Point and Renewal Processes:

① * Point process a set of random points t_i on the time axis

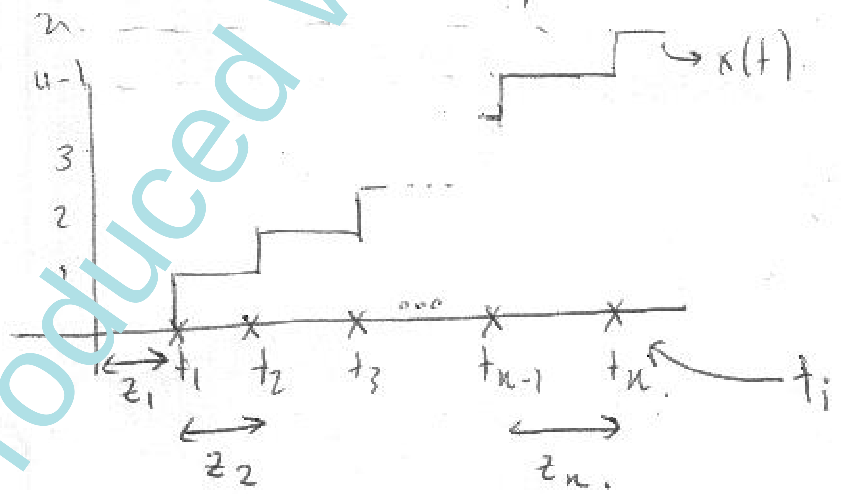
② * To every point process, we can assign a r.p. $x(t)$ such that, $x(t)$ corresponds to number of points in $(0, t)$.

③ *. To every point process, we can also assign a sequence of r.v.'s z_n s.t.

$$z_1 = t_1 \; ; \quad z_2 = t_2 - t_1 \; , \; \text{———} \; , \; z_N = t_N - t_{N-1}$$

z_n : corresponds to time between n^{th} and $(n-1)^{th}$ point.

It's called a renewal process.



t_i: blow-up time of a light bulp. (light bulps are immediately replaced)

$x(t)$: # of bulps that has ~~been~~ ~~replaced~~ blown up up to time t

z_n: life-time of bulp n.

t_i: arrival time of packet i,
$x(t)$: # of packets arrived until time $= t$.
z_n: inter-arrival time between $(n-1)^{st}$ and n^{th} packet.

If $(x(t_k) - x(t_l))$ is independent from $(x(t_m) - x(t_n))$ where (t_k, t_l) and (t_m, t_n) are non-intersecting intervals; then process $x(t)$ is called independent increments.

Stationary Processes:

A process is called stationary if its prob. definition is independent of time-origin.

Strict Sense Stationary:

$$\underset{\underset{\sim}{x}(t_1), \underset{\sim}{x}(t_2), \ldots, \underset{\sim}{x}(t_N)}{f(x_1, x_2, \ldots, x_N)} = \underset{\underset{\sim}{x}(t_1+c), \underset{\sim}{x}(t_2+c), \ldots, \underset{\sim}{x}(t_N+c)}{f(x_1, \ldots, x_N)} \quad \forall c$$

(Should be satisfied for N^{th} order description, when N can be arbitrary large)

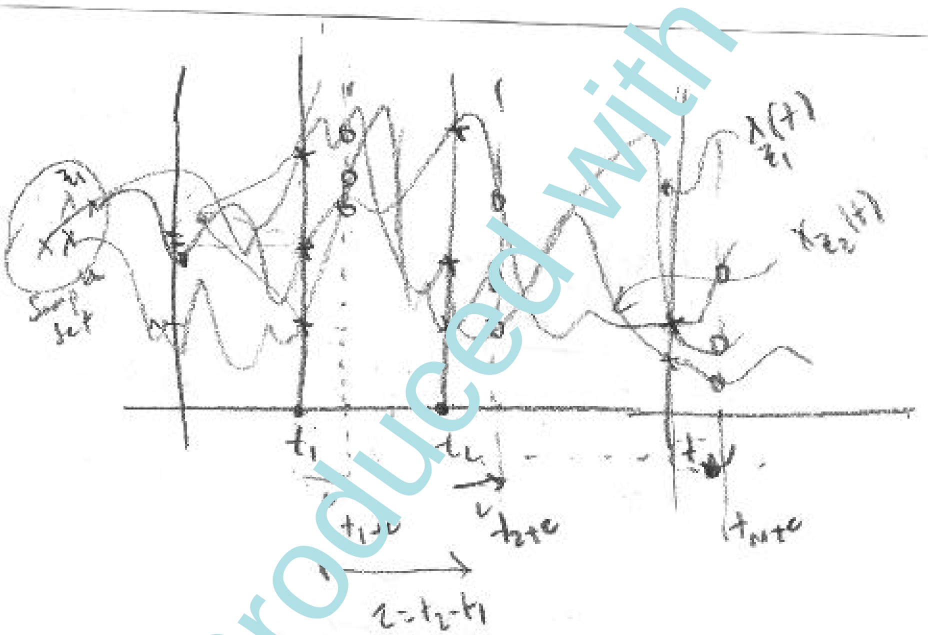
1^{st} Order Stationary: $(N=1)$

$$f_{x(t_1)}(x_1) = f_{\underset{\sim}{x}(t_1+c)}(x_1) = f_{x(t_n)}(x) \quad \forall c, \forall t_1$$

