

EE 201 Lecture Notes

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Chapter 1

Small Signal Analysis

1.1 Introducing Small Signal Approximation

In various engineering problems, there can be a small amplitude AC signal accompanied with a much larger amplitude DC signal. For example, the DC source (a battery) can be used to bias the transistors of a radio. In addition to the DC voltages, the signal received from the antenna having a range on the order of millivolts also appears in the system. The design goal of such a system can be the amplification of the signal of interest to a reasonable voltage level.

In this section, we present the small signal approximation for resistive circuits. This topic is particularly important for the amplifier design applications.

A very good engineering question is how we define the smallness. It should be remembered that, a small voltage for an high voltage engineer is any voltage less than 1000 V; while any voltage greater than 1 V can be beyond the reach of an electronic engineer. Therefore smallness and largeness is only applicable with in a context and we should always keep in mind that the adjectives such as small, large, fast, efficient are used for a comparison with something else. So whenever somebody tosses you an adjective, ask him or her with respect to what!

For our purposes, a signal is considered to be small if the accompanying DC signal is at least an order of magnitude larger. This is just a rule of thumb. A small signal example can be given as $v_s(t) = 3 + 0.1 \cos(\omega t)$ V. Figure 1.1 shows how this signal should like on an oscilloscope screen.

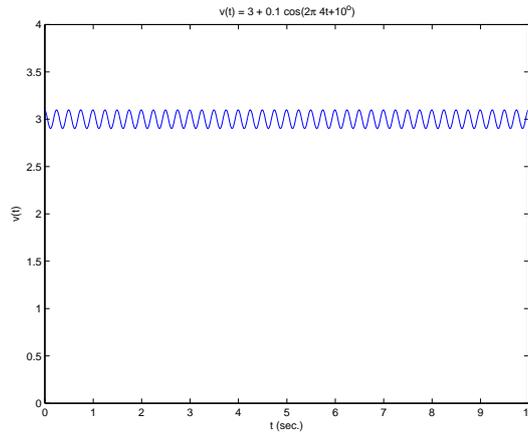


Figure 1.1: A typical input for small signal analysis.

1.2 Small Signal Analysis

We present the topic through an example. We are given the circuit shown in Figure 1.2 for analysis. The circuit contains a non-linear component.

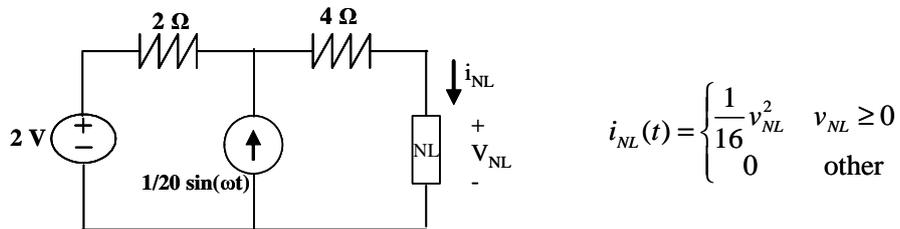


Figure 1.2: An example for small signal analysis

The DC source in the circuit has the amplitude of 2 V and the AC source has the amplitude of 1/20 V. The small signal assumption could be valid for the given circuit. We use the word “could”, since the voltage variation (or the amount of voltage swing) across the non-linear component is critical for the success of small signal approximation. We have more to say on this topic at the later parts of this chapter.

To proceed ahead, we find the Thevenin equivalent circuit seen by the non-linear component and then re-check whether the small signal approximation is applicable or not. Without much difficulty, we can get the Thevenin circuit as in Figure 1.3. The Thevenin circuit shows that the AC component of input seen by the non-linear element is reduced to $0.1 \sin(\omega t)$. Therefore the voltage swing over the non-linear element is limited to 0.2 volts in peak-to-peak.

