $1/T$ centered at the origin, and it approaches 1 for large ω. It acts, therefore, as a high-pass filter suppressing the low frequencies of the input.

Example 10-26 Derivatives. The derivative $\mathbf{x}'(t)$ of a process $\mathbf{x}(t)$ can be considered as the output of a linear system with input $\mathbf{x}(t)$ and system function $j\omega$.

From this and (10-134), it follows that

$$S_{xx'}(\omega) = -j\omega S_{xx}(\omega) \qquad S_{x'x'}(\omega) = \omega^2 S_{xx}(\omega)$$

Hence

$$R_{xx'}(\tau) = -\frac{dR_{xx}(\tau)}{d\tau} \qquad R_{x'x'}(\tau) = -\frac{d^2R_{xx}(\tau)}{d\tau^2}$$

The nth derivative $\mathbf{y}(t) = \mathbf{x}^{(n)}(t)$ of $\mathbf{x}(t)$ is the output of a system with input $\mathbf{x}(t)$ and system function $(j\omega)^n$. Hence

$$S_{yy}(\omega) = |j\omega|^{2n} \qquad R_{yy}(\tau) = (-1)^n R^{(2n)}(\tau) \tag{10-144}$$

Example 10-27. (*a*) The differential equation

$$\mathbf{y}'(t) + c\mathbf{y}(t) = \mathbf{x}(t) \quad \text{all } t$$

specifies a linear system with input $\mathbf{x}(t)$, output $\mathbf{y}(t)$, and system function $1/(j\omega + c)$. We assume that $\mathbf{x}(t)$ is white noise with $R_{xx}(\tau) = q\delta(\tau)$. Applying (10-136), we obtain

$$S_{yy}(\omega) = \frac{S_{xx}(\omega)}{\omega^2 + c^2} = \frac{q}{\omega^2 + c^2} \qquad R_{yy}(\tau) = \frac{q}{2c}e^{-c|\tau|}$$

Note that $E\{\mathbf{y}^2(t)\} = R_{yy}(0) = q/2c$.

(*b*) Similarly, if

$$\mathbf{y}''(t) + b\mathbf{y}'(t) + c\mathbf{y}(t) = \mathbf{x}(t) \qquad S_{xx}(\omega) = q$$

then

$$H(\omega) = \frac{1}{-\omega^2 + jb\omega + c} \qquad S_{yy}(\omega) = \frac{q}{(c - \omega^2)^2 + b^2\omega^2}$$

To find $R_{yy}(\tau)$, we shall consider three cases:

$\underline{b^2 < 4c}$

$$R_{yy}(\tau) = \frac{q}{2bc}e^{-\alpha|\tau|}\left(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta|\tau|\right) \qquad \alpha = \frac{b}{2} \qquad \alpha^2 + \beta^2 = c$$

$\underline{b^2 = 4c}$

$$R_{yy}(\tau) = \frac{q}{2bc}e^{-\alpha|\tau|}(1 + \alpha|\tau|) \qquad \alpha = \frac{b}{2}$$

$\underline{b^2 > 4c}$

$$R_{yy}(\tau) = \frac{q}{4\gamma bc}\left[(\alpha + \gamma)e^{-(\alpha-\gamma)|\tau|} - (\alpha - \gamma)e^{-(\alpha+\gamma)|\tau|}\right]$$

$$\alpha = \frac{b}{2} \qquad \alpha^2 - \gamma^2 = c$$

In all cases, $E\{\mathbf{y}^2(t)\} = q/2bc$.

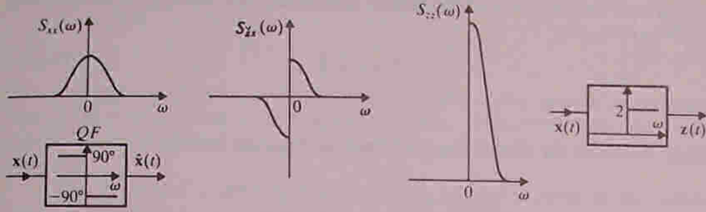

FIGURE 10-14

Example 10-28 Hilbert transforms. A system with system function (Fig. 10-14)

$$H(\omega) = -j\,\mathrm{sgn}\,\omega = \begin{cases} -j & \omega > 0 \\ j & \omega < 0 \end{cases} \tag{10-145}$$

is called a *quadrature filter*. The corresponding impulse response equals $1/\pi t$ (Papoulis, 1977). Thus $H(\omega)$ is all-pass with $-90°$ phase shift; hence its response to $\cos \omega t$ equals $\cos(\omega t - 90°) = \sin \omega t$ and its response to $\sin \omega t$ equals $\sin(\omega t - 90°) = -\cos \omega t$.

The response of a quadrature filter to a real process $\mathbf{x}(t)$ is denoted by $\check{\mathbf{x}}(t)$ and it is called the *Hilbert transform* of $\mathbf{x}(t)$. Thus

$$\check{\mathbf{x}}(t) = \mathbf{x}(t) * \frac{1}{\pi t} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathbf{x}(\alpha)}{t-\alpha}\,d\alpha \tag{10-146}$$

From (10-134) and (10-124) it follows that (Fig. 10-14)

$$\begin{aligned} S_{x\check{x}}(\omega) &= jS_{xx}(\omega)\mathrm{sgn}\,\omega = -S_{\check{x}x}(\omega) \\ S_{\check{x}\check{x}}(\omega) &= S_{xx}(\omega) \end{aligned} \tag{10-147}$$

The complex process

$$\mathbf{z}(t) = \mathbf{x}(t) + j\check{\mathbf{x}}(t)$$

is called the *analytic signal* associated with $\mathbf{x}(t)$. Clearly, $\mathbf{z}(t)$ is the response of the system

$$1 + j(-j\,\mathrm{sgn}\,\omega) = 2U(\omega)$$

with input $\mathbf{x}(t)$. Hence [see (10-136)]

$$S_{zz}(\omega) = 4S_{xx}(\omega)U(\omega) = 2S_{xx}(\omega) + 2jS_{\check{x}x}(\omega) \tag{10-148}$$

$$R_{zz}(\tau) = 2R_{xx}(\tau) + 2jR_{\check{x}x}(\tau) \tag{10-149}$$

THE WIENER–KHINCHIN THEOREM. From (10-121) it follows that

$$E\{\mathbf{x}^2(t)\} = R(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,d\omega \ge 0 \tag{10-150}$$

This shows that the area of the power spectrum of any process is positive. We shall show that

$$S(\omega) \geq 0 \tag{10-151}$$

for every ω.

Proof. We form an ideal bandpass system with system function

$$H(\omega) = \begin{cases} 1 & \omega_1 < \omega < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

and apply $\mathbf{x}(t)$ to its input. From (10-139) it follows that the power spectrum $S_{yy}(\omega)$ of the resulting output $\mathbf{y}(t)$ equals

$$S_{yy}(\omega) = \begin{cases} S(\omega) & \omega_1 < \omega < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$0 \leq E\{\mathbf{y}^2(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(\omega)\, d\omega = \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} S(\omega)\, d\omega \tag{10-152}$$

Thus the area of $S(\omega)$ in any interval is positive. This is possible only if $S(\omega) \geq 0$ everywhere.

We have shown on page 321 that if $S(\omega)$ is a positive function, then we can find a process $\mathbf{x}(t)$ such that $S_{xx}(\omega) = S(\omega)$. From this it follows that a function $S(\omega)$ is a power spectrum iff it is positive. In fact, we can find an exponential with random frequency $\boldsymbol{\omega}$ as in (10-127) with power spectrum an arbitrary positive function $S(\omega)$.

We shall use (10-152) to express the power spectrum $S(\omega)$ of a process $\mathbf{x}(t)$ as the average power of another process $\mathbf{y}(t)$ obtained by filtering $\mathbf{x}(t)$. Setting $\omega_1 = \omega_0 + \delta$ and $\omega_2 = \omega_0 - \delta$, we conclude that if δ is sufficiently small,

$$E\{\mathbf{y}^2(t)\} \simeq \frac{\delta}{\pi} S(\omega_0) \tag{10-153}$$

This shows the *localization* of the average power of $\mathbf{x}(t)$ on the frequency axis.

