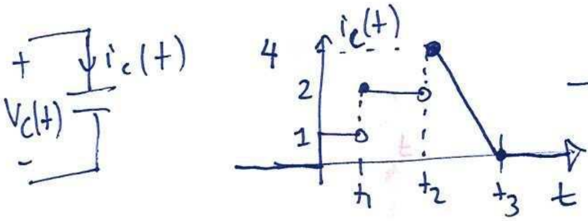
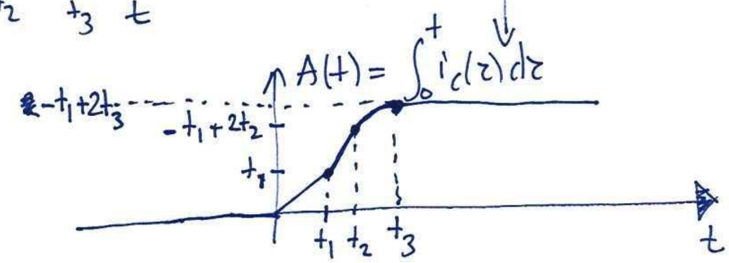
Switching Circuits – Continuity of $v_{cap}(t)$ and $I_L(t)$. ①

Previously, we have disussed that unless there is an impulsive source; the $v_c(t)$ and $I_L(t)$ of the circuit is a continous function of time. Remember that, this is due to the integral operator in $v_c(t) = v_c(t_0^-) + \frac{1}{C}\int_{t_0^-}^{t} i_c(\tau)d\tau$. $\left(\text{or } I_L(t_0) = I_L(t_0^-) + \frac{1}{L}\int_{t_0^-}^{t} v_L(\tau)d\tau\right)$.

It is clear that $i_c(\tau)$ can be discontinuous, as shown below



→ but the area under this curve say in between $[0,t]$ is a continuous function



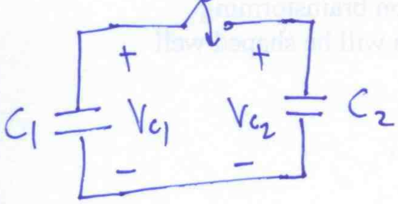
It should be clear that integrator output is "smoother" than its input! Furthermore it should be checked that $\frac{d}{dt}A(t) = i_c(t)$; which is the Fundamental Theorem of Calculus.

So, $v_c(t)$ and $I_L(t)$ are formed by integrating a function; therefore if the integrand does not contain any impulses; we have the continuity of $v_c(t)$ and $I_L(t)$; i.e $V_c(t_x^+) = V_c(t_x^-)$ and $I_L(t_x^+) = I_L(t_x^-)$

Discontinuity of $V_c(t)$ and $I_L(t)$ due to switching:

In the following circuits, we have no sources but dynamic elements and switches. (Note that there are no resistors!)

closes at $t=0$



$V_{c_1}(0^-) = V_1$

$V_{c_2}(0^-) = V_2$

When switch is open $i_{c_1} = i_{c_2} = 0$;

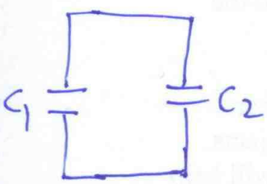
then $\left(v_c(t) = v_c(t_0) + \frac{1}{C}\int_{t_0}^{t} i_c(\tau)d\tau \right)$

no change in cap. voltages. $\rightarrow$

$\rightarrow$ initial cond. are preserved.

Just after switching, i.e. $t = 0^+$

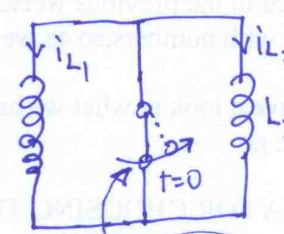
By KVL



$\rightarrow V_{c_1}(0^+) = V_{c_2}(0^+)$

So, unless $v_{c_1}(0^-) = V_{c_2}(0^-)$; we have jump at the voltages after switching.

Q: What is the common voltage after switching?