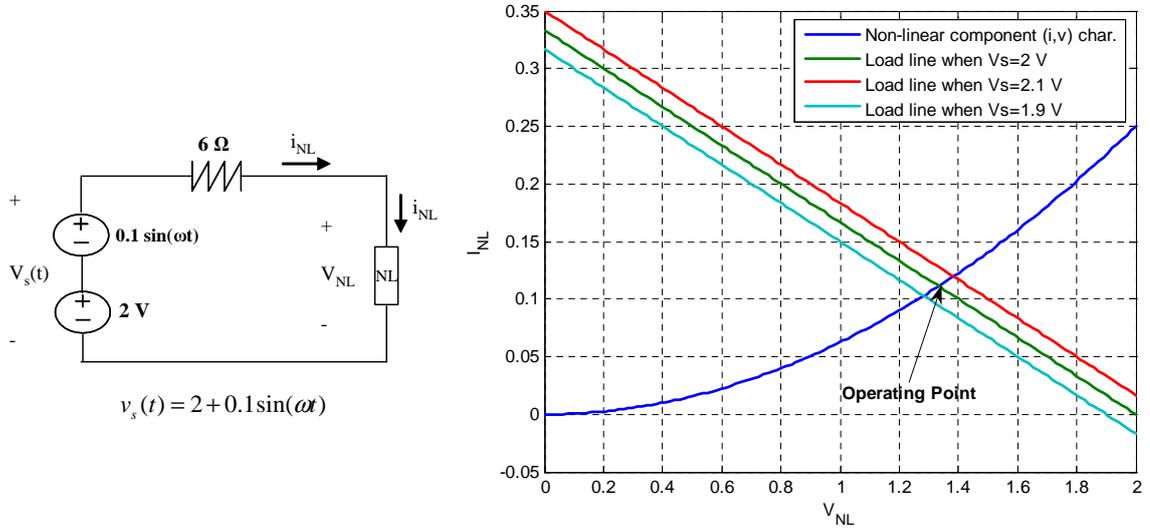


Figure 1.3: Left: The equivalent circuit seen by the nonlinear element. Right: Operating characteristics of the non-linear element and the load lines.

From Figure 1.3, the i_{NL} can be written in two different ways:

$$(1.1) \quad i_{NL} \stackrel{\text{(load line)}}{=} \frac{v_s(t) - V_{NL}}{6} \stackrel{\text{(non-linear comp.)}}{=} \begin{cases} \frac{1}{16} V_{NL}^2, & V_{NL} \geq 0 \\ 0, & \text{other} \end{cases}$$

The first equality (with the label “load line”) is written by noting the circuit configuration. It is clear that for any V_{NL} voltage, the Thevenin circuit outputs i_{NL} amount of current. The second equality follows from the operating curve or the (i, v) characteristics of the non-linear component. As you should remember from earlier chapters (!!!), the (i, v) characteristics show the valid current-voltage pairs of that component.

By solving for V_{NL} from (1.1), we can get the solution of the circuit for any t . At this point, the problem can be considered as solved; but we prefer to spend some more energy on the problem and examine the implications of the smallness on the solution.

In Figure 1.3, the (i, v) characteristic of the non-linear component (blue curve) and three straight lines are given. The straight lines are called the load lines. If the non-linear element is considered as the load then i_{NL} is the load current and the load current can be written as

$$(1.2) \quad i_{NL} \stackrel{\text{(load line)}}{=} \frac{v_s(t) - V_{NL}}{6}$$

Here $v_s(t) = 2 + 0.1 \sin(\omega t)$, as also noted in Figure 1.3. When $v_s(t) = 2$ V, that is at the time instants of $t = \{0, \pi/\omega, 2\pi/\omega, 3\pi/\omega, \dots\}$, the load line is the one in the middle (shown with green). For these time instants, the solution for the (V_{NL}, i_{NL}) is the intersection of the blue curve and the green curve. This point is marked as the operating point.

The input can take the maximum value of $v_s(t) = 2.1$ V. The maximum is achieved for $t = \pi/2 + \{0, 2\pi/\omega, 4\pi/\omega, 6\pi/\omega, \dots\}$. The load line at these time instants is shown with the red color (the top line among the three). The third line with cyan color is the load line for $v_s(t) = 1.9$ V for $t = 3\pi/2 + \{0, 2\pi/\omega, 4\pi/\omega, 6\pi/\omega, \dots\}$.

It is clear that for any time instant, the solution for (V_{NL}, i_{NL}) is a point in between the intersection of (cyan-blue) and (red-blue) lines. The solution is a periodic function with a period identical to the period of the input.

