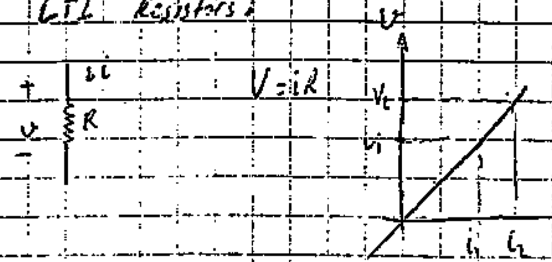
Omer Özdemir, EE201, Fall 2009-10 (Gökgöz (?) ... lecture notes)

Resistors:

Note: First 8 Hours are missing (G.C)

LTI Resistors:

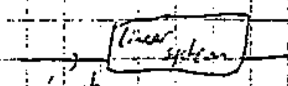


$V = iR$

$V_1 = i_1 R$
$V_2 = i_2 R$

If I have $i_1 + i_2$ as the current through the resistor. $\Rightarrow V_{1+2} = (i_1 + i_2)R = V_1 + V_2$

Linear system

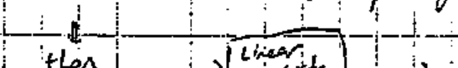


i_1, i_2 as inputs

o_1, o_2 as the corresponding outputs

$\Downarrow$

then



system is linear

$\frac{d}{dt}$

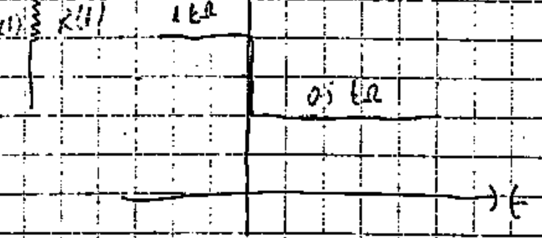
$\int i(t)\,dt$

$3i(t) = v(t)$

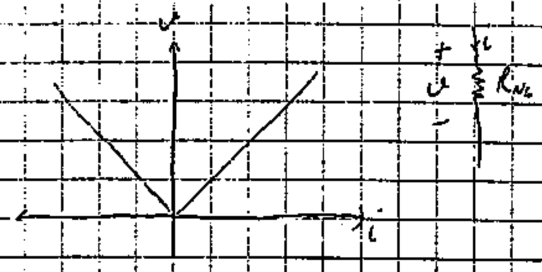
linear systems

Time invariant

$R(t) \rightarrow$ time varying component



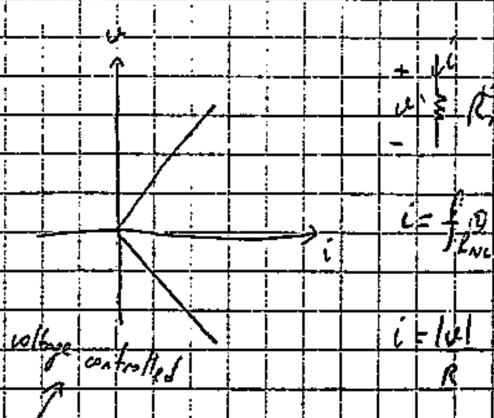
Non-linear resistors



$v = f_{R_{NL}}(i)$

$V = |iR|$

current controlled

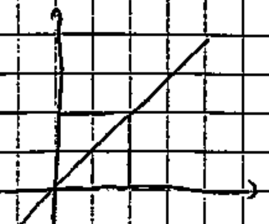


$i = f_{R_{NL}}(v)$

$i = \frac{|v|}{R}$

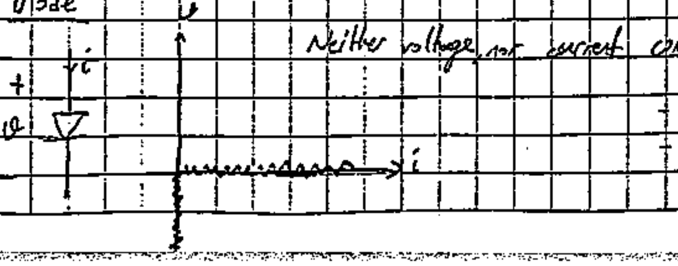
voltage controlled

A component is called current-controlled (voltage-controlled) if its branch voltage (current) is a function of its current (voltage)



Both current and voltage controlled

Diode



Neither voltage nor current controlled

Classification of Circuits

Memoryless Circuits (Resistances and sources)

Dynamic Circuits (Something evolving with time) (R, L, C)

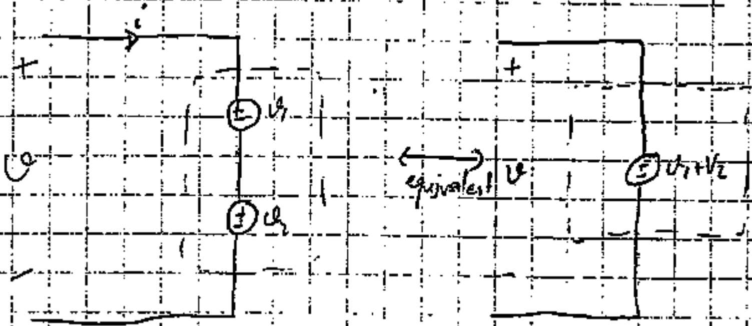
Passive, Active Components

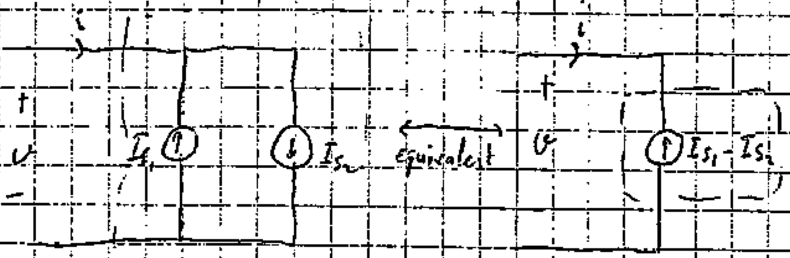
$$\int_{-\infty}^{t} p(\tau)\, d\tau < 0 \quad \exists t \rightarrow \text{component is active (generating energy)}$$

Circuits with active components → active circuits

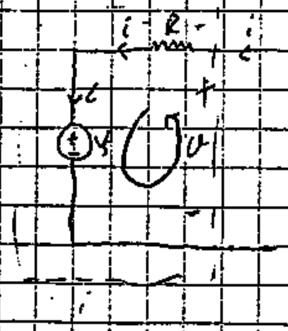
Any circuit with physically realizable components are passive (R, L, C)

Source Addition





Source Transformation

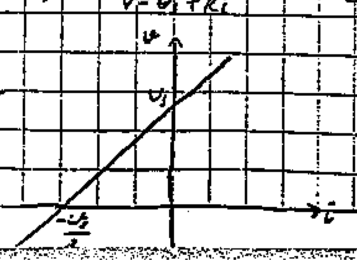


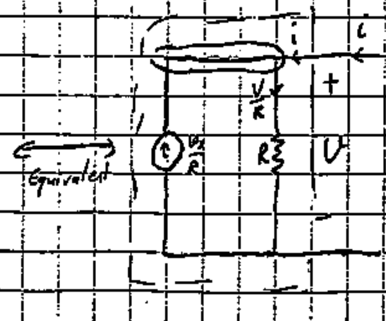
i-v characteristics

KVL

$V_s - V + Ri = 0$

$\Rightarrow V = V_s + Ri$



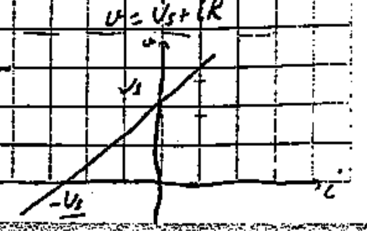


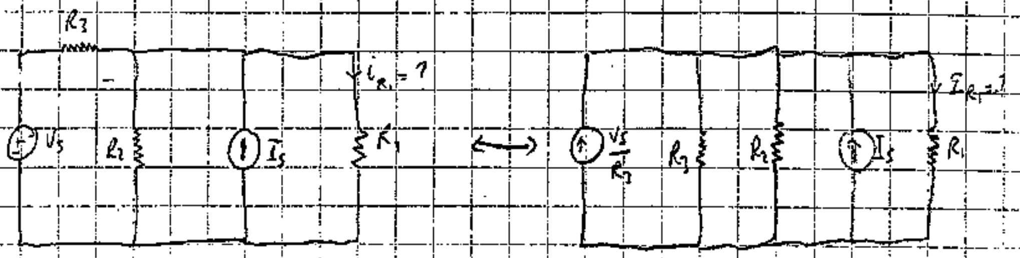
i-v characteristic

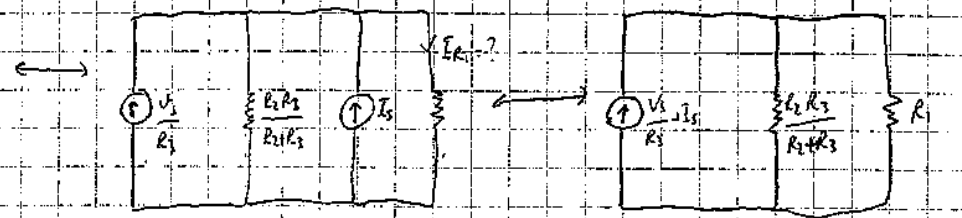
KCL

$\frac{V_s}{R} + i - \frac{V}{R} = 0$

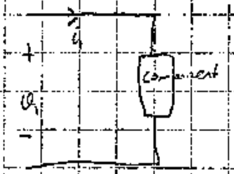
$V = V_s + iR$



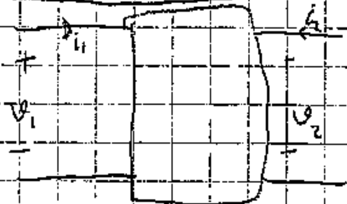




<u>One Port Circuits</u>



<u>Two Port Circuits</u>



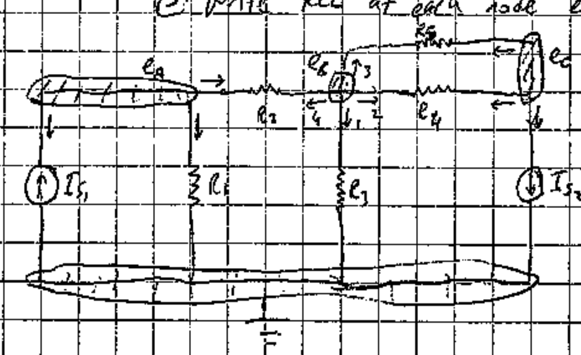
NODE-MESH ANALYSIS TECHNIQUES

<u>Method</u>: ① Introduce Datum Node

② Introduce Node Voltages ($e_A, e_B, e_C \ldots$)

③ Write KCL at each node except Datum node

Ex.