Integrated spectrum. In mathematics, the spectral properties of a process $\mathbf{x}(t)$ are expressed in terms of the integrated spectrum $F(\omega)$ defined as the integral of $S(\omega)$:

$$F(\omega) = \int_{-\infty}^{\omega} S(\alpha)\, d\alpha \tag{10-154}$$

From the positivity of $S(\omega)$, it follows that $F(\omega)$ is a nondecreasing function ω. Integrating the inversion formula (10-121) by parts, we can express the autocorrelation $R(\tau)$ of $\mathbf{x}(t)$ as a Riemann–Stieltjes integral:

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega\tau}\, dF(\omega) \tag{10-155}$$

This approach avoids the use of singularity functions in the spectral representation of $R(\tau)$ even when $S(\omega)$ contains impulses. If $S(\omega)$ contains the terms $\beta_i\delta(\omega - \omega_i)$, then $F(\omega)$ is discontinuous at ω_i and the discontinuity jump equals β_i.

The integrated covariance spectrum $F^c(\omega)$ is the integral of the covariance spectrum. From (10-125) it follows that $F(\omega) = F^c(\omega) + 2\pi\eta^2 U(\omega)$.

Vector spectra. The vector process $\mathbf{X}(t) = [\mathbf{x}_i(t)]$ is WSS if its components $\mathbf{x}_i(t)$ are jointly WSS. In this case, its autocorrelation matrix depends only on $\tau = t_1 - t_2$. From this it follows that [see (10-116)]

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} R_{xx}(\tau + \alpha) H^{\dagger}(\alpha)\, d\alpha \qquad R_{yy}(\tau) = \int_{-\infty}^{\infty} H(\alpha) R_{xy}(\tau - \alpha)\, d\alpha \tag{10-156}$$

The power spectrum of a WSS vector process $\mathbf{X}(t)$ is a square matrix $S_{xx}(\omega) = [S_{ij}(\omega)]$, the elements of which are the Fourier transforms $S_{ij}(\omega)$ of the elements $R_{ij}(\tau)$ of its autocorrelation matrix $R_{xx}(\tau)$. Defining similarly the matrices $S_{xy}(\omega)$ and $S_{yy}(\omega)$, we conclude from (10-156) that

$$S_{xy}(\omega) = S_{xx}(\omega)\overline{H}^{\dagger}(\omega) \qquad S_{yy}(\omega) = \overline{H}(\omega) S_{xy}(\omega) \tag{10-157}$$

where $\overline{H}(\omega) = [H_{ji}(\omega)]$ is an $m \times r$ matrix with elements the Fourier transforms $H_{ji}(\omega)$ of the elements $h_{ji}(t)$ of the impulse response matrix $H(t)$. Thus

$$S_{yy}(\omega) = \overline{H}(\omega) S_{xx}(\omega) \overline{H}^{\dagger}(\omega) \tag{10-158}$$

This is the extension of (10-136) to a multiterminal system.

Example 10-29. The derivatives

$$\mathbf{y}_1(t) = \mathbf{z}^{(m)}(t) \qquad \mathbf{y}_2(t) = \mathbf{w}^{(n)}(t)$$

of two WSS processes $\mathbf{z}(t)$ and $\mathbf{w}(t)$ can be considered as the responses of two differentiators with inputs $\mathbf{z}(t)$ and $\mathbf{w}(t)$ and system functions $H_1(\omega) = (j\omega)^m$ and $H_2(\omega) = (j\omega)^n$. Proceeding as in (10-119), we conclude that the cross-power spectrum of $\mathbf{z}^{(m)}(t)$ and $\mathbf{w}^{(n)}(t)$ equals $(j\omega)^m(-j\omega)^n S_{zw}(\omega)$. Hence

$$E\{\mathbf{z}^{(m)}(t+\tau)\mathbf{z}^{(n)}(t)\} = (-1)^n \frac{d^{m+n} R_{zw}(\tau)}{d\tau^{m+n}} \tag{10-159}$$

PROPERTIES OF CORRELATIONS. If a function $R(\tau)$ is the autocorrelation of a WSS process $\mathbf{x}(t)$, then [see (10-151)] its Fourier transform $S(\omega)$ is positive. Furthermore, if $R(\tau)$ is a function with positive Fourier transform, we can find a process $\mathbf{x}(t)$ as in (10-126) with autocorrelation $R(\tau)$. Thus a necessary and sufficient condition for a function $R(\tau)$ to be an autocorrelation is the positivity of its Fourier transform. The conditions for a function $R(\tau)$ to be an autocorrelation can be expressed directly in terms of $R(\tau)$. We have shown in (10-84) that the autocorrelation $R(\tau)$ of a process $\mathbf{x}(t)$ is p.d., that is,

$$\sum_{i,j} a_i a_j^* R(\tau_i - \tau_j) \ge 0 \tag{10-160}$$

for every a_i, a_j, τ_i, and τ_j. It can be shown that the converse is also true†: If $R(\tau)$ is a p.d. function, then its Fourier transform is positive. Thus a function $R(\tau)$ has a positive Fourier transform iff it is p.d.

A sufficient condition. To establish whether $R(\tau)$ is p.d., we must show either that it satisfies (10-160) or that its transform is positive. This is not, in general, a simple task. The following is a simple sufficient condition.

Polya's criterion. It can be shown that a function $R(\tau)$ is p.d. if it is concave for $\tau > 0$ and it tends to a finite limit as $\tau \to \infty$.

Consider, for example, the function $w(\tau) = e^{-\alpha|\tau|^c}$. If $0 < c < 1$, then $w(\tau) \to 0$ as $\tau \to \infty$ and $w''(\tau) > 0$ for $\tau > 0$; hence $w(\tau)$ is p.d. because it satisfies Polya's criterion. Note, however, that it is p.d. also for $1 \le c \le 2$ even though it does not satisfy this criterion.

Necessary conditions. The autocorrelation $R(\tau)$ of any process $\mathbf{x}(t)$ is maximum at the origin because [see (10-121)]

$$|R(\tau)| \le \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = R(0) \tag{10-161}$$

We show next that if $R(\tau)$ is not periodic, it reaches its maximum only at the origin.

THEOREM. If $R(\tau_1) = R(0)$ for some $\tau_1 \ne 0$, then $R(\tau)$ is periodic with period τ_1:

$$R(\tau + \tau_1) = R(\tau) \quad \text{for all } \tau \tag{10-162}$$

Proof. From Schwarz's inequality

$$E^2\{\mathbf{zw}\} \le E\{\mathbf{z}^2\}E\{\mathbf{w}^2\} \tag{10-163}$$

it follows that

$$\begin{aligned} &E^2\{[\mathbf{x}(t+\tau+\tau_1) - \mathbf{x}(t+\tau)]\mathbf{x}(t)\} \\ &\qquad \le E\{[\mathbf{x}(t+\tau+\tau_1) - \mathbf{x}(t+\tau)]^2\}E\{\mathbf{x}^2(t)\} \end{aligned}$$

Hence

$$[R(\tau+\tau_1) - R(\tau)]^2 \le 2[R(0) - R(\tau_1)]R(0) \tag{10-164}$$

If $R(\tau_1) = R(0)$, then the right side is 0; hence the left side is also 0 for every τ. This yields (10-162).

†S. Bocher: *Lectures on Fourier Integrals*, Princeton Univ. Press, Princeton, NJ, 1959.

COROLLARY. If $R(\tau_1) = R(\tau_2) = R(0)$ and the numbers τ_1 and τ_2 are noncommensurate, that is, their ratio is irrational, then $R(\tau)$ is constant.

Proof. From the theorem it follows that $R(\tau)$ is periodic with periods τ_1 and τ_2. This is possible only if $R(\tau)$ is constant.

Continuity. If $R(\tau)$ is continuous at the origin, it is continuous for every τ.

Proof. From the continuity of $R(\tau)$ at $\tau = 0$ it follows that $R(\tau_1) \to R(0)$; hence the left side of (10-164) also tends to 0 for every τ as $\tau_1 \to 0$.