Let's exactly calculate the solution for $v_s(t) = 2$ V. According to Figure 1.3, the solution should be roughly $V_{NL} \approx 1.3$, $i_{NL} \approx 0.12$. In order to get the exact values, we use (1.1). Let's assume that V_{NL} at the solution is greater than zero. Then the equation (1.1) reduces to

$$(1.3) \quad i_{NL} = \frac{2 - V_{NL}}{6} = \frac{1}{16} V_{NL}^2, \quad \text{provided that } V_{NL} \geq 0.$$

From the last equation, we can write $\frac{3}{8} V_{NL}^2 + V_{NL} - 2 = 0$ and find the candidate solutions as $V_{NL} = \{-4, \frac{4}{3}\}$. Since it is assumed that $V_{NL} \geq 0$, $V_{NL} = \frac{4}{3}$ is the only possible solution.

When $V_{NL} = 4/3$, i_{NL} becomes $\frac{1}{9}$. Our initial guesses from read from graph turned out to be quite good. By repeating the process for every possible V_s in $[1.9, 2.1]$, we can get the solution for all t values. Even tough, this approach is feasible with a digital calculator; it is not very informative especially when you need to have a rough understanding of the problem which is highly desired in the design problems.

Small Signal Approximation: The approximation is based on the following idea. The input $v_s(t)$ has a narrow voltage swing, for that voltage swing it can be possible to approximate the non-linear function with a simpler function.

The small signal approximation uses the tangent line at the operating point as the approximation. In other words, the non-linear characteristics is approximated with a line tangent to the characteristics at the operating point.

The tangent approximation can be calculated as follows:

$$(1.4) \quad \hat{i}_{NL} = i_{NL}^{OP} + \left. \frac{df(V_{NL})}{dV_{NL}} \right|_{V_{NL}=V_{NL}^{OP}} (\hat{V}_{NL} - V_{NL}^{OP})$$

The variables with hat show the (i, v) characteristics of the approximation. It should

be that by calculus veterans that this equation is the equation of the line tangent to the curve at the point of $(V_{NL}^{OP}, i_{NL}^{OP})$.

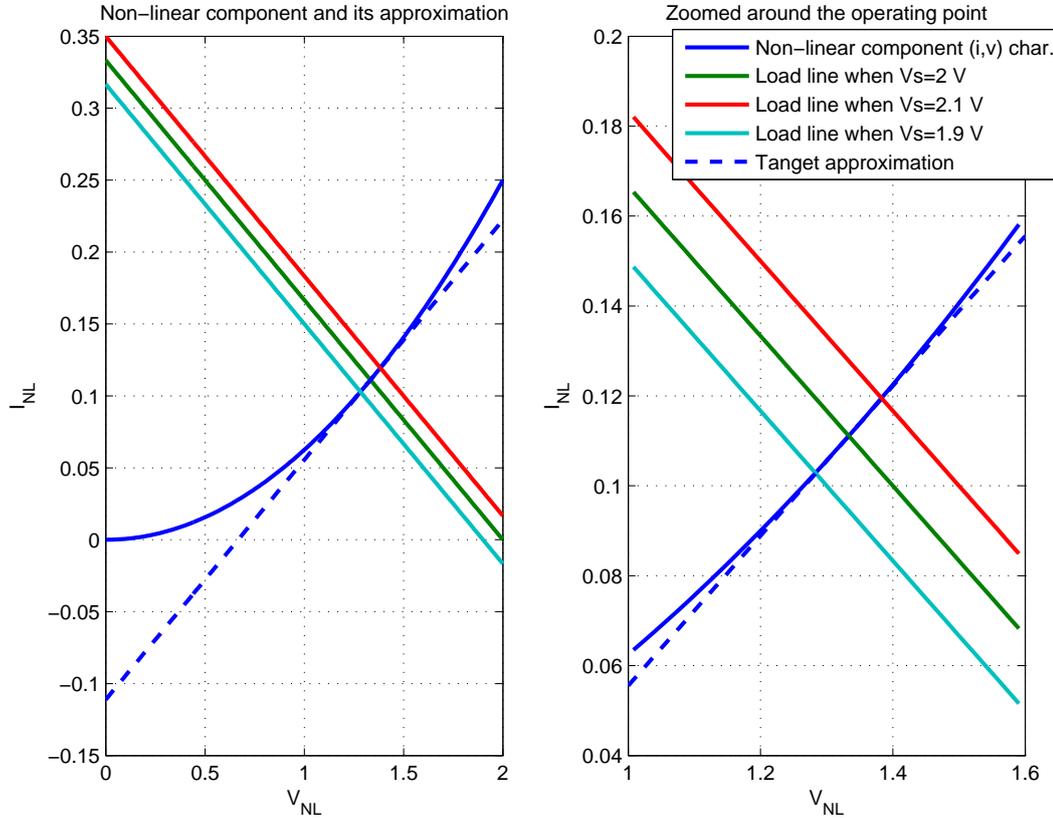


Figure 1.4: Left: The non-linear element and its approximation. Right: The approximation around the operating point.