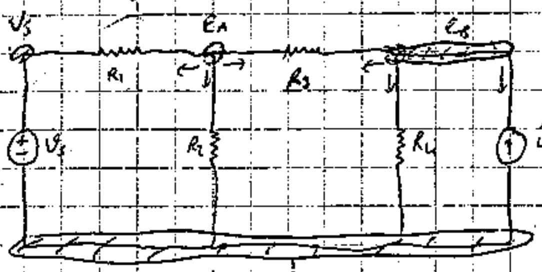
KCL at e_A: $\underbrace{-I_s}_{\substack{\text{exiting current}\\ \#1}} + \underbrace{\frac{e_A-0}{R_1}}_{\substack{\text{e.c.}\\ \#2}} + \underbrace{\frac{e_A-e_B}{R_2}}_{\substack{\text{e.c.}\\ \#3}} = 0$

KCL at e_B: $\frac{e_B-0}{R_3} + \frac{e_B-e_C}{R_4} + \frac{e_B-e_C}{R_5} + \frac{e_B-e_A}{R_2} = 0$

KCL at e_C: $I_{s_1} + \frac{e_C-e_B}{R_4} + \frac{e_C-e_B}{R_5}$

$$\begin{bmatrix} \frac{1}{R_1}+\frac{1}{R_2} & -\frac{1}{R_2} & 0 \\ -\frac{1}{R_2} & \frac{1}{R_2}+\frac{1}{R_4}+\frac{1}{R_3}+\frac{1}{R_5} & -\frac{1}{R_4}-\frac{1}{R_5} \\ 0 & -\frac{1}{R_4}-\frac{1}{R_5} & \frac{1}{R_4}+\frac{1}{R_5} \end{bmatrix} \begin{bmatrix} e_A \\ e_B \\ e_C \end{bmatrix} = \begin{bmatrix} I_{s_1} \\ 0 \\ -I_{s_2} \end{bmatrix}$$

Node Analysis with Voltage Source

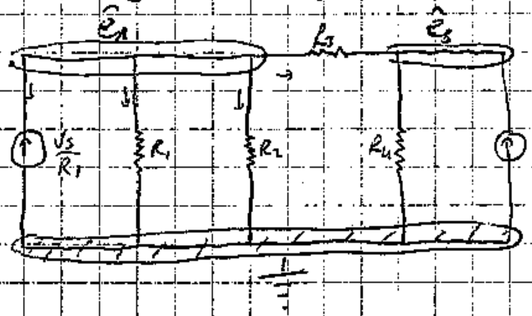


KCL at e_A : $\frac{e_A - V_s}{R_1} + \frac{e_A - 0}{R_2} + \frac{e_A - e_B}{R_3} = 0$

KCL at e_B : $-i_s + \frac{e_B}{R_4} + \frac{e_B - e_A}{R_3}$

OR

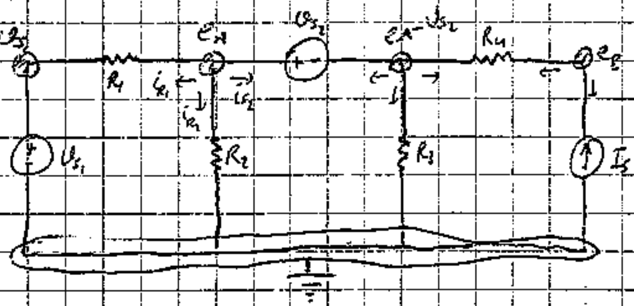
Source transformation



KCL at $\hat{e}_A$: $\frac{\hat{e}_A}{R_1} + \frac{\hat{e}_A}{R_2} - \frac{V_s}{R_1} + \frac{\hat{e}_A - \hat{e}_B}{R_3} = 0$

KCL at $\hat{e}_B$: $\frac{\hat{e}_B}{R_4} + \frac{\hat{e}_B - \hat{e}_A}{R_3} - i_s = 0$

Supernode:

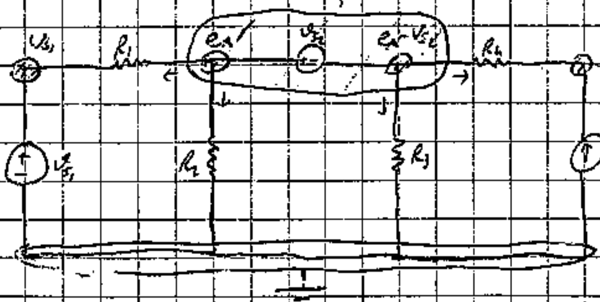


① KCL at e_A : $\frac{e_A - 0}{R_2} + \frac{e_A - V_s}{R_1} + i_{s_2} = 0$

② KCL at e_B : $\frac{e_B - (e_A - v_{s_2})}{R_4} - I_s = 0$

Add ① + ③ = supernode eqn.

③ KCL at "$e_A - V_{s_2}$" : $\frac{e_A - V_{s_2}}{R_3} + \frac{e_A - V_{s_2} - e_B}{R_4} - i_{s_2} = 0$

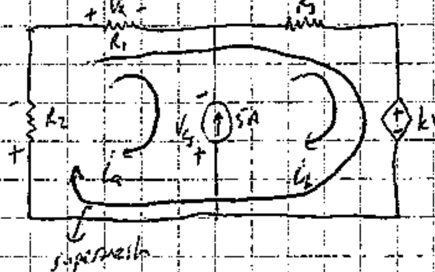


KCL at e_B : $\frac{e_B - (e_A - V_{s_2})}{R_4} - I_s = 0$

KCL at supernode : $\frac{e_A - V_{s_1}}{R_1} + \frac{e_A}{R_2} + \frac{e_A - V_{s_2}}{R_3} + \frac{e_A - V_{s_2} - e_B}{R_4} = 0$

2 eq. , 2 unknowns

Supermesh: Problem with current sources in the interior of the circuit!



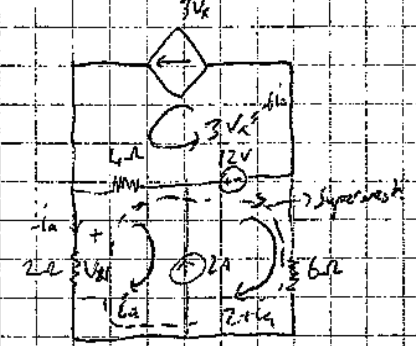
$i_b - i_a = 5$

KVL around i_a loop: $R_2 i_a + R_1 i_a - V_{cs} = 0$

KVL around i_b : $kV_x + V_{cs} + R_3 i_b = 0$ ($V_x = i_a R_1$)

KVL around supermesh : $R_2 i_a + R_1 i_a + R_3 i_b + kV_x = 0$ ($V_x = i_a R_1$)

Ex//



$V_x = -2i_a$

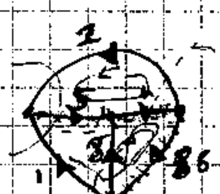
KVL around supermesh

$2i_a + 4(i_a - 6i_a) - 12 + 6(2 + i_a) = 0$

$-12 i_a + 24 = 0$

$i_a = 2$

Tree

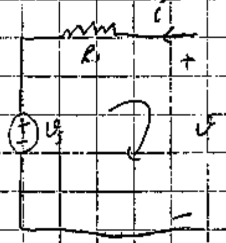


$\underline{B}\underline{v} = 0 \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} v = 0$

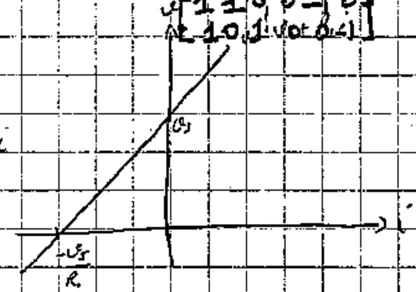
$\underline{Q}\underline{i} = 0 \rightarrow \begin{bmatrix} 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$

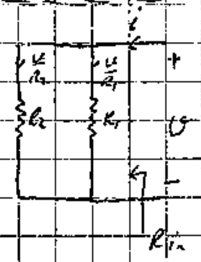
Input Resistance Calculation

$v = \underline{Q}^T \underline{e}$

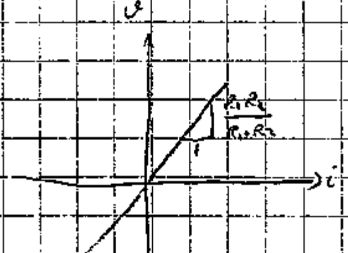


$\rightarrow -v_s - R_1 i + v = 0 \Rightarrow v = v_s + R_1 i$

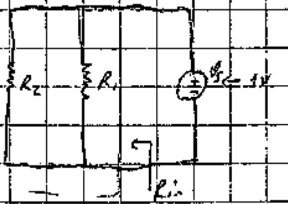




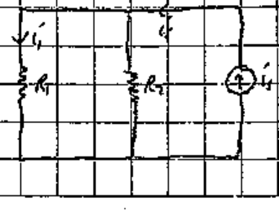
$i = \frac{v}{R_1} + \frac{v}{R_2} \rightarrow v = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) i$



Another method to find i-v char (when you have only resistors)

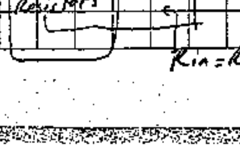


$i = \frac{1}{R_1} + \frac{1}{R_2}$ when $v_s = 1\,V \rightarrow \frac{1V}{\frac{1}{R_1} + \frac{1}{R_2}} = R_{in} \rightarrow R_{in} = R_1 \parallel R_2$



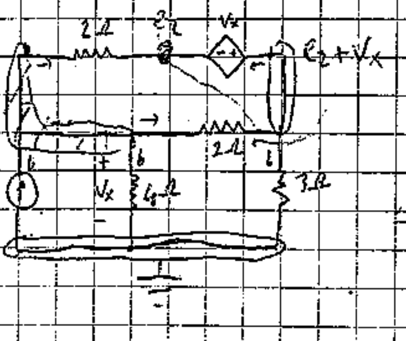
$i_1 = \frac{R_2}{R_1 + R_2} I_s \rightarrow i_1 = \frac{R_2}{R_1 + R_2}$ for $i_s = 1\,A$

$v = R_1 i_1 = \frac{R_1 R_2}{R_1 + R_2} I_s \rightarrow v = \frac{R_1 R_2}{R_1 + R_2}$



$R_{in} = R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$

Ex.