Example 10-30. Using the theorem, we shall show that the truncated parabola

$$w(\tau) = \begin{cases} a^2 - \tau^2 & |\tau| < a \\ 0 & |\tau| > a \end{cases}$$

is not an autocorrelation.

If $w(\tau)$ is the autocorrelation of some process $\mathbf{x}(t)$, then [see (10-144)] the function

$$-w''(\tau) = \begin{cases} 2 & |\tau| < a \\ 0 & |\tau| > a \end{cases}$$

is the autocorrelation of $\mathbf{x}'(t)$. This is impossible because $-w''(\tau)$ is continuous for $\tau = 0$ but not for $\tau = a$.

MS continuity and periodicity. We shall say that the process $\mathbf{x}(t)$ is MS continuous if

$$E\{[\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)]^2\} \to 0 \qquad \text{as} \quad \varepsilon \to 0 \tag{10-165}$$

Since $E\{[\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)]^2\} = 2[R(0) - R(\varepsilon)]$, we conclude that if $\mathbf{x}(t)$ is MS continuous, $R(0) - R(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus a WSS process $\mathbf{x}(t)$ is MS continuous iff its autocorrelation $R(\tau)$ is continuous for all τ.

We shall say that the process $\mathbf{x}(t)$ is MS periodic with period τ_1 if

$$E\{[\mathbf{x}(t+\tau_1) - \mathbf{x}(t)]^2\} = 0 \tag{10-166}$$

Since the left side equals $2[R(0) - R(\tau_1)]$, we conclude that $R(\tau_1) = R(0)$; hence [see (10-162)] $R(\tau)$ is periodic. This leads to the conclusion that a WSS process $\mathbf{x}(t)$ is MS periodic iff its autocorrelation is periodic.

Cross-correlation. Using (10-163), we shall show that the cross-correlation $R_{xy}(\tau)$ of two WSS processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ satisfies the inequality

$$R_{xy}^2(\tau) \le R_{xx}(0)\, R_{yy}(0) \tag{10-167}$$

Proof. From (10-163) it follows that

$$E^2\{\mathbf{x}(t+\tau)\mathbf{y}^*(t)\} \le E\{|\mathbf{x}(t+\tau)|^2\} E\{|\mathbf{y}(t)|^2\} = R_{xx}(0)\, R_{yy}(0)$$

and (10-167) results.

COROLLARY. For any a and b,

$$\left|\int_a^b S_{xy}(\omega)\,d\omega\right|^2 \le \int_a^b S_{xx}(\omega)\,d\omega \int_a^b S_{yy}(\omega)\,d\omega \qquad (10\text{-}168)$$

Proof. Suppose that $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are the inputs to the ideal filters

$$H_1(\omega) = H_2(\omega) = \begin{cases} 1 & a < \omega < b \\ 0 & \text{otherwise} \end{cases}$$

Denoting by $\mathbf{z}(t)$ and $\mathbf{w}(t)$ respectively the resulting outputs, we conclude that

$$R_{zz}(0) = \frac{1}{2\pi}\int_a^b S_{xx}(\omega)\,d\omega \qquad R_{ww}(0) = \frac{1}{2\pi}\int_a^b S_{yy}(\omega)\,d\omega$$

$$R_{zw}(0) = \frac{1}{2\pi}\int_a^b S_{zw}(\omega)\,d\omega$$

and (10-168) follows because $R_{zw}^2(0) \le R_{zz}(0)R_{ww}(0)$.

10-4 DIGITAL PROCESSES

A digital (or discrete-time) process is a sequence $\mathbf{x}_n$ or RVs. To avoid double subscripts, we shall use also the notation $\mathbf{x}[n]$ where the brackets will indicate that n is an integer. Most results involving analog (or continuous-time) processes can be readily extended to digital processes. We outline the main concepts.

The autocorrelation and autocovariance of $\mathbf{x}[n]$ are given by

$$R[n_1, n_2] = E\{\mathbf{x}[n_1]\mathbf{x}^*[n_2]\} \qquad C[n_1, n_2] = R[n_1, n_2] - \eta[n_1]\eta^*[n_2] \qquad (10\text{-}169)$$

respectively where $\eta[n] = E\{\mathbf{x}[n]\}$ is the mean of $\mathbf{x}[n]$.

A process $\mathbf{x}[n]$ is SSS if its statistical properties are invariant to a shift of the origin. It is WSS if $\eta[n] = \eta =$ constant and

$$R[n+m, n] = E\{\mathbf{x}[n+m]\mathbf{x}^*[n]\} = R[m] \qquad (10\text{-}170)$$

A process $\mathbf{x}[n]$ is strictly white noise if the RVs $\mathbf{x}[n_i]$ are independent. It is white noise if the RVs $\mathbf{x}[n_i]$ are uncorrelated. The autocorrelation of a white-noise process with zero mean is thus given by

$$R[n_1, n_2] = q[n_1]\delta[n_1 - n_2] \qquad \text{where} \quad \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \ne 0 \end{cases} \qquad (10\text{-}171)$$

and $q[n] = E\{\mathbf{x}^2[n]\}$. If $\mathbf{x}[n]$ is also stationary, then $R[m] = q\delta[m]$. Thus a WSS white noise is a sequence of i.i.d. RVs with variance q.

The delta response $h[n]$ of a linear system is its response to the delta sequence $\delta[n]$. Its system function is the z transform of $h[n]$:

$$\mathbf{H}(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \qquad (10\text{-}172)$$

If $\mathbf{x}[n]$ is the input to a digital system, the resulting output is the digital convolution of $\mathbf{x}[n]$ with $h[n]$:

$$\mathbf{y}[n] = \sum_{k=-\infty}^{\infty} \mathbf{x}[n-k]h[k] = \mathbf{x}[n] * h[n] \tag{10-173}$$

From this it follows that $\eta_y[n] = \eta_x[n] * h[n]$. Furthermore,

$$R_{xy}[n_1, n_2] = \sum_{k=-\infty}^{\infty} R_{xx}[n_1, n_2 - k]h^*[k] \tag{10-174}$$

$$R_{yy}[n_1, n_2] = \sum_{r=-\infty}^{\infty} R_{xy}[n_1 - r, n_2]h[r] \tag{10-175}$$

If $\mathbf{x}[n]$ is white noise with average intensity $q[n]$ as in (10-171), then, [see (10-90)],

$$E\{\mathbf{y}^2[n]\} = q[n] * |h[n]|^2 \tag{10-176}$$

If $\mathbf{x}[n]$ is WSS, then $\mathbf{y}[n]$ is also WSS with $\eta_y = \eta_x = \mathbf{H}(1)$. Furthermore,

$$R_{xy}[m] = R_{xx}[m] * h^*[-m] \qquad R_{yy}[m] = R_{xy}[m] * h[m]$$

$$R_{yy}[m] = R_{xx}[m] * \rho[m] \qquad \rho[m] = \sum_{k=-\infty}^{\infty} h[m+k]h^*[k] \tag{10-177}$$

as in (10-133) and (10-135).

THE POWER SPECTRUM. Given a WSS process $\mathbf{x}[n]$, we form the z transform $\mathbf{S}(z)$ of its autocorrelation $R[m]$:

$$\mathbf{S}(z) = \sum_{m=-\infty}^{\infty} R[m]z^{-m} \tag{10-178}$$

The power spectrum of $\mathbf{x}[n]$ is the function

$$S(\omega) = \mathbf{S}(e^{j\omega}) = \sum_{m=-\infty}^{\infty} R[m]e^{-jm\omega} \tag{10-179}$$

Thus $\mathbf{S}(e^{j\omega})$ is the DFT of $R[m]$. The function $\mathbf{S}(e^{j\omega})$ is periodic with period 2π and Fourier series coefficients $R[m]$. Hence

$$R[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{S}(e^{j\omega})e^{jm\omega}\,d\omega \tag{10-180}$$

It suffices, therefore, to specify $\mathbf{S}(e^{j\omega})$ for $|\omega| < \pi$ only (see Fig. 10-15).