$i_{L_1}(0^-) = I_1$

$i_{L_2}(0^-) = I_2$

opens at $t=0$

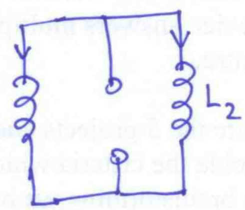
When switch is closed, $V_{L_1} = V_{L_2} = 0 \rightarrow$

$\rightarrow$ No change in $i_{L_1} = i_{L_2} \rightarrow$

initial conditions are preserved.

At $t = 0^+$

By KCL



$i_{L_1}(0^+) = -i_{L_2}(0^+)$

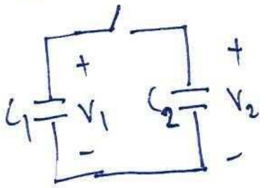
So unless $I_{L_1}(0^-) = I_{L_2}(0^-) \rightarrow$ we have a jump in inductor currents.

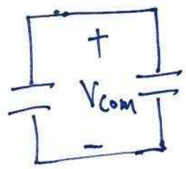
Q: What is the common current after switching?

Answer:

Method 1 Conservation of Charge

At $t=0^-$ | At $t=0^+$





$\varphi(0^-) = C_1V_1 + C_2V_2$ | $\varphi(0^+) = C_1 V_{com} + C_2 V_{com}$

$\varphi(0^-)$: total charge in two caps

Then $\varphi(0^-) = \varphi(0^+)$

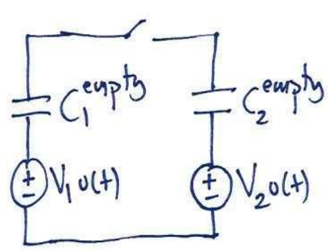
$$C_1V_1 + C_2V_2 = (C_1 + C_2)V_{com}$$

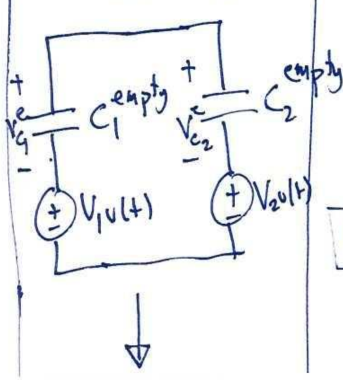
$$V_{com} = \frac{C_1}{C_1+C_2}V_1 + \frac{C_2}{C_1+C_2}V_2$$

$\boxed{V_{C_1}(0^+) = V_{C_2}(0^+)}$

Method 2 Initial Cond. Models.

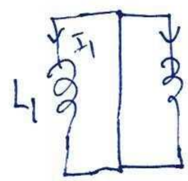
At $t=0^-$ | $t=0^+$

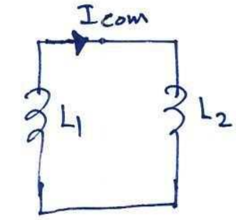




Method 1 Conservation of flux

$t=0^-$ | $t=0^+$





$\varphi(0^-) = -L_1I_1 + L_2I_2$ | $\varphi(0^+) = (L_1+L_2)I_{com}$

Tot. flux

~~Then~~

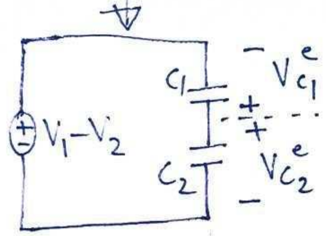
$$\boxed{I_{com} = \frac{-L_1I_1 + L_2I_2}{L_1+L_2}}$$

Note: I_{com} is in the same direction with I_2 and it is in the opposite direction with I_1. This is the reason of "−" sign on RHS of $\varphi(0^-)$.

Comment: +/− issues related to conservation principles is too tricky and may not be required if you know initial cond. models.