Find the power dissipated by 3Ω resistor

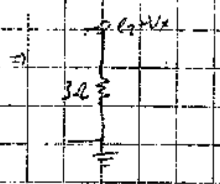
Node Analysis

KCL at e_1: $-5+\frac{e_1-0}{4}+\frac{e_1-e_2}{2}+\frac{e_1-(e_2+V_x)}{2}=0$

KCL at supernode: $\frac{e_2-e_1}{2}+\frac{e_2+V_x}{3}+\frac{e_2+V_x-e_1}{2}=0$

$\Rightarrow \begin{array}{l} 3e_1-4e_2=20 \\ -e_1+8e_2=0 \end{array} \Bigg\} \begin{array}{l} e_1=8V \\ e_2=1V \end{array}$

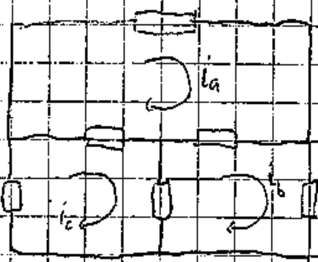
$\Rightarrow$



$V_{3\Omega}=e_2+e_1=9V$

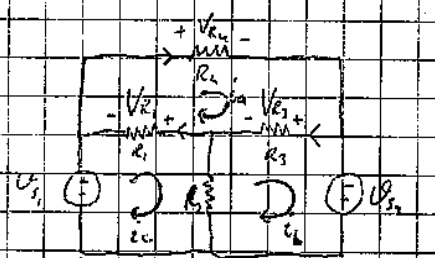
$P_{3\Omega}=V_{3\Omega}\frac{V_{3\Omega}}{3}=27\ W$

MESH ANALYSIS



Only applicable to planar circuits.

KVL for i_a, i_b, i_c meshes.



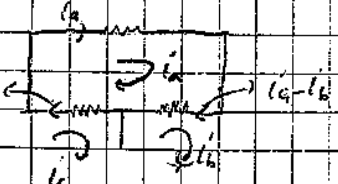
KVL for i_a mesh

$V_{R_3}+V_{R_1}+V_{R_4}=0$

$i_{R_3}R_3+i_{R_1}R_1+i_{R_4}R_4=0$

* Express in terms mesh currents

KVL for mesh i_a



$R_4i_a+R_3(i_a-i_b)+R_1(i_a-i_c)=0$

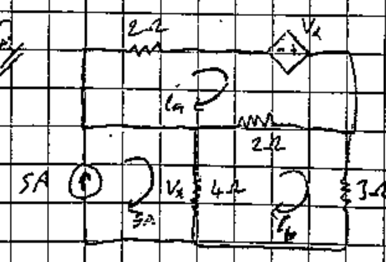
For i_b

$-V_{s_2}+R_2(i_b-i_c)+R_3(i_b-i_a)=0$

For i_c

$-V_{s_1}+R_1(i_c-i_a)+R_2(i_c-i_b)=0$

Ex.



KVL for

i_a: $2(i_a-i_b)+2i_a-V_x=0$ $\quad (5-i_b).4$

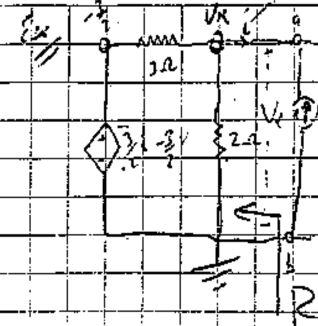
i_b: $3i_b+4(i_b-5)+2(i_b-i_a)=0$

$\begin{bmatrix} 4 & 2 \\ -2 & 9 \end{bmatrix}\begin{bmatrix} i_a \\ i_b \end{bmatrix}=\begin{bmatrix} 20 \\ 20 \end{bmatrix} \Rightarrow \begin{bmatrix} i_a \\ i_b \end{bmatrix}=\frac{1}{40}\begin{bmatrix} 9 & -2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 20 \\ 20 \end{bmatrix}=\begin{bmatrix} \frac{7}{2} \\ 3 \end{bmatrix}$

$\Rightarrow i_a=\frac{7}{2}$

$i_b=3$

Ex



find $V_x \rightarrow \frac{V_x}{1} = R_{in}$

KCL at V_x: $\frac{V_x - 0}{2} - 1 + \frac{V_x - (-\frac{3}{2}V_x)}{3} = 0$

$V_x = \frac{6-3}{5} = 0.6\,V$ $\rightarrow R_{in} = \frac{V_x}{1} = 0.6\,\Omega$

$R_{in} = ?$

Linearity

Operator: A mapping between input and output

Linear Systems

$f_{out} = \mathcal{L}\{f_{in}\}$ → A linear system

(0) $f_{out}^{(1)} = \mathcal{L}\{f_{in}^{(1)}\}$ $\quad f_{out}^{(2)} = \mathcal{L}\{f_{in}^{(2)}\}$

(1) $\mathcal{L}\{a f_{in}^{(1)}\} = a f_{out}^{(1)}$

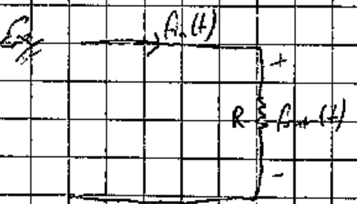
(2) $\mathcal{L}\{f_{in}^{(1)} + f_{in}^{(2)}\} = f_{out}^{(1)} + f_{out}^{(2)}$

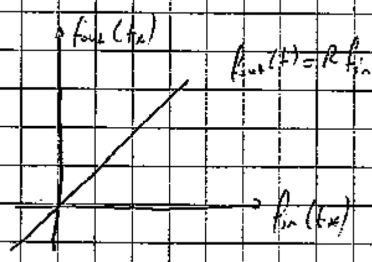
From (1)-(2)

$\mathcal{L}\{0\} = 0$ $\qquad \mathcal{L}\{f_{in}^{(1)} + (-f_{in}^{(1)})\} = f_{out}^{(1)} - f_{out}^{(1)} \Rightarrow \mathcal{L}\{0\} = 0$

$\mathcal{L}\{-f_{in}^{(1)}\} = -f_{out}^{(1)}$

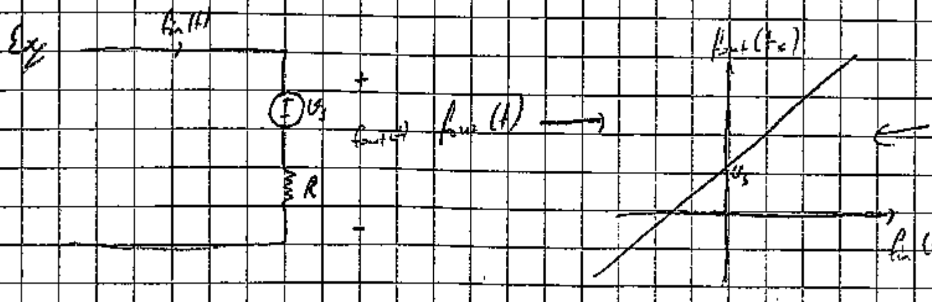
Ex





$f_{out}(t) = R f_{in}(t)$

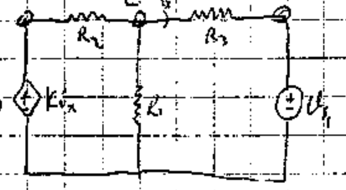
Ex



← Not a linear relation because of V_s source

Circuit with A Single Source

dependent source



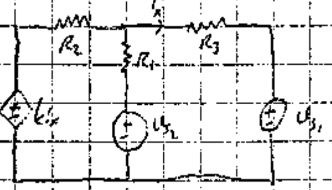
When v_s is doubled → all circuit variables are also doubled

$$\frac{e}{R_1} + \frac{e - ki_x}{R_2} + \frac{e - v_{s_1}}{R_3} = 0 \qquad i_x = \frac{e - v_{s_1}}{R_3}$$

$$(\ldots\ldots)e = \alpha v_{s_1}$$

!!! Never treat dependent sources as an independent source.

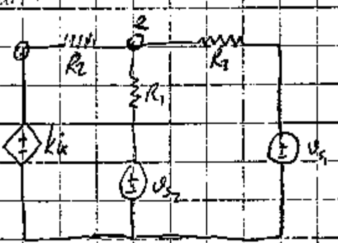
Multiple sources



When v_{s_1} and v_{s_2} are doubled → all circuit variables are also doubled.

Superposition Principle

A linear system with multiple inputs can be decomposed into several circuits with single input and addition of all output variables in the decomposition results in the solution of multiple input circuit.