If $\mathbf{x}[n]$ is a real process, then $R[-m] = R[m]$ and (10-179) yields

$$\mathbf{S}(e^{j\omega}) = R[0] + 2\sum_{m=0}^{\infty} R[m]\cos m\omega \tag{10-181}$$

This shows that the power spectrum of a real process is a function of $\cos\omega$ because $\cos m\omega$ is a function of $\cos\omega$.

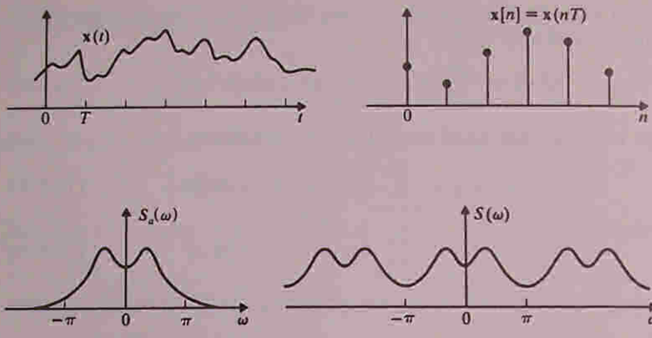

FIGURE 10-15

Example 10-31. If $R[m] = a^{|m|}$, then

$$\mathbf{S}(z) = \sum_{m=-\infty}^{-1} a^{-m} z^{-m} + \sum_{m=0}^{\infty} a^m z^{-m} = \frac{az}{1-az} + \frac{z}{z-a}$$

$$= \frac{a^{-1} - a}{(a^{-1} + a) - (z^{-1} + z)}$$

Hence

$$\mathbf{S}(e^{j\omega}) = \frac{a^{-1} - a}{a^{-1} + a - 2\cos\omega}$$

Example 10-32. Proceeding as in the analog case, we can show that the process

$$\mathbf{x}[n] = \sum_i \mathbf{c}_i e^{j\omega_i n}$$

is WSS iff the coefficients $\mathbf{c}_i$ are uncorrelated with zero mean. In this case,

$$R[m] = \sum_i \sigma_i^2 e^{j\beta_i |m|} \qquad S(\omega) = 2\pi \sum_i \sigma_i^2 \delta(\omega - \beta_i) \qquad |\omega| < \pi \quad (10\text{-}182)$$

where $\sigma_i^2 = E\{\mathbf{c}_i^2\}$, $\omega_i = 2\pi k_i + \beta_i$, and $|\beta_i| < \pi$.

From (10-177) and the convolution theorem, it follows that if $\mathbf{y}[n]$ is the output of a linear system with input $\mathbf{x}[n]$, then

$$\mathbf{S}_{xy}(e^{j\omega}) = \mathbf{S}_{xx}(e^{j\omega})\mathbf{H}^*(e^{j\omega})$$
$$\mathbf{S}_{yy}(e^{j\omega}) = \mathbf{S}_{xy}(e^{j\omega})\mathbf{H}(e^{j\omega}) \qquad (10\text{-}183)$$
$$\mathbf{S}_{yy}(e^{j\omega}) = \mathbf{S}_{xx}(e^{j\omega})|\mathbf{H}(e^{j\omega})|^2$$

If $h[n]$ is real, $\mathbf{H}^*(e^{j\omega}) = \mathbf{H}(e^{-j\omega})$. In this case

$$\mathbf{S}_{yy}(z) = \mathbf{S}_{xx}(z)\mathbf{H}(z)\mathbf{H}(1/z) \qquad (10\text{-}184)$$

Example 10-33. The first difference

$$\mathbf{y}[n] = \mathbf{x}[n] - \mathbf{x}[n-1]$$

of a process $\mathbf{x}[n]$ can be considered as the output of a linear system with input $\mathbf{x}[n]$ and system function $\mathbf{H}(z) = 1 - z^{-1}$. Applying (10-184), we obtain

$$\mathbf{S}_{yy}(z) = \mathbf{S}_{xx}(z)(1 - z^{-1})(1 - z) = \mathbf{S}_{xx}(z)(2 - z - z^{-1})$$
$$R_{yy}[m] = -R_{xx}[m+1] + 2R_{xx}[m] - R_{xx}[m-1]$$

If $\mathbf{x}[n]$ is white noise with $\mathbf{S}_{xx}(z) = q$, then

$$\mathbf{S}_{yy}(e^{j\omega}) = q(2 - e^{j\omega} - e^{-j\omega}) = 2q(1 - \cos\omega)$$

Example 10-34. The recursion equation

$$\mathbf{y}[n] - a\mathbf{y}[n-1] = \mathbf{x}[n]$$

specifies a linear system with input $\mathbf{x}[n]$ and system function $\mathbf{H}(z) = 1/(1 - az^{-1})$. If $\mathbf{S}_{xx}(z) = q$, then (see Example 10-31)

$$\mathbf{S}_{yy}(z) = \frac{q}{(1 - az^{-1})(1 - az)} \qquad R_{yy}[m] = \frac{q}{a^{-1} - a} a^{|m|}$$

From (10-183) it follows that

$$E\{|\mathbf{y}[n]|^2\} = R_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{S}_{xx}(e^{j\omega})|\mathbf{H}(e^{j\omega})|^2 d\omega \qquad (10\text{-}185)$$

Using this identity, we shall show that the power spectrum of a process $\mathbf{x}[n]$ real or complex is a positive function:

$$\mathbf{S}_{xx}(e^{j\omega}) \geq 0 \qquad (10\text{-}186)$$

Proof. We form an ideal bandpass filter with center frequency ω_0 and bandwidth 2Δ and apply (10-185). For small Δ,

$$E\{|\mathbf{y}[n]|^2\} = \frac{1}{2\pi}\int_{\omega_0-\Delta}^{\omega_0+\Delta} \mathbf{S}_{xx}(e^{j\omega})\,d\omega \simeq \frac{\Delta}{\pi}\mathbf{S}_{xx}(e^{j\omega_0})$$

and (10-186) results because $E\{\mathbf{y}^2[n]\} \geq 0$ and ω_0 is arbitrary.

SAMPLING. In many applications, the digital processes under consideration are obtained by sampling various analog processes. We relate next the corresponding correlations and spectra.

Given an analog process $\mathbf{x}(t)$, we form the digital process

$$\mathbf{x}[n] = \mathbf{x}(nT)$$

where T is a given constant. From this it follows that

$$\eta[n] = \eta_a(nT) \qquad R[n_1, n_2] = R_a(n_1 T, n_2 T) \tag{10-187}$$

where $\eta_a(t)$ is the mean and $R_a(t_1, t_2)$ the autocorrelation of $\mathbf{x}(t)$. If $\mathbf{x}(t)$ is a stationary process, then $\mathbf{x}[n]$ is also stationary with mean $\eta = \eta_a$ and autocorrelation

$$R[m] = R_a(mT)$$

From this it follows that the power spectrum of $\mathbf{x}[n]$ equals (Fig. 10-15)

$$\mathbf{S}(e^{j\omega}) = \sum_{m=-\infty}^{\infty} R_a(mT)e^{-jm\omega} = \frac{1}{T}\sum_{n=-\infty}^{\infty} S_a\left(\frac{\omega + 2\pi n}{T}\right) \tag{10-188}$$

where $S_a(\omega)$ is the power spectrum of $\mathbf{x}(t)$. The above is a consequence of Poisson's sum formula [see (11A-1)].

Example 10-35. Suppose that $\mathbf{x}(t)$ is a WSS process consisting of M exponentials as in (10-130):

$$\mathbf{x}(t) = \sum_{i=1}^{M} \mathbf{c}_i e^{j\omega_i t} \qquad S_a(\omega) = 2\pi \sum_{i=1}^{M} \sigma_i^2 \delta(\omega - \omega_i)$$

where $\sigma_i^2 = E\{\mathbf{c}_i^2\}$. We shall determine the power spectrum $\mathbf{S}(e^{j\omega})$ of the process $\mathbf{x}[n] = \mathbf{x}(nT)$. From (10-188) it follows that

$$\mathbf{S}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \sum_{i=1}^{M} \sigma_i^2(\omega - \omega_i + 2\pi n)$$

In the interval $(-\pi, \pi)$, this consists of M lines:

$$\mathbf{S}(e^{j\omega}) = \sum_{i=1}^{M} \sigma_i^2 \delta(\omega - \beta_1) \qquad |\omega| < \pi \qquad |\beta_i| < \pi$$

where β_i are such that $\omega_i = 2\pi n_i + \beta_i$.