The dotted line in Figure 1.4 shows the tangent approximation. In our example, the operating point is $(V_{NL}^{OP}, i_{NL}^{OP}) = (\frac{4}{3}, \frac{1}{9})$. The tangent to the function $\frac{1}{16}V_{NL}^2$ at the point of $V_{NL} = \frac{4}{3}$ has the slope of $\frac{1}{6}$. Hence, the dotted line can be formed by writing the equation of a line having a slope of $\frac{1}{6}$ and passing through $(\frac{4}{3}, \frac{1}{9})$.

It should be noted from the right hand side of Figure 1.4 that the approximation is remarkably accurate. This is due to smoothness of the non-linear function. Polynomials such as quadratic functions are “nice” functions in terms of approximations. The only other function nicer than the polynomials is the constant function in the sense of approximation. (Here is the pedestrian way of sorting functions in the increasing order of smoothness: \mathcal{L}^p functions, bounded functions, continuous functions, functions satisfying Lipschitz condition, differentiable functions, n-times differentiable functions, infinitely differentiable functions, analytic functions, entire functions, polynomials of restricted degree and constants, [1, Sec1.4].)

With the tangent approximation, it is possible to write the solution by finding the intersection of the straight lines. This is much easier than solving non-linear equations, but we suggest even a better approach. To show this approach, we first find an equivalent circuit component for the line approximating the non-linearity.

Figure 1.5 shows the non-linear (i, v) characteristics and its approximation. The question is “What is the circuit component having the (i, v) characteristics shown with the dotted lines ?” This should be familiar question for the readers of earlier chapters. The component is a resistor in series with a voltage source. The value of the voltage source is the x-axis intercept. (Why?)

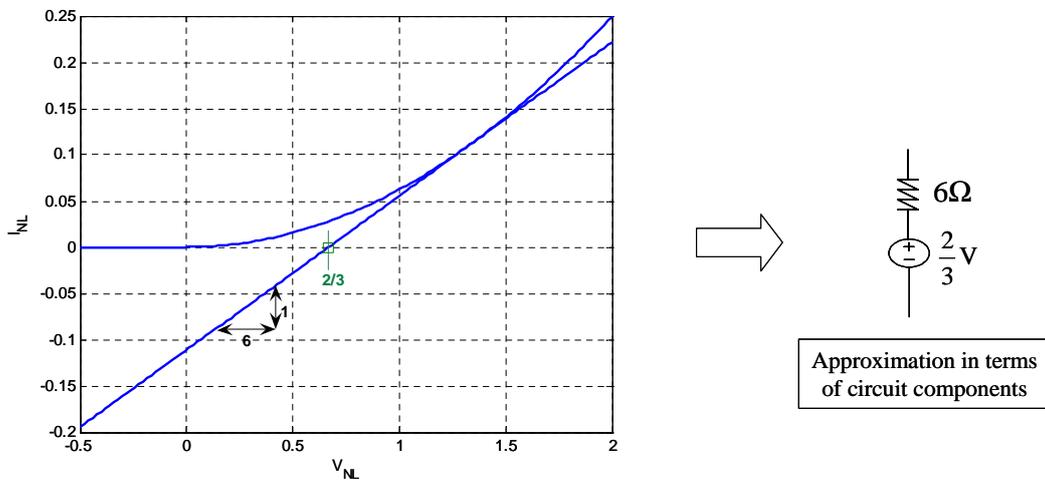


Figure 1.5: Left: The non-linearity and its tangent approximation. Right: The circuit representation of the approximation

Now we are done. Figure 1.6 shows the small signal analysis steps. At the first step, we replace the non-linear element with its small signal approximation (the tangent approximation). Once the non-linearity is replaced, we are left with a good looking circuit. This circuit can be analyzed using any method, but we would like to emphasize the application of superposition principle as shown in Figure 1.6.