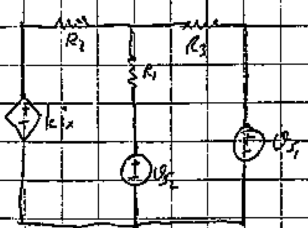
$$(\underline{\text{a number}})\, e^{③} = (\ldots) V_{s_1} + (\ldots) V_{s_2}$$

$$\Sigma \to (\quad)\, e^{①} = (\ldots) V_{s_1}$$

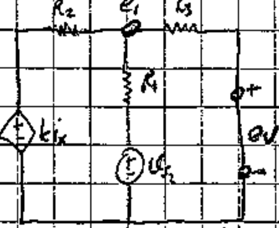
$$(\quad)\, e^{②} = (\ldots) V_{s_2}$$

$$\Rightarrow e^{③} = e^{①} + e^{②}$$

Turning OFF the sources (killing the sources)

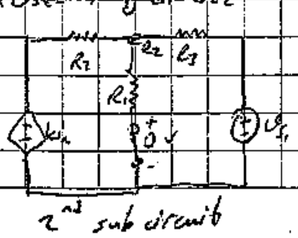


kill v_{s_2} ($v_{s_2} = 0\,V$)



1^{st} sub-circuit

($v_{s_2} = 0$) kill v_{s_2}

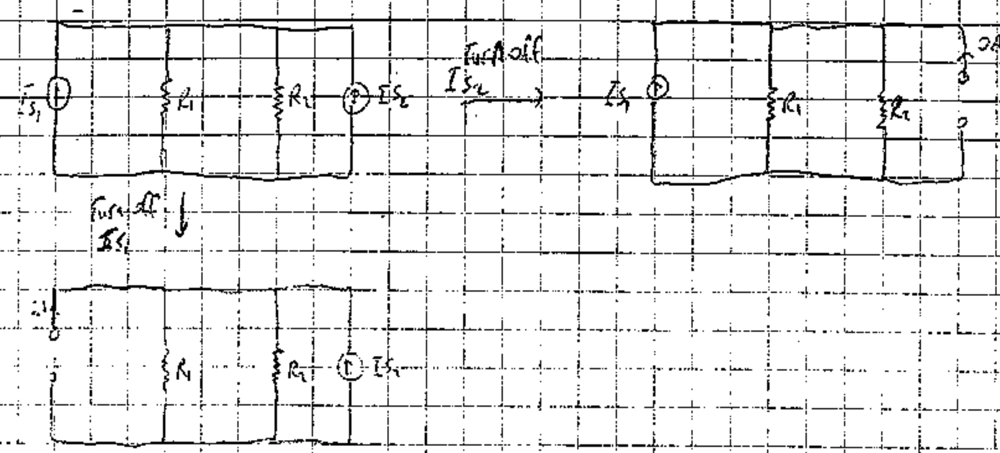


2^{nd} sub circuit

① Solve both circuits for $e^{①}$ and $e^{②}$

② $e^{③} = e^{①} + e^{②}$

Independent Current Sources



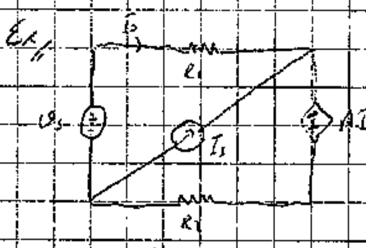
Superposition principle applies to all linear systems hence all linear circuits. To turn-off

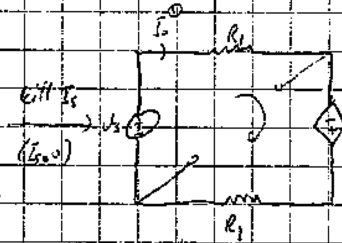
1) Voltage source → Replace with short circuit ($V_s = 0$)

2) Current source → Replace with open circuit ($I_s = 0$)

Never treat dependent sources as independent sources. (Do not try to turn off dependent sources)

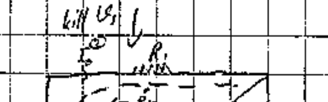
Ex/





$$I_o^{(1)}(R_1+R_2) - A I_o^{(1)} - V_s = 0$$

$$I_o^{(1)} = \frac{V_s}{R_1+R_2-A}$$



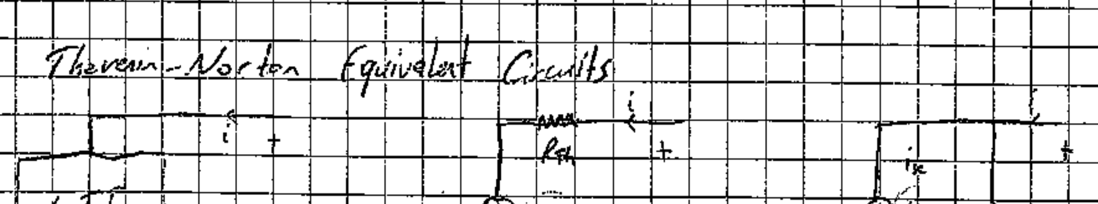
$$R_1 i_x + R_2(i_x - I_s) + A I_o^{(2)} = 0$$

$$i_x = \frac{R_2 I_s}{R_1+R_2-A} \qquad I_o^{(2)} = \frac{-R_2 I_s}{R_1+R_2-A}$$

$$I_o = I_o^{(1)} + I_o^{(2)}$$

$$I_o = \frac{V_s - R_2 I_s}{R_1+R_2-A}$$

Thevenin-Norton Equivalent Circuits



Thevenin equivalent

V_{Th}: Thevenin voltage

R_{Th}: Thevenin resistance

Norton equivalent

V_{oc}: Open circuit voltage

I_{sc}: Short circuit current

Procedure ①

① Find V_{oc}

② Find R_{Th} → Kill all the independent sources and find R_{eq} seen from a-b terminals

↘ Kill all the independent sources and apply 1V and find i_s of the 1V source → $R_{eq} = \frac{1V}{i_s}$

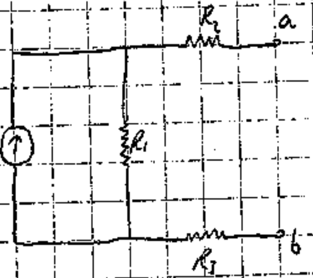
Procedure ②

① Find V_{oc}

② Find i_{sc}

③ $\frac{V_{oc}}{i_{sc}} = R_{Th}$

Ex) Find equivalent Thevenin-Norton circuit.



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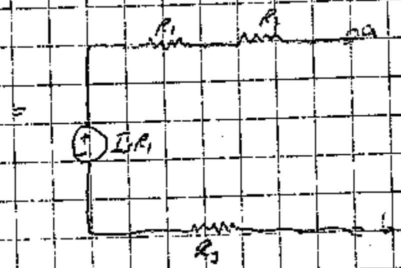

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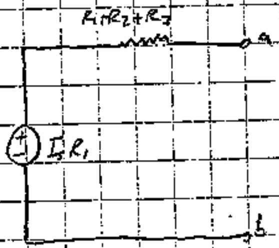
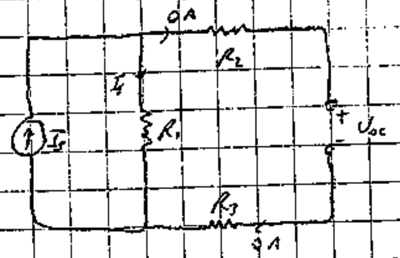


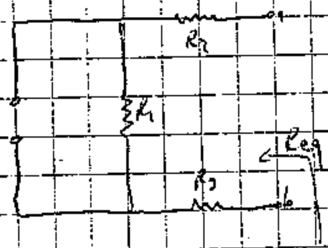
Procedure ①

①



$V_{oc} = I_s R_1$

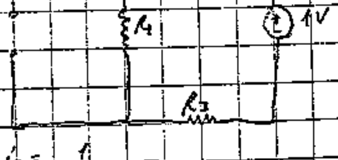
② Kill all independent sources ($v_s = 0V$, $I_s = 0A$)



$R_{eq} = R_1+R_2+R_3$

or

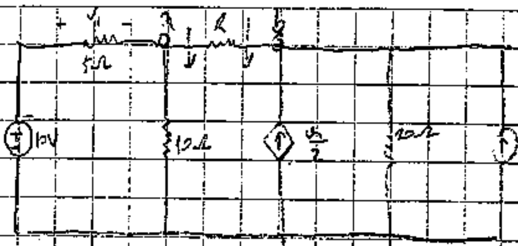
Kill all independent sources apply 1V source



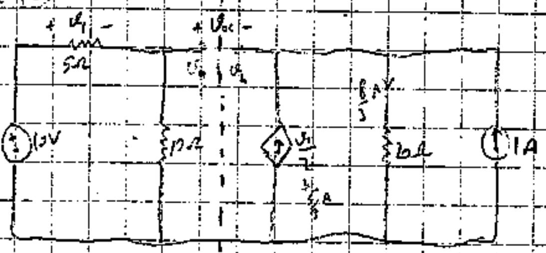
$i_s = \frac{1}{R_1+R_2+R_3}$

$R_{eq} = \frac{1}{i_s} = R_1+R_2+R_3$

Ex



Procedure (2)

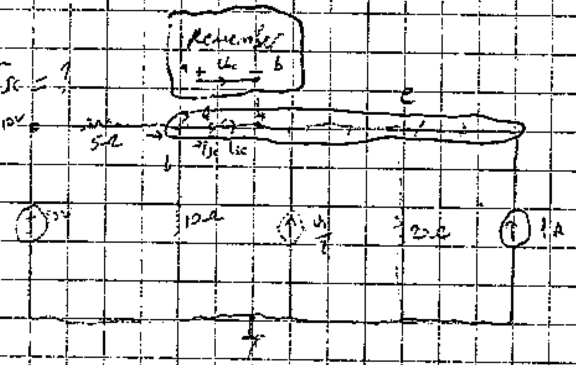


① $v_1 = 10 \cdot \frac{5}{5+10} = \frac{10}{3}$ V ;

$v_{20\Omega} = 20 \cdot \frac{8}{3} = \frac{160}{3}$ V

$V_{oc} = V_a - V_b = \left(10 - \frac{10}{3}\right) - \left(\frac{160}{3}\right) = -\frac{140}{3}$ V

② $I_{sc} = ?$



N/

$\frac{e-10}{5} + \frac{e}{10} - \frac{v_1}{2} + \frac{e}{20} - 1 = 0$

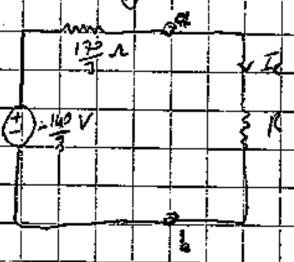
$(4+2+10+1)e = 40+100-20$

$e = \frac{160}{17}$ V

$\Rightarrow i_{sc} = \frac{10-e}{5} - \frac{e}{10}$

$= \frac{20-3e}{10} = 2 - 0.3e = 2 - \frac{48}{17} = \boxed{-\frac{14}{17}}$ A

$R_{Th} = \frac{V_{oc}}{I_{sc}} = \frac{-140/3}{-14/17} = \frac{170}{3}\ \Omega$



a) When $R = \infty$

$V_R = v_{oc} = -\frac{140}{3}$ V

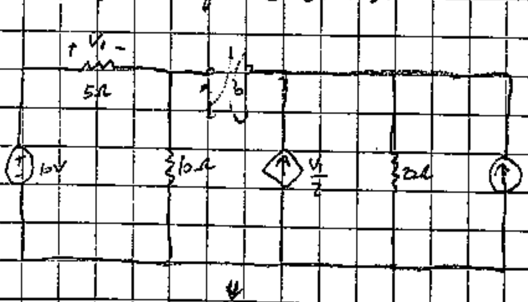
b) When $R = 0$

$I_R = I_{sc}$

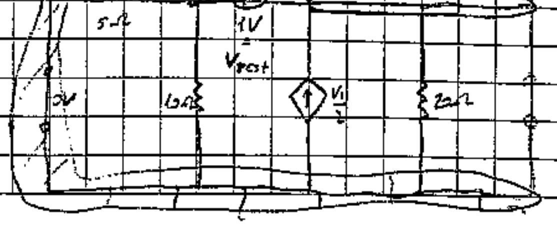
d) For any R

$i_R = \frac{-140/3}{\frac{170}{3} + R}$

Ex $R_{Th} = ?$ Using external source



Kill ind. sources.