APPENDIX 10A
CONTINUITY, DIFFERENTIATION, INTEGRATION

In the earlier discussion, we routinely used various limiting operations involving stochastic processes, with the tacit assumption that these operations hold for every sample involved. This assumption is, in many cases, unnecessarily restrictive. To give some idea of the notion of limits in a more general case, we discuss next conditions for the existence of MS limits and we show that these conditions can be phrased in terms of second-order moments (see also Sec. 8-4).

STOCHASTIC CONTINUITY. A process $\mathbf{x}(t)$ is called MS continuous if

$$E\{[\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)]^2\} \underset{\varepsilon \to 0}{\longrightarrow} 0 \tag{10A-1}$$

THEOREM. We maintain that $\mathbf{x}(t)$ is MS continuous if its autocorrelation is continuous.

Proof. Clearly,

$$E\{[\mathbf{x}(t+\varepsilon)-\mathbf{x}(t)]^2\} = R(t+\varepsilon,t+\varepsilon) - 2R(t+\varepsilon,t) + R(t,t)$$

If, therefore, $R(t_1,t_2)$ is continuous, then the right side tends to 0 as $\varepsilon \to 0$ and (10A-1) results.

Note Suppose that (10A-1) holds for *every* t in an interval I. From this it follows that [see (10-1)] almost all samples of $\mathbf{x}(t)$ will be continuous at a *particular* point of I. It does not follow, however, that these samples will be continuous for *every* point in I. We mention as illustrations the Poisson process and the Wiener process. As we see from (10-14) and (11-5), both processes are MS continuous. However, the samples of the Poisson process are discontinuous at the points $\mathbf{t}_i$, whereas almost all samples of the Wiener process are continuous.

COROLLARY. If $\mathbf{x}(t)$ is MS continuous, then its mean is continuous

$$\eta(t+\varepsilon) \to \eta(t) \qquad \varepsilon \to 0 \tag{10A-2}$$

Proof. As we know

$$E\{[\mathbf{x}(t+\varepsilon)-\mathbf{x}(t)]^2\} \ge E^2\{[\mathbf{x}(t+\varepsilon)-\mathbf{x}(t)]\}$$

Hence (10A-2) follows that (10A-1).

The above shows that

$$\lim_{\varepsilon\to 0} E\{\mathbf{x}(t+\varepsilon)\} = E\Big\{\lim_{\varepsilon\to 0}\mathbf{x}(t+\varepsilon)\Big\} \tag{10A-3}$$

STOCHASTIC DIFFERENTIATION. A process $\mathbf{x}(t)$ is MS differentiable if

$$\frac{\mathbf{x}(t+\varepsilon)-\mathbf{x}(t)}{\varepsilon} \xrightarrow[\varepsilon\to 0]{} \mathbf{x}'(t) \tag{10A-4}$$

in the MS sense, that is, if

$$E\left\{\left[\frac{\mathbf{x}(t+\varepsilon)-\mathbf{x}(t)}{\varepsilon} - \mathbf{x}'(t)\right]^2\right\} \xrightarrow[\varepsilon\to 0]{} 0 \tag{10A-5}$$

THEOREM. The process $\mathbf{x}(t)$ is MS differentiable if $\partial^2 R(t_1,t_2)/\partial t_1\,\partial t_2$ exists.

Proof. It suffices to show that (Cauchy criterion)

$$E\left\{\left[\frac{\mathbf{x}(t+\varepsilon_1)-\mathbf{x}(t)}{\varepsilon_1} - \frac{\mathbf{x}(t+\varepsilon_2)-\mathbf{x}(t)}{\varepsilon_2}\right]^2\right\} \xrightarrow[\varepsilon_1,\varepsilon_2\to 0]{} 0 \tag{10A-6}$$

We use this criterion because, unlike (10A-5), it does not involve the unknown

$\mathbf{x}'(t)$. Clearly,

$$E\{[\mathbf{x}(t+\varepsilon_1)-\mathbf{x}(t)][\mathbf{x}(t+\varepsilon_2)-\mathbf{x}(t)]\}$$
$$= R(t+\varepsilon_1, t+\varepsilon_2) - R(t+\varepsilon_1, t) - R(t, t+\varepsilon_2) + R(t,t)$$

The right side divided by $\varepsilon_1\varepsilon_2$ tends to $\partial^2 R(t,t)/\partial t\,\partial t$ which, by assumption, exists. Expanding the square in (10A-6), we conclude that its left side tends to

$$\frac{\partial^2 R(t,t)}{\partial t\,\partial t} - 2\frac{\partial^2 R(t,t)}{\partial t\,\partial t} + \frac{\partial^2 R(t,t)}{\partial t\,\partial t} = 0$$

COROLLARY. The above yields

$$E\{\mathbf{x}'(t)\} = E\left\{\lim_{\varepsilon\to 0}\frac{\mathbf{x}(t+\varepsilon)-\mathbf{x}(t)}{\varepsilon}\right\} = \lim_{\varepsilon\to 0} E\left\{\frac{\mathbf{x}(t+\varepsilon)-\mathbf{x}(t)}{\varepsilon}\right\}$$

Note The autocorrelation of a Poisson process $\mathbf{x}(t)$ is discontinuous at the points $\mathbf{t}_i$; hence $\mathbf{x}'(t)$ does not exist at these points. However, as in the case of deterministic signals, it is convenient to introduce random impulses and to interpret $\mathbf{x}'(t)$ as in (10-98).

STOCHASTIC INTEGRALS. A process $\mathbf{x}(t)$ is MS integrable if the limit

$$\int_a^b \mathbf{x}(t)\,dt = \lim_{\Delta t_i \to 0}\sum_i \mathbf{x}(t_i)\,\Delta t_i \tag{10A-7}$$

exists in the MS sense.

THEOREM. The process $\mathbf{x}(t)$ is MS integrable if

$$\int_a^b\int_a^b |R(t_1,t_2)|\,dt_1\,dt_2 < \infty \tag{10A-8}$$

Proof. Using again the Cauchy criterion, we must show that

$$E\left\{\left|\sum_i \mathbf{x}(t_i)\,\Delta t_i - \sum_k \mathbf{x}(t_k)\,\Delta t_k\right|^2\right\} \xrightarrow[\Delta t_i,\,\Delta t_k\to 0]{} 0$$

This follows if we expand the square and use the identity

$$E\left\{\sum_i \mathbf{x}(t_i)\,\Delta t_i \sum_k \mathbf{x}(t_k)\,\Delta t_k\right\} = \sum_{i,k} R(t_i,t_k)\,\Delta t_i, \Delta t_k$$

because the right side tends to the integral of $R(t_1,t_2)$ as Δt_i and Δt_k tend to 0.

COROLLARY. From the above it follows that

$$E\left\{\left|\int_a^b \mathbf{x}(t)\,dt\right|^2\right\} = \int_a^b\int_a^b R(t_1,t_2)\,dt_1\,dt_2 \tag{10A-9}$$

as in (10-11).

APPENDIX 10B
SHIFT OPERATORS AND STATIONARY PROCESSES

An SSS process can be generated by a succession of shifts $T\mathbf{x}$ of a single RV $\mathbf{x}$ where T is a one-to-one measure preserving transformation (mapping) of the probability space $\mathscr{S}$ into itself. This difficult topic is of fundamental importance in mathematics. In the following, we give a brief explanation of the underlying concept, limiting the discussion to the discrete-time case.

A *transformation* T of $\mathscr{S}$ into itself is a rule for assigning to each element ζ_i of $\mathscr{S}$ another element of $\mathscr{S}$:

$$\bar{\zeta}_i = T\zeta_i \tag{10B-1}$$

called the *image* of ζ_i. The images $\bar{\zeta}_i$ of all elements ζ_i of a subset $\mathscr{A}$ of $\mathscr{S}$ form another subset

$$\bar{\mathscr{A}} = T\mathscr{A}$$

of $\mathscr{S}$ called the image of $\mathscr{A}$.