Method 2 Initial Cond. Models

⋮



$$V_{C_1}^{emp} = -(V_1-V_2)\,\frac{1/C_1}{1/C_1+1/C_2} = (V_2-V_1)\,\frac{C_2}{C_1+C_2}$$

$$V_{C_2}^{emp} = (V_1-V_2)\,\frac{1/C_2}{1/C_1+1/C_2} = (V_1-V_2)\,\frac{C_1}{C_1+C_2}$$

These are voltages of the empty cap.'s; we want the voltages of actual cap.'s of the circuit.

$$V_{C_1}(0^+) = V_{C_1}^{emp}(0^+) + V_1 = \frac{C_1V_1+C_2V_2}{C_1+C_2}$$

$$V_{C_2}(0^+) = V_{C_2}^{emp}(0^+) + V_2 = \frac{C_1V_1+C_2V_2}{C_1+C_2}.$$

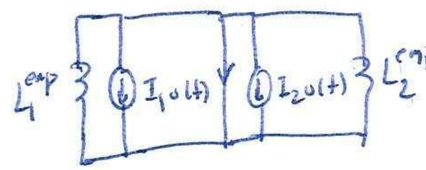
So,

$$\boxed{V_{C_1}(0^+) = V_{C_2}(0^+) = \frac{C_1}{C_1+C_2}V_1 + \frac{C_2}{C_1+C_2}V_2}$$

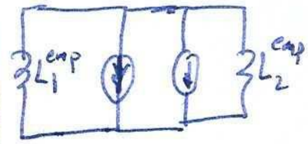
Same as before

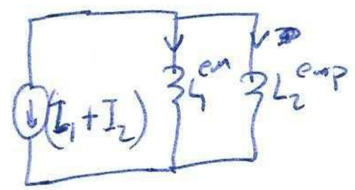
Let's try to use the other initial cond. model

$t=0^-$



$t=0^+$





$$I_{L_1}^{emp}(0^+) = (I_1+I_2)\cdot\frac{L_2}{L_1+L_2}$$

$$I_{L_2}^{emp}(0^+) = -(I_1+I_2)\,\frac{L_1}{L_1+L_2}$$

Actual Inductor currents:

$$I_{L_1}(0^+) = I_{L_1}^{emp}(0^+) + I_1 = -\frac{(-L_1I_1+L_2I_2)}{L_1+L_2}$$

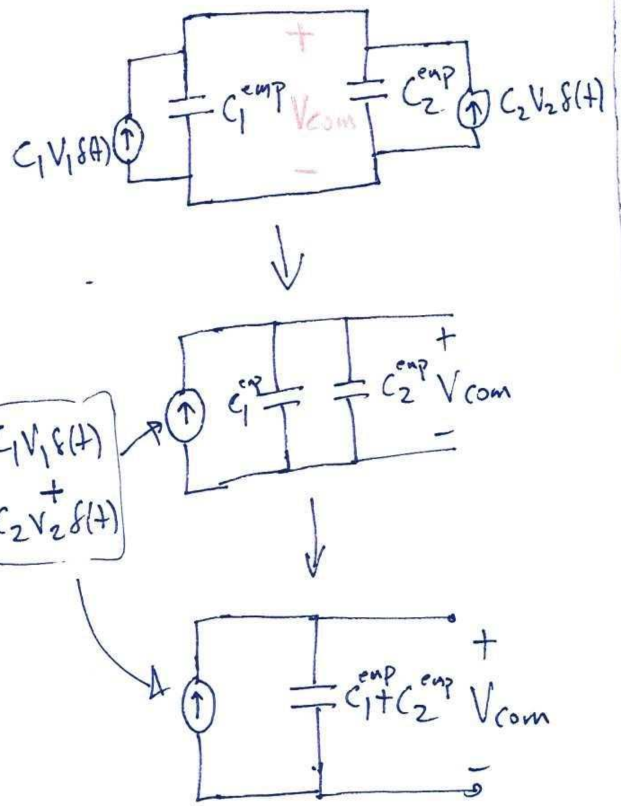
$$I_{L_2}(0^+) = I_{L_2}^{emp}(0^+) + I_2 = \frac{-L_1I_1+L_2I_2}{L_1+L_2}$$

I_{com} as before.