$v_1 = e - 1$

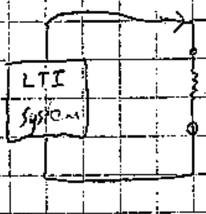
$\frac{e-1}{5} + \frac{e-1}{10} + \frac{e-1}{2} + \frac{e}{20} = 0 \qquad e = \frac{16}{17}$

$I_s = \frac{1-e}{5} + \frac{1-e}{10} = 0.3 - 0.3e = 0.3 - \frac{4.8}{17} = \frac{0.3}{17} = \frac{3}{170}$

$R_{Th} = \frac{V_{test}}{I_s} = \frac{170}{3}\ \Omega$

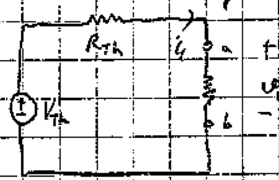
Maximum Power Transfer

In many situations, you would like to maximize energy/power delivered to a special component.



R_L: Load resistance

Power received by R_L



$i_L = \frac{V_{Th}}{(R_{Th}+R_L)}$; $V_L = R_L i_L$ $P_{RL} = i_L V_L = (i_L)^2 R_L = V_{Th}^2 \frac{R_L}{(R_{Th}+R_L)^2}$

To maximize $P(R_L) \rightarrow \frac{dP(R_L)}{dR_L} = 0$

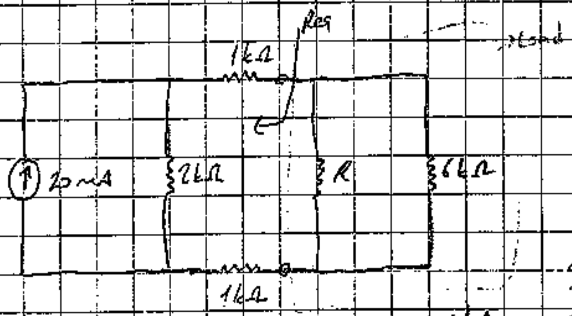
$$\frac{dP(R_L)}{dR_L} = V_{Th}^2 \frac{(R_{Th}+R_L)^2 - R_L \cdot 2 \cdot (R_{Th}+R_L)}{(R_{Th}+R_L)^4} = 0 \Rightarrow (R_{Th}+R_L)(R_{Th}+R_L-2R_L) = 0$$

$$\Downarrow$$

$$R_L = \{-R_{Th}, R_{Th}\}$$

To maximize the power transferred: $R_L = R_{Th}$

ex/

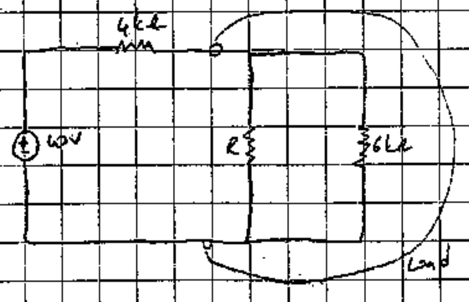


The resistance is adjusted to maximize power transfer to the load. Find the voltage and power delivered to load

$R_{eq} = 1+2+1 = 4\,k\Omega$

$V_{oc} = 40\,V$

⇒

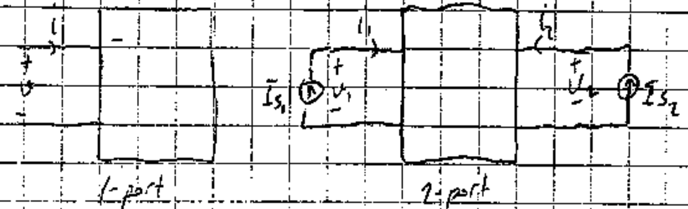


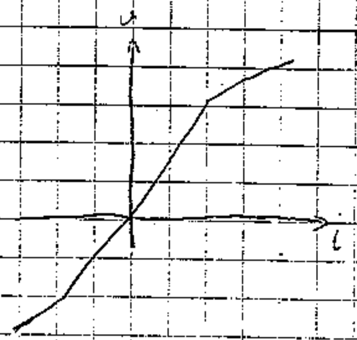
$R \parallel 6\,k\Omega = 4\,k\Omega \Rightarrow R = 12\,k\Omega$

$V_{load} = 20\,V$

$P_{load} = \frac{20^2}{4\,k\Omega} = \frac{400}{4000} = 0.1$ watt

One Port - Two Ports





R: Resistance Parameters

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\text{output circuit variables}} = \underbrace{\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}}_{R} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \rightarrow \text{current sources} \rightarrow \text{input sources}$$

G: Conductance Parameters $(G^{-1} = R)$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \underbrace{\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}}_{G} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Hybrid Parameters (I)

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \underbrace{\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}}_{H} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$$

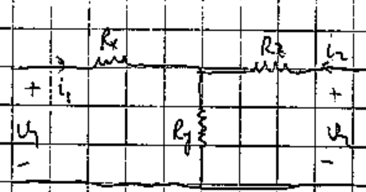
Hybrid Parameters (II)

$$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = H^{-1} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$$

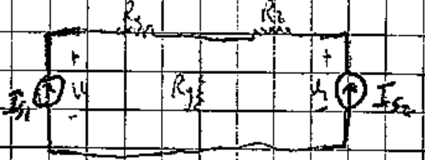
Transfer Parameters (ABCD) (Chain Parameters)

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix} \rightarrow \text{not } i_2 \text{ but } -i_2$$

Ex) Find R parameters



$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \longrightarrow$$



$v_1 = I_{s_1} R_x + (I_{s_1} + I_{s_2}) R_y$

$v_1 = \begin{bmatrix} (R_x + R_y) & R_y \end{bmatrix} \begin{bmatrix} I_{s_1} \\ I_{s_2} \end{bmatrix}$

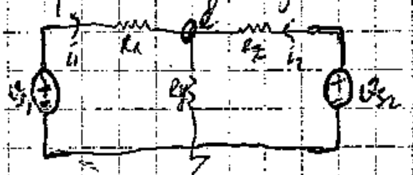
$v_2 = I_{s_2} R_z + (I_{s_1} + I_{s_2}) R_y$

$v_2 = \begin{bmatrix} R_y & (R_y + R_z) \end{bmatrix} \begin{bmatrix} I_{s_1} \\ I_{s_2} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \underbrace{\begin{bmatrix} (R_x + R_y) & R_y \\ R_y & (R_y + R_z) \end{bmatrix}}_{R} \begin{bmatrix} I_{s_1} \rightarrow i_1 \\ I_{s_2} \rightarrow i_2 \end{bmatrix}$$

$$G = R^{-1} = \begin{bmatrix} (R_y + R_z) & -R_y \\ -R_y & (R_x + R_y) \end{bmatrix} \cdot \frac{1}{(R_x + R_y)(R_y + R_z) - (R_y)^2}$$

G parameters using analysis



$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \rightarrow \text{source}$$

unknowns

$$\frac{e - v_{s_1}}{R_x} + \frac{e}{R_y} + \frac{e - v_{s_2}}{R_z} = 0 \quad \text{Find } e\,? \qquad i_1 = \frac{v_{s_1} - e}{R_x} \qquad i_2 = \frac{v_{s_2} - e}{R_z}$$

$$\left(\frac{1}{R_x} + \frac{1}{R_y} + \frac{1}{R_z}\right) e = \frac{v_{s_1}}{R_x} + \frac{v_{s_2}}{R_z}$$

$$i_1 = \frac{v_{s_1}}{R_x} - \frac{e}{R_x}$$

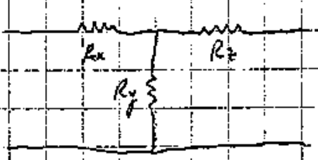
$$e = \frac{v_{s_1}/R_x + v_{s_2}/R_z}{\dfrac{R_y R_z + R_y R_z + R_x R_z}{R_x R_y R_z}}$$

$$i_1 = \underbrace{\left(\frac{1}{R_x} - \frac{R_y R_z}{R R_x}\right)}_{g_{11}} v_{s_1} - \underbrace{\left(\frac{R_y}{R}\right)}_{g_{12}} v_{s_2}$$

$$e = \frac{v_{s_1} R_y R_z + v_{s_2} R_x R_y}{\underbrace{R_x R_y + R_x R_z + R_y R_z}_{R}}$$

$$i_1 = g_{11} v_{s_1} + g_{12} v_{s_2}$$

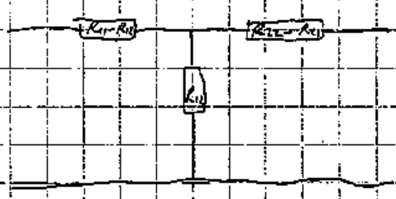
T Network



$R_{11} = R_x + R_y$

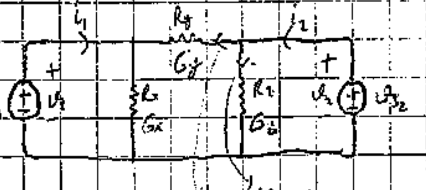
$R_{21} = R_{12} = R_y$

$R_{22} = R_y + R_z$

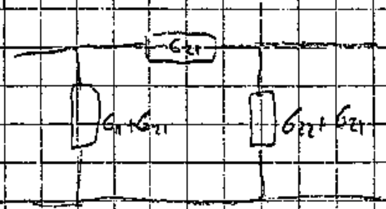


A given $\underline{R}$ parameters can be used to form a T-network

G Parameters and π Network



$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$



$G_y(v_{s_2} - v_{s_1})$ $\quad v_{s_2} G_z$

$$i_2 = v_{s_2} G_z + (v_{s_2} - v_{s_1}) G_y$$

$$= (-G_y) v_{s_1} + (G_y + G_z) v_{s_2}$$

MORE ON TWO PORTS

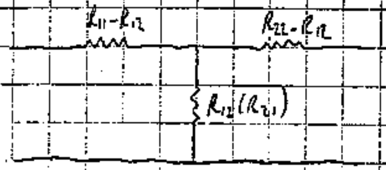
① Symmetry : $\underline{R}$ and $\underline{G}$ has to be symmetric : ?