We shall assume that the transformation T has the following properties.

P_1: It is one-to-one. This means that

$$\text{if} \quad \zeta_i \neq \zeta_j \qquad \text{then} \quad \bar{\zeta}_i \neq \bar{\zeta}_j$$

P_2: It is measure preserving. This means that if $\mathscr{A}$ is an event, then its image $\bar{\mathscr{A}}$ is also an event and

$$P(\bar{\mathscr{A}}) = P(\mathscr{A}) \tag{10B-2}$$

Suppose that $\mathbf{x}$ is an RV and that T is a transformation as above. The expression $T\mathbf{x}$ will mean another RV

$$\mathbf{y} = T\mathbf{x} \qquad \text{such that} \quad \mathbf{y}(\bar{\zeta}_i) = \mathbf{x}(\zeta_i) \tag{10B-3}$$

where ζ_i is the unique inverse of $\bar{\zeta}_i$. This specifies $\mathbf{y}$ for every element of $\mathscr{S}$ because (see P_1) the set of elements $\bar{\zeta}_i$ equals $\mathscr{S}$.

The expression $\mathbf{z} = T^{-1}\mathbf{x}$ will mean that $\mathbf{x} = T\mathbf{z}$. Thus

$$\mathbf{z} = T^{-1}\mathbf{x} \qquad \text{iff} \quad \mathbf{z}(\zeta_i) = \mathbf{x}(\bar{\zeta}_i)$$

We can define similarly $T^2\mathbf{x} = T(T\mathbf{x}) = T\mathbf{y}$ and

$$T^n\mathbf{x} = T(T^{n-1}\mathbf{x}) = T^{-1}(T^{n+1}\mathbf{x})$$

for any n positive or negative.

From (10B-3) it follows that if, for some ζ_i, $\mathbf{x}(\zeta_i) \le w$, then $\mathbf{y}(\bar{\zeta}_i) = \mathbf{x}(\zeta_i) \le w$. Hence the event $\{\mathbf{y} \le w\}$ is the image of the event $\{\mathbf{x} \le w\}$. This yields [see (10B-2)]

$$P\{\mathbf{x} \le w\} = P\{\mathbf{y} \le w\} \qquad \mathbf{y} = T\mathbf{x} \tag{10B-4}$$

for any w. We thus conclude that the RVs $\mathbf{x}$ and $T\mathbf{x}$ have the same distribution $F_x(x)$.

Given an RV $\mathbf{x}$ and a transformation T as above, we form the random process

$$\mathbf{x}_0 = \mathbf{x} \qquad \mathbf{x}_n = T^n\mathbf{x} \qquad n = -\infty, \ldots, \infty \tag{10B-5}$$

It follows from (10B-4) that the random variables $\mathbf{x}_n$ so formed have the same distribution. We can similarly show that their joint distributions of any order are invariant to a shift of the origin. Hence the process $\mathbf{x}_n$ so formed is SSS.

It can be shown that the converse is also true: Given an SSS process $\mathbf{x}_n$, we can find an RV $\mathbf{x}$ and a one-to-one measuring preserving transformation of the space $\mathscr{S}$ into itself such that for all essential purposes, $\mathbf{x}_n = T^n\mathbf{x}$. The proof of this difficult result will not be given.

PROBLEMS

10-1. In the fair-coin experiment, we define the process $\mathbf{x}(t)$ as follows: $\mathbf{x}(t) = \sin \pi t$ if heads shows, $\mathbf{x}(t) = 2t$ if tails shows. (*a*) Find $E\{\mathbf{x}(t)\}$. (*b*) Find $F(x, t)$ for $t = 0.25$, $t = 0.5$, and $t = 1$.

10-2. The process $\mathbf{x}(t) = e^{\mathbf{a}t}$ is a family of exponentials depending on the RV $\mathbf{a}$. Express the mean $\eta(t)$, the autocorrelation $R(t_1, t_2)$, and the first-order density $f(x, t)$ of $\mathbf{x}(t)$ in terms of the density $f_a(a)$ of $\mathbf{a}$.

10-3. Suppose that $\mathbf{x}(t)$ is a Poisson process as in Fig. 10-3 such that $E\{\mathbf{x}(9)\} = 6$. (*a*) Find the mean and the variance of $\mathbf{x}(8)$. (*b*) Find $P\{\mathbf{x}(2) \le 3\}$. (*c*) Find $P\{\mathbf{x}(4) \le 5 | \mathbf{x}(2) \le 3\}$.

10-4. The RV $\mathbf{c}$ is uniform in the interval $(0, T)$. Find $R_x(t_1, t_2)$ if (*a*) $\mathbf{x}(t) = U(t - \mathbf{c})$, (*b*) $\mathbf{x}(t) = \delta(t - \mathbf{c})$.

10-5. The RVs $\mathbf{a}$ and $\mathbf{b}$ are independent $N(0; \sigma)$ and p is the probability that the process $\mathbf{x}(t) = \mathbf{a} - \mathbf{b}t$ crosses the t axis in the interval $(0, T)$. Show that $\pi p = \arctan T$.

Hint: $p = P\{0 \le \mathbf{a}/\mathbf{b} \le T\}$.

10-6. Show that if

$$R_v(t_1, t_2) = q(t_1)\delta(t_1 - t_2)$$

$\mathbf{w}''(t) = \mathbf{v}(t)U(t)$ and $\mathbf{w}(0) = \mathbf{w}'(0) = 0$, then

$$E\{\mathbf{w}^2(t)\} = \int_0^t (t - \tau)q(\tau)\,d\tau$$

10-7. The process $\mathbf{x}(t)$ is real with autocorrelation $R(\tau)$. (*a*) Show that

$$P\{|\mathbf{x}(t + \tau) - \mathbf{x}(t)| \ge a\} \le 2[R(0) - R(\tau)]/a^2$$

(*b*) Express $P\{|\mathbf{x}(t + \tau) - \mathbf{x}(t)| \ge a\}$ in terms of the second-order density $f(x_1, x_2; \tau)$ of $\mathbf{x}(t)$.

10-8. The process $\mathbf{x}(t)$ is WSS and normal with $E\{\mathbf{x}(t)\} = 0$ and $R(\tau) = 4e^{-2|\tau|}$. (*a*) Find $P\{\mathbf{x}(t) \le 3\}$. (*b*) Find $E\{[\mathbf{x}(t + 1) - \mathbf{x}(t - 1)]^2\}$.

10-9. Show that the process $\mathbf{x}(t) = \mathbf{c}w(t)$ is WSS iff $E\{\mathbf{c}\} = 0$ and $w(t) = e^{j(\omega t + \theta)}$.

10-10. The process $\mathbf{x}(t)$ is WSS and $E\{\mathbf{x}(t)\} = 0$. Show that if $\mathbf{z}(t) = \mathbf{x}^2(t)$, then $C_{zz}(\tau) = 2C_{xx}^2(\tau)$.

10-11. Find $E\{y(t)\}$, $E\{y^2(t)\}$, and $R_{yy}(\tau)$ if

$$y''(t) + 4y(t) + 13y(t) = 26 + \nu(t) \qquad R_{\nu\nu}(\tau) = 10\delta(\tau)$$

Find $P\{y(t) \le 3\}$ if $\nu(t)$ is normal.

10-12. Show that: If $x(t)$ is a process with zero mean and autocorrelation $f(t_1)f(t_2)w(t_1 - t_2)$, then the process $y(t) = x(t)/f(t)$ is WSS with autocorrelation $w(\tau)$. If $x(t)$ is white noise with autocorrelation $q(t_1)\delta(t_1 - t_2)$, then the process $z(t) = x(t)/\sqrt{q(t)}$ is WSS white noise with autocorrelation $\delta(\tau)$.

10-13. Show that $|R_{xy}(\tau)| \le \frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$.