$\uparrow$ any t

2^{nd} Order Stationary: $(N=2)$

$$f_{x(t_1)x(t_2)}(x_1, x_2) = f_{x(t_1+c)\,x(t_2+c)}(x_1, x_2) \quad \forall c, \ \forall t_1, t_2.$$



2.5

1st - 2nd Order Stationarity

3

$$f_{x(t_1)x(t_2)}(x_1,x_2) = f_{x(0)x(\underbrace{t_2-t_1}_{\tau})}(x_1,x_2) \qquad \forall\, t_1, t_2$$

<u>Wide Sense Stationarity:</u> (Stationarity of moments of 2nd order description)

① First order moment.

$$E\{x(t)\} = E\{x(t+\tau)\} = m_x.$$

② Second order moment

$$E\{x(t_1)x(t_2)\} = E\{x(0)\,x(\underbrace{t_2-t_1}_{\tau})\} = R_x(\tau)$$

a function of time difference $t_2 - t_1 = \text{lag}$.

From these two, we can define

i) $C_x(\tau) = R_x(\tau) - m_x^2 \leftarrow$ Auto Covariance

ii) $r(\tau) = \dfrac{C(\tau)}{C(0)} \leftarrow$ correlation coef for the lag $=\tau$.

If a process is SSS $\longrightarrow$ WSS.
$\not\longleftarrow$

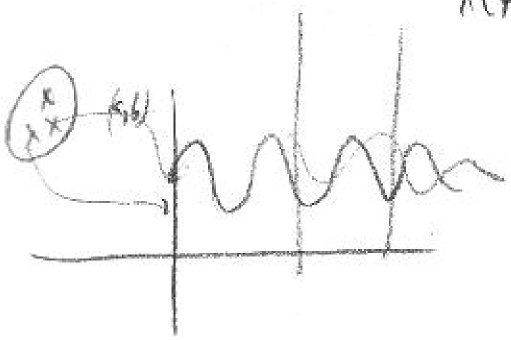
Important Exception:

If a normal process (→ N: Gaussian) WSS $\longrightarrow$ SSS.

Since mean and auto-covariance functions completely define a normal process; if the process is invariant to time shifts on mean and on auto-covariance (WSS) $\Rightarrow$ SSS.

Ex: Examine the conditions for WSS and SSS of the following process:

$$x(t) = \underset{\sim}{a}\cos\omega t + \underset{\sim}{b}\sin\omega t$$



(0) $E\{x(t)\} \Rightarrow$ should be independent of time.

$$E\{x(t_\Delta)\} = E\{a\}\cos\omega t_\Delta + E\{b\}\sin\omega t_\Delta \longrightarrow E\{a\} = E\{b\} = 0$$

* We assume that $E\{a\} = E\{b\} = 0$ for both stationarity cases to be examined.

(1) **WSS:** $\xrightarrow{\text{from}}$ (0) $\rightarrow m_x(t)$ independent of time.

$\rightarrow$ $R(0) = E\{(x(0))^2\} = E\left\{\left[x\left(\frac{\pi}{2\omega}\right)\right]^2\right\}$

$$R(0) = E\{a^2\} = E\{b^2\} \xrightarrow[\text{be satisfied}]{\text{should}} E\{a^2\} = E\{b^2\} \triangleq \sigma^2$$

Then.

$$R_x(\tau) = E\{x(t+\tau)\,x(t)\}$$

$$= E\{[a\cos(\omega(t+\tau)) + b\sin(\omega(t+\tau))][a\cos\omega t + b\sin\omega t]\}$$

$$= \overbrace{E\{a^2\}}^{\sigma^2}\cos(\omega(t+\tau))\cos(\omega t) + \overbrace{E\{b^2\}}^{\sigma^2}\sin(\omega(t+\tau))\sin\omega t +$$

$$E\{ab\}[\cos(\omega(t+\tau))\sin\omega t + \cos\omega t\,\sin(\omega(t+\tau))]$$

$$= \sigma^2[\overbrace{\cos(\omega(t+\tau))\cos\ldots + \sin(\omega(t+\tau))\sin\omega t}^{\cos(\omega\tau)}]$$

$$+$$

$$E\{ab\}[\sin(\omega(2t+\tau))]$$

$$R_x(\tau) = \sigma^2\cos(\omega\tau) + E\{ab\}\sin(\omega(2t+\tau))$$

t dependence should be eliminated for WSS.

then $E\{ab\}=0$.