⇒ Symmetry occurs if we do not have dependent sources (only resistances → symmetry)

(Inductors, capacitors → symmetry)
(RLC)

8

$\underline{R}: (R_{11}, R_{12}, R_{21}, R_{22})$; $R_{12} = R_{21}$

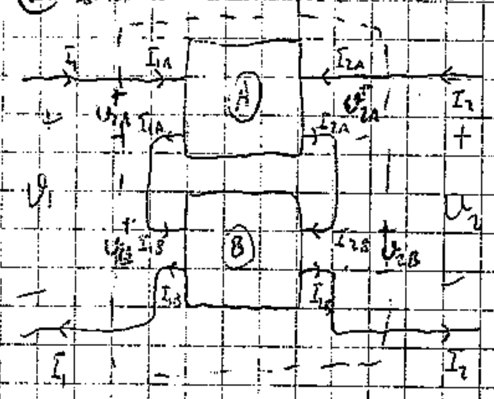


$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

when ① $i_1 = 0$; $i_2 = I_s \rightarrow (V_1, V_2) = (R_{12} I_s, R_{22} I_s)$

② $i_1 = I_s$; $i_2 = 0 \rightarrow (V_1, V_2) = (R_{11} I_s, R_{21} I_s)$

② Interconnection of Two Ports



Series Form

$\underline{R}$ parameters $\rightarrow \underline{R}_A, \underline{R}_B$

$$\underline{R}^{combined} = \underline{R}_A + \underline{R}_B$$

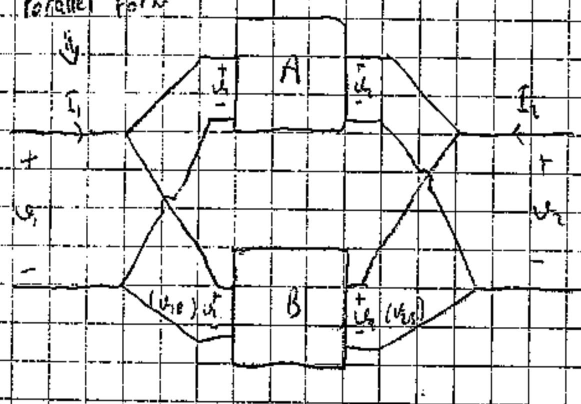
$\Rightarrow v_2 = v_{2A} + v_{2B}$

$v_1 = v_{1A} + v_{1B}$

$I_{2A} = I_{2B} = I_2$

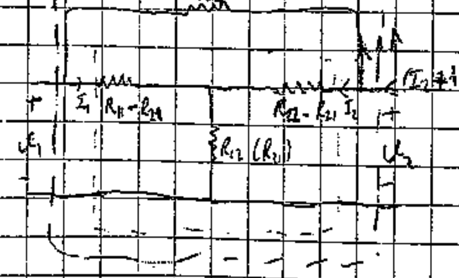
$I_{1A} = I_{1B} = I_1$

Parallel form



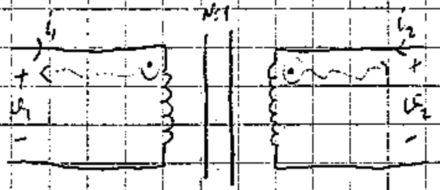
$$\underline{G}^{combined} = \underline{G}_A + \underline{G}_B$$

③ Two Ports With Feedback Connection



$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

④ Transformers

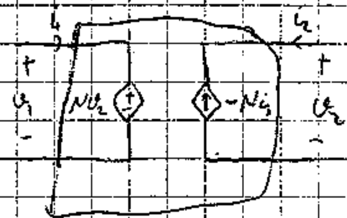


$$\frac{v_1}{v_2} = N \qquad \frac{i_1}{i_2} = -\frac{1}{N}$$

because both i_1 and i_2 is entering

N:1 (Turns ratio)

Ideal transformer



Note: Physically transformers can only operate with AC input

BUT

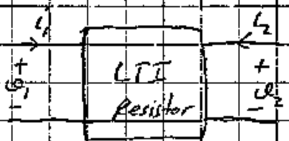
In EE201, we use transformers in DC circuit analysis to practice analysis with transformers.

$P_1 = v_1 i_1$, $P_2 = v_2 i_2$

$P_1 + P_2 = 0 \iff$ Ideal transformer

2n Parameter Calculation

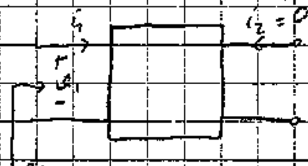
R parameters



$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

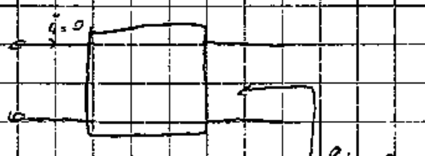
$v_1 = r_{11} i_1 + r_{12} i_2$

$v_2 = r_{21} i_1 + r_{22} i_2$



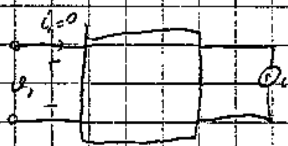
$R_{in} = r_{11}$

$$r_{11} = \frac{v_1}{i_1}\Big|_{i_2 = 0}$$

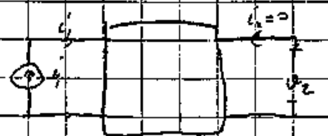


$R_{in} = r_{22}$

$$r_{22} = \frac{v_2}{i_2}\Big|_{i_1 = 0}$$

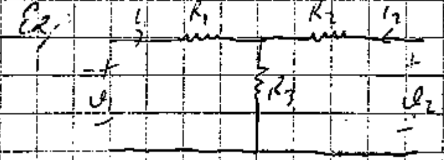


$$r_{12} = \frac{v_1}{i_2}\Big|_{i_1 = 0}$$



$$r_{21} = \frac{v_2}{i_1}\Big|_{i_2 = 0}$$

Ex/



$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

$r_{11} = \frac{v_1}{i_1}\Big|_{i_2=0} = R_1 + R_3$ | $r_{12} = \frac{v_1}{i_2}\Big|_{i_1=0} = R_3$

$r_{22} = \frac{v_2}{i_2}\Big|_{i_1=0} = R_2 + R_3$ | $r_{21} = \frac{v_2}{i_1}\Big|_{i_2=0} = R_3$

$$\underline{R} = \begin{bmatrix} R_1+R_3 & R_3 \\ R_3 & R_2+R_3 \end{bmatrix}$$

Note: $\underline{R}$ is symmetric ($r_{21} = r_{12}$) as expected since the network consists of only resistances.

Ex/ Express $\underline{H}$ parameters in terms of $\underline{R}$ parameters

$v_1 = r_{11} i_1 + r_{12} i_2 \quad (1) \qquad v_1 = h_{11} i_1 + h_{12} v_2$

$v_2 = r_{21} i_1 + r_{22} i_2 \quad (2) \qquad i_2 = h_{21} i_1 + h_{22} v_2$

$h_{11} = \frac{v_1}{i_1}\Big|_{v_2=0} \rightarrow (2) \rightarrow \underset{(3)}{i_2 = \frac{-r_{21} i_1}{r_{22}}} \xrightarrow{\text{substitute in } (1)} v_1 = r_{11} i_1 - \frac{r_{12} r_{21}}{r_{22}} i_1$

$$h_{11} = r_{11} - \frac{r_{12} r_{21}}{r_{22}} = \frac{r_{11} r_{22} - r_{12} r_{21}}{r_{22}} = \boxed{\frac{|\underline{R}|}{r_{22}}}$$

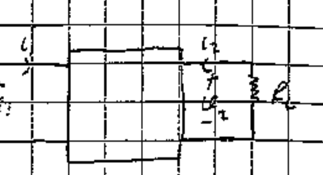
$h_{21} = \frac{i_2}{i_1}\Big|_{v_2=0} = \boxed{-\frac{r_{21}}{r_{22}}}$ from (3)

$h_{22} = \frac{i_2}{v_2}\Big|_{i_1=0} = \boxed{\frac{1}{r_{22}}}$

$h_{12} = \frac{v_1}{v_2}\Big|_{i_1=0} = \boxed{\frac{r_{12}}{r_{22}}}$

If $\underline{R}$ matrix is symmetric ($r_{12} = r_{21}$) $\rightarrow \underline{H}$ has the property that ($h_{12} = -h_{21}$)

Terminated Two Ports



$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

2 equations from two port equations
1 equation from load

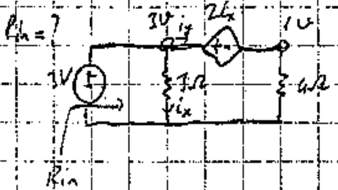
$\boxed{v_2 = -R_L i_2}$

$$-R_L i_2 = r_{21} i_1 + r_{22} i_2 \Rightarrow i_2 = \frac{-r_{21}}{R_L + r_{22}} i_1$$

$$v_1 = r_{11} i_1 + r_{12} i_2$$

$$\frac{v_1}{i_1} = r_{11} - \frac{r_{12} r_{21}}{r_{22} + R_L} \Rightarrow R_{in} = \frac{r_{11} r_{22} + r_{11} R_L - r_{12} r_{21}}{R_{22} + R_L}$$

ZPS-II (14-a) Ex// Find R:



$i_x = \frac{3}{3} = 1\,A$, $i_y = \frac{1}{4}\,A$ $\Rightarrow i_{3V} = 1 + \frac{1}{4} = 1.75\,A$