10-14. Show that if the processes $x(t), y(t)$ are WSS and $E\{|x(0) - y(0)|^2\} = 0$, then $R_{xx}(\tau) \equiv R_{xy}(\tau) \equiv R_{yy}(\tau)$.

Hint: Set $z = x(t + \tau)$, $w = x^*(t)) - y^*(t)$ in (10-163).

10-15. Show that if $x(t)$ is a complex WSS process, then

$$E\{|x(t + \tau) - x(t)|^2\} = 2\,\text{Re}[R(0) - R(\tau)]$$

10-16. Show that if φ is an RV with $\Phi(\lambda) = E\{e^{j\lambda\varphi}\}$ and $\Phi(1) = \Phi(2) = 0$, then the process $x(t) = \cos(\omega t + \varphi)$ is WSS. Find $E\{x(t)\}$ and $R_x(\tau)$ if φ is uniform in the interval $(-\pi, \pi)$.

10-17. Given a process $x(t)$ with orthogonal increments and such that $x(0) = 0$, show that (a) $R(t_1, t_2) = R(t_1, t_1)$ for $t_1 \le t_2$, and (b) if $E\{[x(t_1) - x(t_2)]^2\} = q|t_1 - t_2|$ then the process $y(t) = [x(t + \varepsilon) - x(t)]/\varepsilon$ is WSS and its autocorrelation is a triangle with area q and base 2ε.

10-18. Show that if $R_{xx}(t_1, t_2) = q(t_1)\delta(t_1 - t_2)$ and $y(t) = x(t) * h(t)$ then

$$E\{x(t)y(t)\} = h(0)q(t)$$

10-19. The process $x(t)$ is normal with $\eta_x = 0$ and $R_x(\tau) = 4e^{-3|\tau|}$. Find a memoryless system $g(x)$ such that the first-order density $f_y(y)$ of the resulting output $y(t) = g[x(t)]$ is uniform in the interval $(6, 9)$.

Answer: $g(x) = 3\mathbb{G}(x/2) + 6$.

10-20. Show that if $x(t)$ is an SSS process and ε is an RV independent of $x(t)$, then the process $y(t) = x(t - \varepsilon)$ is SSS.

10-21. Show that if $x(t)$ is a stationary process with derivative $x'(t)$, then for a given t the RVs $x(t)$ and $x'(t)$ are orthogonal and uncorrelated.

10-22. Given a normal process $x(t)$ with $\eta_x = 0$ and $R_x(\tau) = 4e^{-2|\tau|}$, we form the RVs $z = x(t + 1)$, $w = x(t - 1)$, (a) find $E\{zw\}$ and $E\{(z + w)^2\}$, (b) find

$$f_z(z) \qquad P\{z < 1\} \qquad f_{zw}(z, w)$$

10-23. Show that if $x(t)$ is normal with autocorrelation $R(\tau)$, then

$$P\{x'(t) \le a\} = \mathbb{G}\left[\frac{a}{\sqrt{-R''(0)}}\right]$$

10-24. Show that if $x(t)$ is a normal process with zero mean and $y(t) = \text{sgn}\,x(t)$, then

$$R_y(\tau) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[J_0(n\pi) - (-1)^n\right]\sin\left[n\pi\frac{R_x(\tau)}{R_x(0)}\right]$$

where $J_0(x)$ is the Bessel function.

Hint: Expand the arcsine in (10-71) into a Fourier series.

10-25. Show that if $\mathbf{x}(t)$ is a normal process with zero mean and $\mathbf{y}(t) = Ie^{a\mathbf{x}(t)}$, then

$$\eta_y = = I\exp\left\{\frac{a^2}{2}R_x(0)\right\} \qquad R_y(\tau) = I^2\exp\{a^2[R_x(0) + R_x(\tau)]\}$$

10-26. Show that (*a*) if

$$\mathbf{y}(t) = a\mathbf{x}(ct) \qquad \text{then} \quad R_y(\tau) = a^2R_x(c\tau)$$

(*b*) if $R_x(\tau) \to 0$ as $\tau \to \infty$ and

$$\mathbf{z}(t) = \lim_{\varepsilon\to\infty}\sqrt{\varepsilon}\,\mathbf{x}(\varepsilon t) \qquad \text{then} \quad R_z(\tau) = q\delta(\tau) \qquad q = \int_{-\infty}^{\infty}R_x(\tau)\,d\tau$$

10-27. Show that if $\mathbf{x}(t)$ is white noise, $h(t) = 0$ outside the interval $(0, T)$, and $\mathbf{y}(t) = \mathbf{x}(t) * h(t)$ then $R_{yy}(t_1, t_2) = 0$ for $|t_1 - t_2| > T$.

10-28. Show that if

$$R_{xx}(t_1, t_2) = q(t_1)\delta(t_1 - t_2) \qquad E\{\mathbf{y}^2(t)\} = I(t)$$

and

$$(a) \qquad \mathbf{y}(t) = \int_0^t h(t,\alpha)\mathbf{x}(\alpha)\,d\alpha \qquad \text{then} \quad I(t) = \int_0^t h^2(t,\alpha)q(\alpha)\,d\alpha$$

$$(b) \qquad \mathbf{y}'(t) + c(t)\mathbf{y}(t) = \mathbf{x}(t) \qquad \text{then} \quad I'(t) + 2c(t)I(t) = q(t)$$

10-29. Find $E\{\mathbf{y}^2(t)\}$ (*a*) if $R_{xx}(t) = 5\delta(\tau)$ and

$$\mathbf{y}'(t) + 2\mathbf{y}(t) = \mathbf{x}(t) \qquad \text{all} \quad t \tag{i}$$

(*b*) if (i) holds for $t > 0$ only and $\mathbf{y}(t) = 0$ for $t \le 0$.

Hint: Use (10-90).

10-30. The input to a linear system with $h(t) = Ae^{-\alpha t}U(t)$ is a process $\mathbf{x}(t)$ with $R_x(\tau) = N\delta(\tau)$ applied at $t = 0$ and disconnected at $t = T$. Find and sketch $E\{\mathbf{y}^2(t)\}$.

Hint: Use (10-90) with $q(t) = N$ for $0 < t < T$ and 0 otherwise.

10-31. Show that if

$$\mathbf{s} = \int_0^{10}\mathbf{x}(t)\,dt \qquad \text{then} \quad E\{\mathbf{s}^2\} = \int_{-10}^{10}(10 - |\tau|)R_x(\tau)\,d\tau$$

Find the mean and variance of $\mathbf{s}$ if $E\{\mathbf{x}(t)\} = 8$, $R_x(\tau) = 64 + 10e^{-2|\tau|}$.

10-32. The process $\mathbf{x}(t)$ is WSS with $R_{xx}(\tau) = 5\delta(\tau)$ and

$$\mathbf{y}'(t) + 2\mathbf{y}(t) = \mathbf{x}(t) \tag{i}$$

Find $E\{\mathbf{y}^2(t)\}$, $R_{xy}(t_1, t_2)$, $R_{yy}(t_1, t_2)$ (*a*) if (i) holds for all t, (*b*) if $\mathbf{y}(0) = 0$ and (i) holds for $t \ge 0$.

10-33. Find $S(\omega)$ if (*a*) $R(\tau) = e^{-\alpha\tau^2}$, (*b*) $R(\tau) = e^{-\alpha\tau^2}\cos\omega_0\tau$.

10-34. Show that the power spectrum of an SSS process $\mathbf{x}(t)$ equals

$$S(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1x_2G(x_1, x_2; \omega)\,dx_1\,dx_2$$

where $G(x_1, x_2; \omega)$ is the Fourier transform in the variable τ of the second-order density $f(x_1, x_2; \tau)$ of $\mathbf{x}(t)$.