$$\boxed{I_{com} = I_{L_2}(0^+) = -I_{L_2}(0^+)}$$

Let's use the other initial cond. model

$t=0^+$

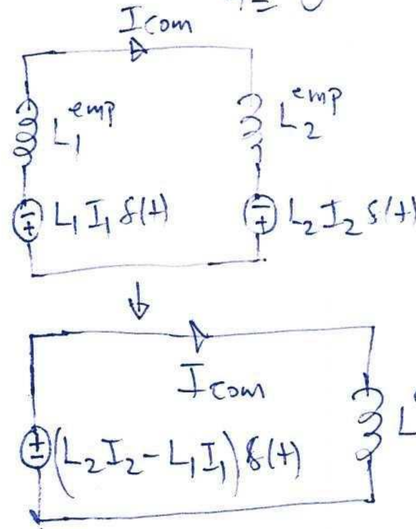


$$\boxed{V_{com}(0^+)} = \cancel{V_{com}(0^-)}^{\,0} + \frac{1}{C_1+C_2}\int_{0^-}^{0^+}(C_1V_1 + C_2V_2)\delta(t')\,dt'$$

$$= \frac{1}{C_1+C_2}(C_1V_1 + C_2V_2)$$

Same as before

$t=0^+$

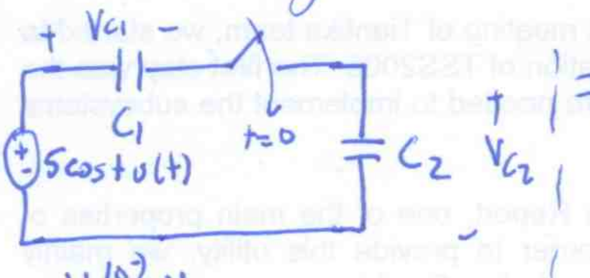


$$I_{com}(0^+) = \frac{1}{L_1+L_2}\int_{0^-}^{0^+} V_{L_1+L_2}(z)\,dz$$

$$= \frac{1}{L_1+L_2}(L_2I_2 - L_1I_1)$$

The situation can be further interesting, when we have a source in the system

Ex

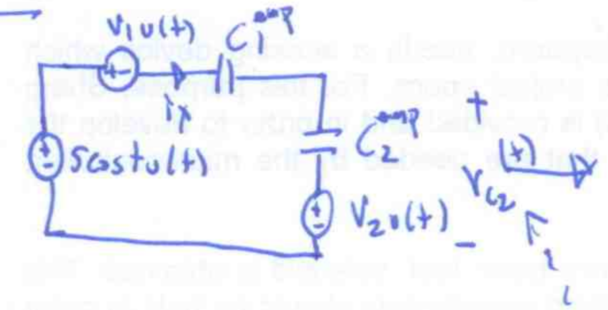


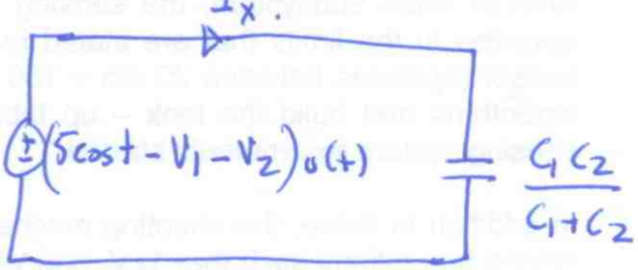
Find $V_{C2}(t)$ for $t \geq 0$

$V_{C_1}(0^-) = V_1$

$V_{C_2}(0^-) = V_2$

At $t=0^+$





$$I_x = \frac{C_1 C_2}{C_1+C_2}\frac{d}{dt}\left\{5\cos t - V_1 - V_2\right\}u(t)$$

$$= \frac{C_1 C_2}{C_1+C_2}\left(-5\sin t\, u(t) + (5 - V_1 - V_2)\delta(t)\right)$$

$$\boxed{V_{C_2}(t)} = V_2 + \frac{1}{C_2}\int_{0^-}^{t} i_x(t')\,dt'$$

$$= V_2 + \frac{C_1}{C_1+C_2}(5\cos t - 5) + \frac{C_1}{C_1+C_2}(5 - V_1 - V_2)$$

$t \geq 0$

$$= V_2 + \frac{C_1}{C_1+C_2}5\cos t - \frac{C_1}{C_1+C_2}(V_1 + V_2)$$