Then: $E\{a\}=E\{b\}=E\{ab\}=0 \longrightarrow x(t)$ WSS.

$E\{a^2\}=E\{b^2\}=\sigma^2$

SSS: The process $x(t)$ is SSS, iff $\longleftrightarrow$ the joint p.d.f of a, b has circular symmetry.

$$f_{\underset{\sim}{a}\,\underset{\sim}{b}}(a,b) = f\left(\sqrt{a^2+b^2}\right)$$

two functions are different

Proof:

① If $x(t)$ SSS $\longrightarrow$ circular symmetric $f(a,b)$.

$$\begin{bmatrix} x(t) \\ x(t+\pi/2\omega) \end{bmatrix} \begin{array}{l} z_1 \\ z_2 \end{array} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$x(t) = a\cos\underset{(\omega t+\frac{\pi}{2})}{\omega t} + b\sin\underset{\omega t+\frac{\pi}{2}}{\omega t}$

$x(t+\frac{\pi}{2\omega}) = -a\sin\omega t + b\cos\omega t$

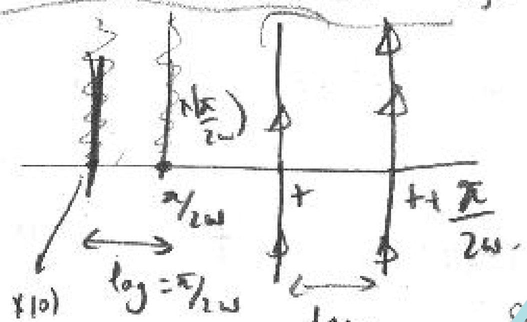
4.7

Since
$x(t)$ is SSS ; the joint p.d.f of $x(t), x(t+\frac{\pi}{2w})$
should be independent of time. (t independent variable).

$$f_{z_1 z_2}(z_1, z_2) = f_{a,b}(a,b) \qquad (2)$$

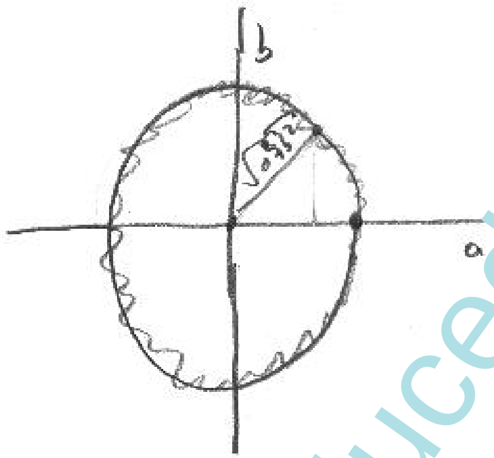
→ joint dist at $x(0), x(\frac{\pi}{2w})$ (shaded lines)

$x(0) = a$
$x(\pi/2w) = b$



$$f_{z_1,z_2}(z_1,z_2) = f_{ab}\left(\begin{bmatrix} \cos wt & -\sin wt \\ \sin wt & \cos wt \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right) \qquad (1)$$

Rotation matrix



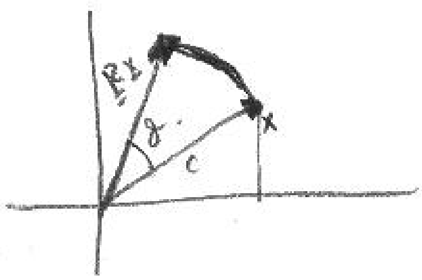
$$\Rightarrow f_{z_1,z_2}(1,0) = f_{ab}(\cos wt, \sin wt)$$

Since (1) is the rotation of f_{ab} by "wt" radians ; and (1) says that after rotation the distribution ($f_{z_1 z_2} = f_{a,b}$) is equal to the before rotation dist. → Circularly Symmetric

Matrices :

$$\underline{R} = \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

$\underline{R} \cdot \underline{x} =$
↑ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.



$$\underline{R}\,\underline{x} = \begin{bmatrix} x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta \end{bmatrix}.$$

$$\|\underline{R}\underline{x}\| = \|\underline{x}\|$$

2) If circularly symmetric $\longrightarrow$ SSS?

4.8

Call

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} \cos w\tau & \sin w\tau \\ -\sin w\tau & \cos w\tau \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$x_1(t) = a_1 \cos wt + b_1 \sin wt = x(t+\tau)$ (by $\cos(A+B)$ expansion)
$\sin(A+B)$

then statistics of $x(t+\tau)$ and $x(t)$ is the same for $\forall \tau$.

then joint p.d.f is the same for all lags τ.