10-35. Show that if $\mathbf{y}(t) = \mathbf{x}(t + a) - \mathbf{x}(t - a)$, then

$$R_y(\tau) = 2R_x(\tau) - R_x(\tau + 2a) - R_x(\tau - 2a) \qquad S_y(\omega) = 4S_x(\omega)\sin^2 a\omega$$

10-36. Using (10-122), show that

$$R(0) - R(\tau) \geq \frac{1}{4^n}[R(0) - R(2^n\tau)]$$

Hint:

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2} \geq 2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} = \frac{1}{4}(1 - \cos 2\theta)$$

10-37. The process $\mathbf{x}(t)$ is normal with zero mean and $R_x(\tau) = Ie^{-\alpha|\tau|}\cos\beta\tau$. Show that if $\mathbf{y}(t) = \mathbf{x}^2(t)$, then $C_y(\tau) = I^2e^{-2\alpha|\tau|}(1 + \cos 2\beta\tau)$. Find $S_y(\omega)$.

10-38. Show that if $R(\tau)$ is the inverse Fourier transform of a function $S(\omega)$ and $S(\omega) \geq 0$, then, for any a_i,

$$\sum_{i,k} a_i a_k^* R(\tau_i - \tau_k) \geq 0$$

Hint:

$$\int_{-\infty}^{\infty} S(\omega)\left|\sum_i a_i e^{j\omega\tau_i}\right|^2 d\omega \geq 0$$

10-39. Find $R(\tau)$ if (*a*) $S(\omega) = 1/(1 + \omega^4)$, (*b*) $S(\omega) = 1/(4 + \omega^2)^2$.

10-40. Show that, for complex systems, (10-136) and (10-181) yield

$$\mathbf{S}_{yy}(s) = \mathbf{S}_{xx}(s)\mathbf{H}(s)\mathbf{H}^*(-s^*) \qquad \mathbf{S}_{yy}(z) = \mathbf{S}_{xx}(z)\mathbf{H}(z)\mathbf{H}^*(1/z^*)$$

10-41. The process $\mathbf{x}(t)$ is normal with zero mean. Show that if $\mathbf{y}(t) = \mathbf{x}^2(t)$, then

$$S_y(\omega) = 2\pi R_x^2(0)\delta(\omega) + 2S_x(\omega) * S_x(\omega)$$

Plot $S_y(\omega)$ if $S_x(\omega)$ is (*a*) ideal LP, (*b*) ideal BP.

10-42. The process $\mathbf{x}(t)$ is WSS with $E\{\mathbf{x}(t)\} = 5$ and $R_{xx}(\tau) = 25 + 4e^{-2|\tau|}$. If $\mathbf{y}(t) = 2\mathbf{x}(t) + 3\mathbf{x}'(t)$, find η_y, $R_{yy}(\tau)$, and $S_{yy}(\omega)$.

10-43. The process $\mathbf{x}(t)$ is WSS and $R_{xx}(\tau) = 5\delta(\tau)$. (*a*) Find $E\{\mathbf{y}^2(t)\}$ and $S_{yy}(\omega)$ if $\mathbf{y}'(t) + 3\mathbf{y}(t) = \mathbf{x}(t)$. (*b*) Find $E\{\mathbf{y}^2(t)\}$ and $R_{xy}(t_1, t_2)$ if $\mathbf{y}'(t) + 3\mathbf{y}(t) = \mathbf{x}(t)U(t)$. Sketch the functions $R_{xy}(2, t_1)$ and $R_{xy}(t_1, 3)$.

10-44. Given a complex process $\mathbf{x}(t)$ with autocorrelation $R(\tau)$, show that if $|R(\tau_1)| = 1$, then

$$R(\tau) = e^{j\omega\tau}w(\tau) \qquad \mathbf{x}(t) = e^{j\omega t}\mathbf{y}(t)$$

where $w(\tau)$ is a periodic function with period τ_1 and $\mathbf{y}(t)$ is an MS periodic process with the same period.

10-45. Show that (*a*) $E\{\mathbf{x}(t)\check{\mathbf{x}}(t)\} = 0$, (*b*) $\check{\check{\mathbf{x}}}(t) = -\mathbf{x}(t)$.

10-46. (*Stochastic resonance*) The input to the system

$$\mathbf{H}(s) = \frac{1}{s^2 + 2s + 5}$$

is a WSS process $\mathbf{x}(t)$ with $E\{\mathbf{x}^2(t)\} = 10$. Find $S_x(\omega)$ such that the average power $E\{\mathbf{y}^2(t)\}$ of the resulting output $\mathbf{y}(t)$ is maximum.

Hint: $|\mathbf{H}(j\omega)|$ is maximum for $\omega = \sqrt{3}$.

10-47. Show that if $R_x(\tau) = Ae^{j\omega_0\tau}$, then $R_{xy}(\tau) = Be^{j\omega_0\tau}$ for any $\mathbf{y}(t)$.

Hint: Use (10-167).

10-48. Given a system $H(\omega)$ with input $\mathbf{x}(t)$ and output $\mathbf{y}(t)$, show that (*a*) if $\mathbf{x}(t)$ is WSS and $R_{xx}(\tau) = e^{j\alpha\tau}$, then

$$R_{yx}(\tau) = e^{j\alpha\tau}H(\alpha) \qquad R_{yy}(\tau) = e^{j\alpha\tau}|H(\alpha)|^2$$

(*b*) if $R_{xx}(t_1, t_2) = e^{j(\alpha t_1 - \beta t_2)}$, then

$$R_{yx}(t_1, t_2) = e^{j(\alpha t_1 - \beta t_2)}H(\alpha) \qquad R_{yy}(t_1, t_2) = e^{j(\alpha t_1 - \beta t_2)}H(\alpha)H^*(\beta)$$

10-49. Show that if $S_{xx}(\omega)S_{yy}(\omega) \equiv 0$, then $S_{xy}(\omega) \equiv 0$.

10-50. Show that if $\mathbf{x}[n]$ is WSS and $R_x[1] = R_x[0]$, then $R_x[m] = R_x[0]$ for every m.

10-51. Show that if $R[m] = E\{\mathbf{x}[n+m]\mathbf{x}[n]\}$, then

$$R[0]R[2] > 2R^2[1] - R^2[0]$$

10-52. Given an RV $\boldsymbol{\omega}$ with density $f(\omega)$ such that $f(\omega) = 0$ for $|\omega| > \pi$, we form the process $\mathbf{x}[n] = Ae^{jn\boldsymbol{\omega}\pi}$. Show that $S_x(\omega) = 2\pi A^2 f(\omega)$ for $|\omega| < \pi$.

10-53. (*a*) Find $E\{\mathbf{y}^2(t)\}$ if $\mathbf{y}(0) = \mathbf{y}'(0) = 0$ and

$$\mathbf{y}''(t) + 7\mathbf{y}'(t) + 10\mathbf{y}(t) = \mathbf{x}(t) \qquad R_x(\tau) = 5\delta(\tau)$$

(*b*) Find $E\{\mathbf{y}^2[n]\}$ if $\mathbf{y}[-1] = \mathbf{y}[-2] = 0$ and

$$8\mathbf{y}[n] - 6\mathbf{y}[n-1] + \mathbf{y}[n-2] = \mathbf{x}[n] \qquad R_x[m] = 5\delta[m]$$

10-54. The process $\mathbf{x}[n]$ is WSS with $R_{xx}[m] = 5\delta[m]$ and

$$\mathbf{y}[n] - 0.5\mathbf{y}[n-1] = \mathbf{x}[n] \qquad \text{(i)}$$

Find $E\{\mathbf{y}^2[n]\}$, $R_{xy}[m_1, m_2]$, $R_{yy}[m_1, m_2]$ (*a*) if (i) holds for all n, (*b*) if $\mathbf{y}[-1] = 0$ and (i) holds for $n \ge 0$.

10-55. Show that (*a*) if $R_x[m_1, m_2] = q[m_1]\delta[m_1 - m_2]$ and

$$\mathbf{s} = \sum_{n=0}^{N} a_n\mathbf{x}[n] \quad \text{then} \quad E\{\mathbf{s}^2\} = \sum_{n=0}^{N} a_n^2 q[n]$$

(*b*) If $R_{xx}(t_1, t_2) = q(t_1)\delta(t_1 - t_2)$ and

$$\mathbf{s} = \int_0^T a(t)\mathbf{x}(t)\,dt \quad \text{then} \quad E\{\mathbf{s}^2\} = \int_0^T a^2(t)q(t)\,dt$$