The superposition is applied for the DC and AC sources. The solution for the DC part is $i_{NL}(\text{DC}) = \frac{(2-2/3)}{12} = 1/9$ and $v_{NL}(\text{DC}) = 2/3 + 6 \times 1/9 = 4/3$. Hence the DC part of the solution is identical of the operating point. This should not be surprising. (Why?)

Let’s find the AC part of the solution. The superposition circuit for the AC part immediately lends the solution as

$$(1.5) \quad v_{NL}(\text{AC}) = 0.05 \sin(\omega t).$$

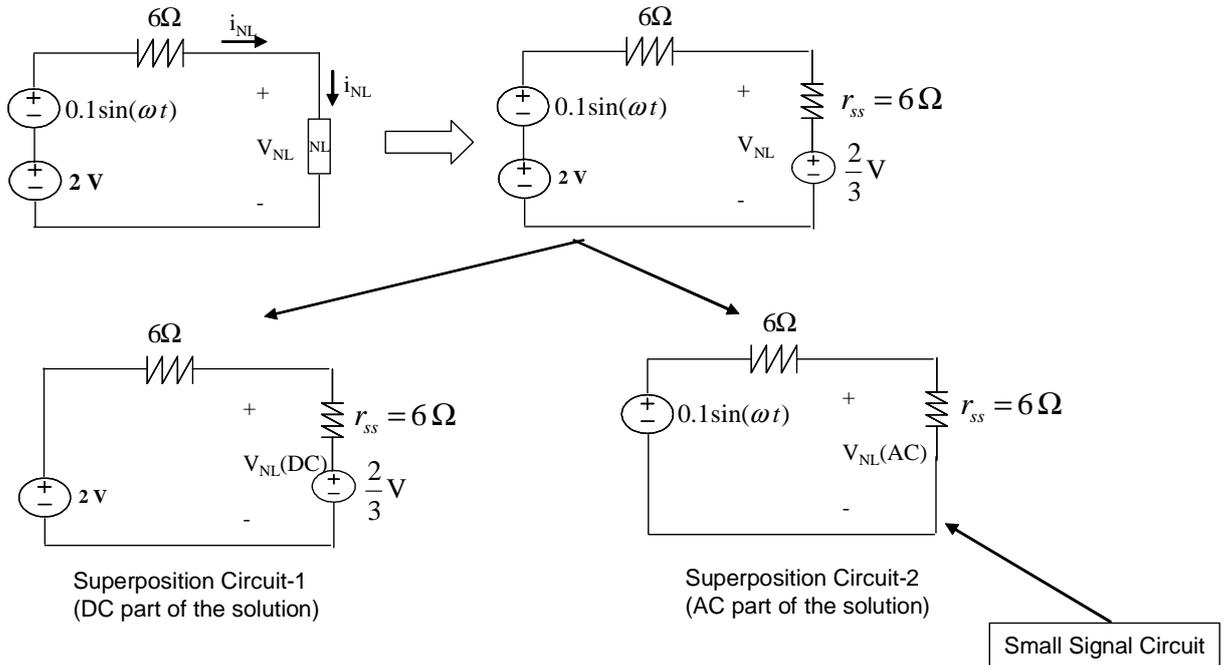


Figure 1.6: Stages of Small Signal Analysis

The complete solution of the problem via small signal approximation is:

$$(1.6) \quad v_{NL}(t) = \frac{4}{3} + 0.05 \sin(\omega t)V$$

Before concluding, we would like to note that the superposition circuit for the AC solution is called the *small signal circuit*. In the small signal circuit, the non-linear component is approximated with a resistor whose value is determined from the slope of non-linear component at the operating point. This resistance is called the *small signal resistance*. This is the effective resistance that the AC part of the input confronts. It should be clear that the small signal resistance is set by the DC part of the input in conjunction with the non-linearity and the other circuit components. Hence if you change the operating point, the small signal resistance also changes.

This chapter presents the concept of small signal analysis. In many other problems, the smallness of a parameter or a signal can be well utilized to present an intuitive and informative solution. Hence, it can be a good practice to consider the approximation of a non-linearity with a suitable linear function, i.e. the Taylor series. This is how the small signal resistance concept has been formulated.

Bibliography

- [1] P. J. Davis, *Interpolation and approximation*. Blaisdell Pub. Co, New York, 1963.