$R_{in} = \frac{3}{1.75} = \frac{12}{5}\,\Omega = 2.4\,\Omega$

it is given in question $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$

$v_2 = -2.4 i_2$

$v_2 = 2i_1 + 4i_2$

$-2.4 i_2 = 2i_1 + 4i_2$

$\Rightarrow i_1 = -3.2 i_2$

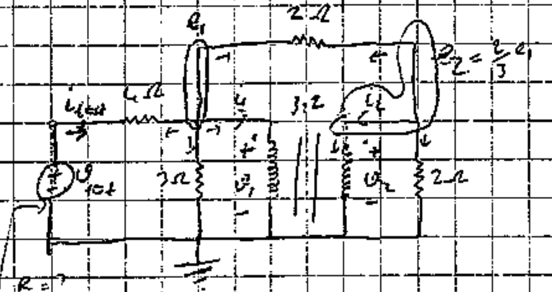
$v_1 = 3i_1 + 2i_2$

$v_1 = 3i_1 - 2\frac{i_1}{3.2}$

$v_1 = \frac{3.8}{1.6} i_1$

$$R = \frac{v_1}{i_1} = \frac{19}{8}\,\Omega = 2.375\,\Omega$$

ZPS-II (4a) Ex



KCL at e_1: $\frac{e_1}{3} + \frac{e_1 - v_{test}}{4} + \frac{e_1 - \frac{2}{3}e_1}{2} + i_1 = 0 \xrightarrow{\times 12} (4+3+2)e_1 + 12 i_1 = 3v_{test}$

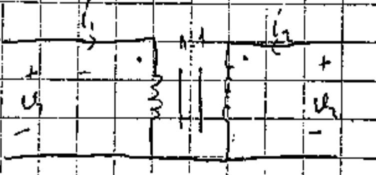
KCL at $e_2 = \frac{2}{3}e_1$: $\frac{\frac{2}{3}e_1 - e_1}{2} + \frac{\frac{2}{3}e_1}{2} + i_2 = 0$, $i_2 = -\frac{3}{2} i_1$ $\xrightarrow{\times 6} (-1+2)e_1 - 9i_1 = 0$

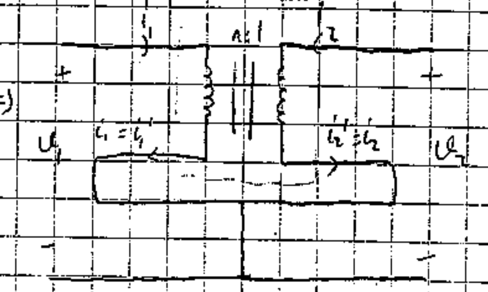
$$\begin{bmatrix} 9 & 12 \\ 1 & -9 \end{bmatrix} \begin{bmatrix} e_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} 3v_{test} \\ 0 \end{bmatrix}$$

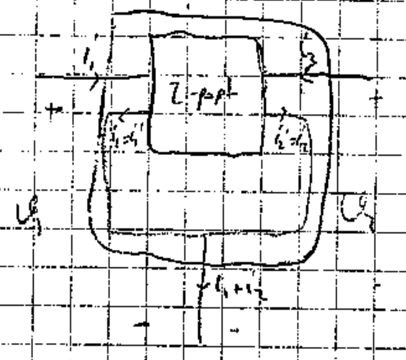
$$\rightarrow \begin{bmatrix} e_1 \\ i_1 \end{bmatrix} = \frac{1}{-81-12} \begin{bmatrix} -9 & -12 \\ -1 & 9 \end{bmatrix} \begin{bmatrix} 3v_{test} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{9}{31} v_{test} \\ \frac{1}{31} v_{test} \end{bmatrix}$$

$$i_{test} = \frac{v_{test} - e_1}{4}$$

$$i_{test} = \frac{v_{test} - \frac{9}{31} v_{test}}{4} \Rightarrow i_{test} = \frac{11}{62} v_{test} \Rightarrow R = \frac{v_{test}}{i_{test}} = \frac{62}{11}\,\Omega$$







3 Terminal - 2 port

for

Then 3-terminal 2 port

$i_1' = i_1 \quad i_2' = i_2$

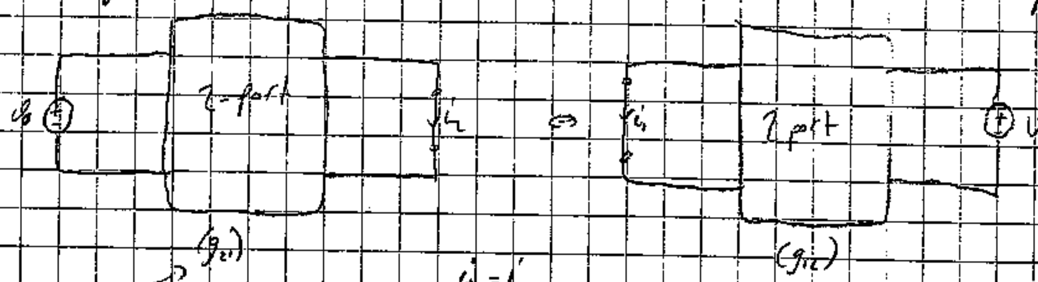
Reciprocity:

① R, G, H parameters have some special properties (symmetry or anti-symmetry (for H)) when 2 port in question does include only Resistors, (Inductors, Capacitors)

i.e. $r_{12} = r_{21}$, $g_{12} = g_{21}$, $h_{12} = -h_{21}$

② Reciprocity makes use of observation ① and is used also in measurements or experiments.

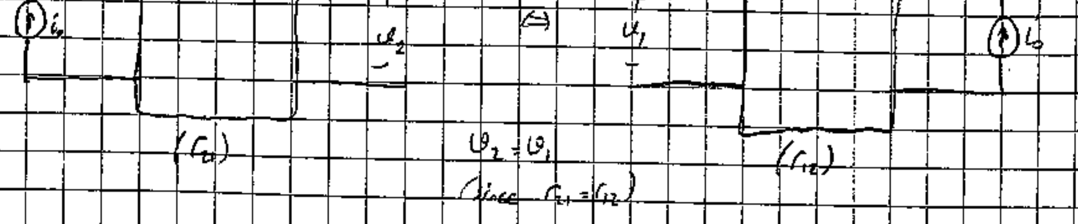
①



$\rightarrow \quad i_2 = g_{21} v_1 + g_{22} v_2$

$i_1 = i_2$
(since $g_{21} = g_{12}$)

②

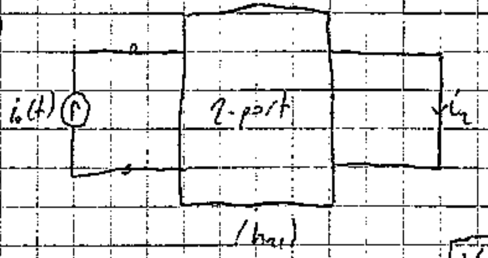


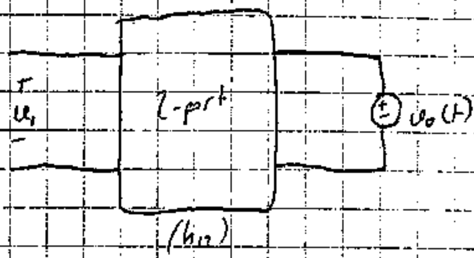
$v_2 = v_1$
(since $r_{21} = r_{12}$)

$\rightarrow \quad V_1 = r_{11} i_1 + r_{12} i_2$

$\rightarrow \quad V_2 = r_{21} i_1 + r_{22} i_2$

(3)



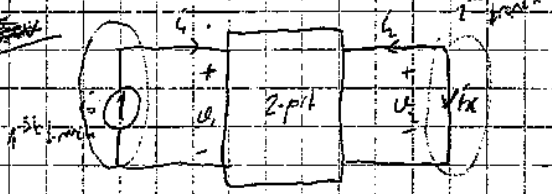


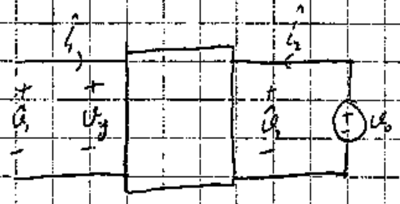
$\boxed{V_1 = i_2}$

Sources: $i_0(t) = V_0(t)$

Proofs of Reciprocity laws

3rd law





we would like to prove $i_x = v_y$

$$\sum_{k=1}^{\#\text{branches}} V_k \hat{i}_k = \sum_{k=1}^{\#\text{ of branches}} \hat{V}_k i_k = 0$$

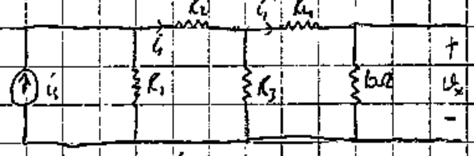
$$V_1(-\hat{i}_1) + V_2(-\hat{i}_2) + \sum_{k=3}^{\#\text{2 port branches}} \underbrace{V_k}_{(R_k i_k)} \hat{i}_k = \hat{v}_1\underbrace{(-i_1)}_{(-i_0)} + \underbrace{\hat{v}_2}_{v_0}(-i_2) + \cancel{\sum_{k=3}^{\#\text{2 port branches}} \underbrace{\hat{V}_k}_{(R_k \hat{i}_k)} i_k}$$

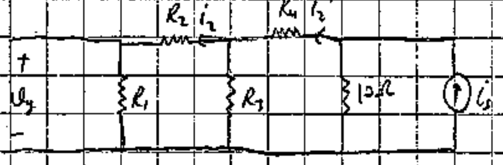
$0 = \hat{V}_1(-i_0) + V_0(-i_x)$

$i_0(t) = V_0(t)$, i.e. $i_0 = V_0$

$\Rightarrow 0 = \hat{V}_1 + i_2 \Rightarrow \hat{V}_1 = -i_2 \Rightarrow \boxed{i_x = \hat{v}_1}$

Ex: Two sets of measurements are made. Calculate R_1





$i_1 = 0.6\, i_s$

$i_1' = 0.3\, i_s$

$i_2 = 0.2\, i_s$

$i_2' = 0.5\, i_s$

From reciprocity:

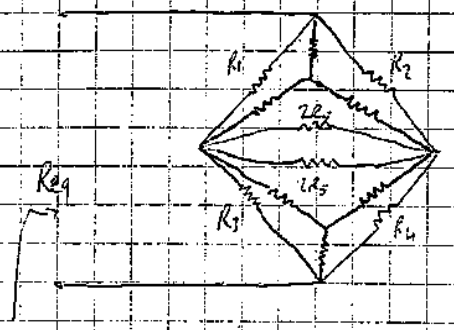
$V_x = V_y$

$i_1' \, 10\Omega = i_2 R_1$

$(0.3\, i_s) 10\Omega = (0.2\, i_s) R_1$

$R_1 = 15\,\Omega$

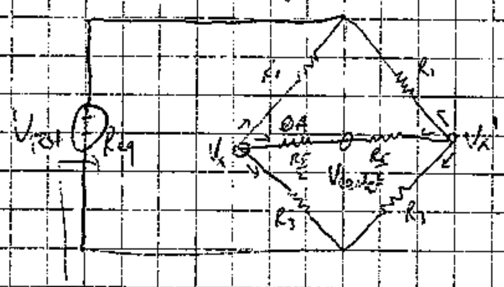
Symmetric Circuits

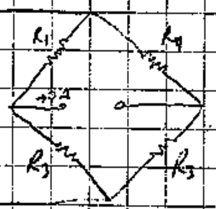


① $R_1 = R_2$ and $R_3 = R_4$, $R_{eq} = ?$

Write the node equations for V_x, V_x', V_0

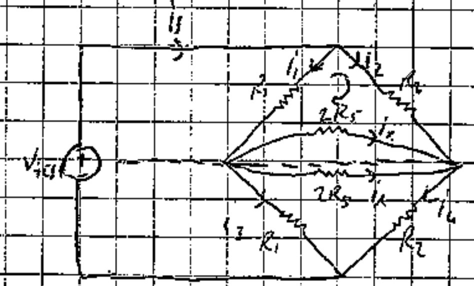
If you exchange $V_x \longleftrightarrow V_x'$, you have the same equation.

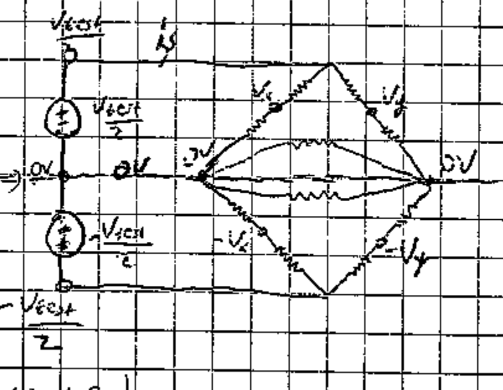




$R_{eq} = \frac{R_1 + R_3}{2}$

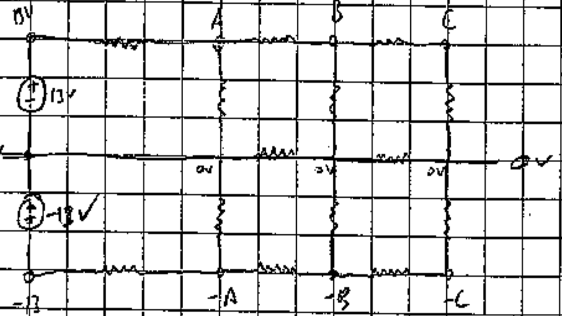
② $R_1 = R_3$ and $R_2 = R_4$ $R_{eq} = ?$



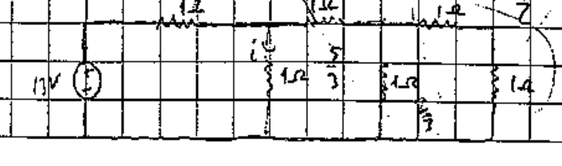


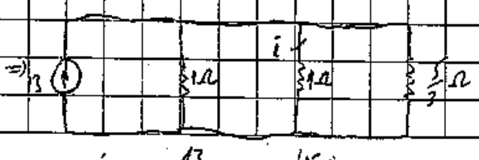
$i_1 = \frac{V_{test}}{2(R_1 \| R_2)} \Rightarrow R_{eq} = \frac{V_{test}}{i_1} = 2(R_1 \| R_2)$

Ex/



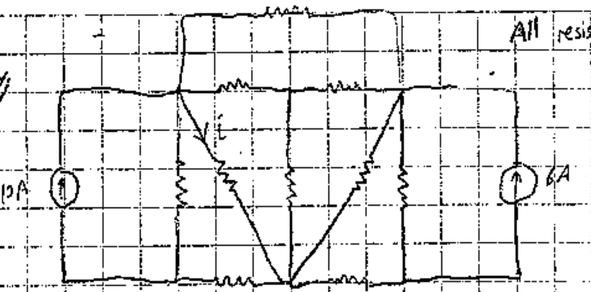
All resistors are 1Ω. $i = ?$



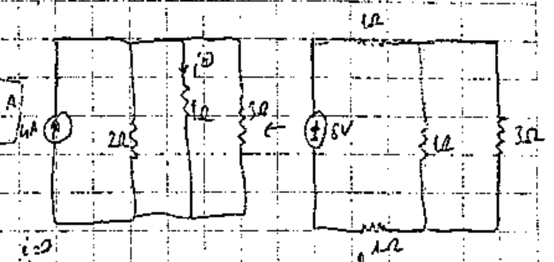


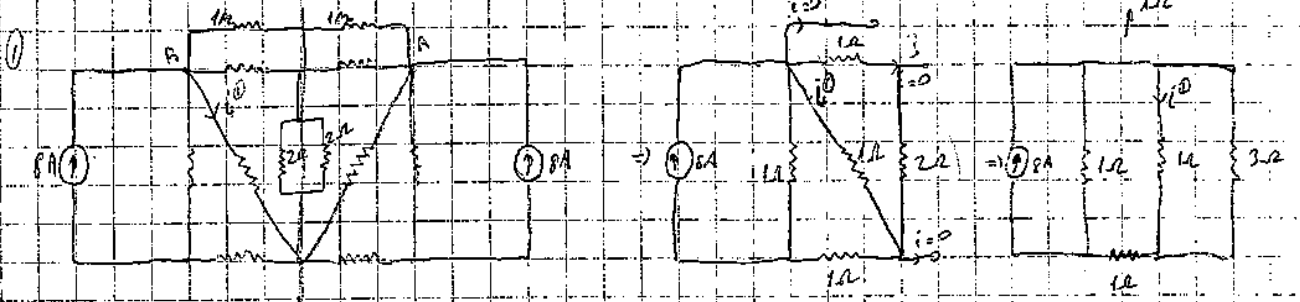
$i = \frac{13}{1+1+\frac{5}{3}} = \boxed{5A}$

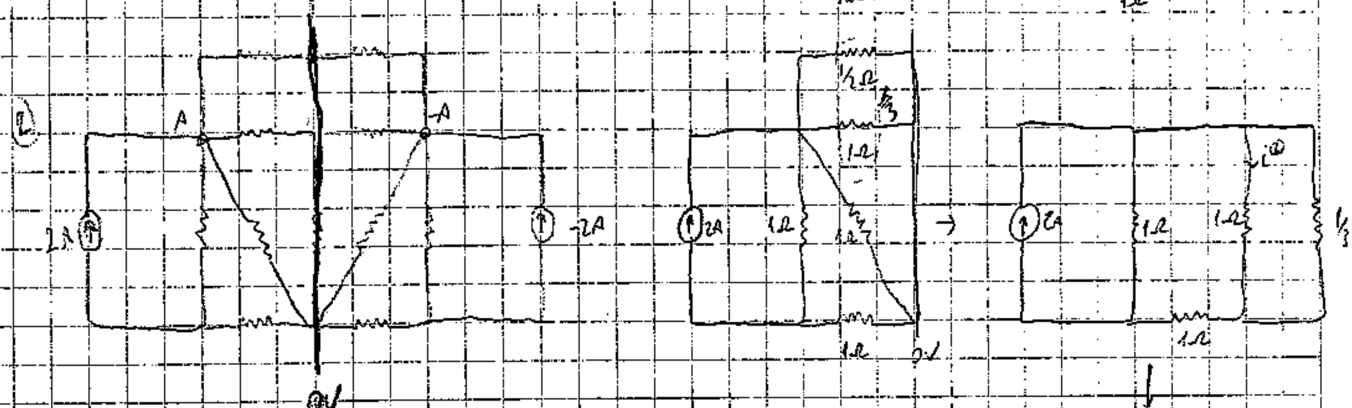
All resistances 1Ω, $i=?$



$\boxed{i^{(1)} = \frac{24}{11}\,A}$

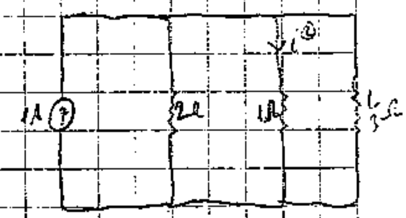


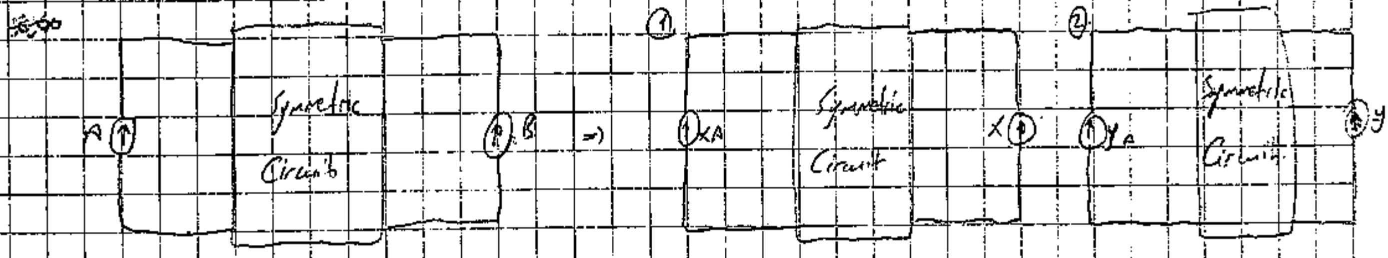




$$i = i^{(1)} + i^{(2)} = \frac{24}{11} + \frac{2}{9} = \boxed{\frac{238}{99}}$$

$$i^{(2)} = \frac{1}{\frac{1}{2}+1+3} = \frac{2}{9}$$

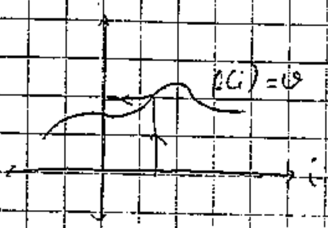


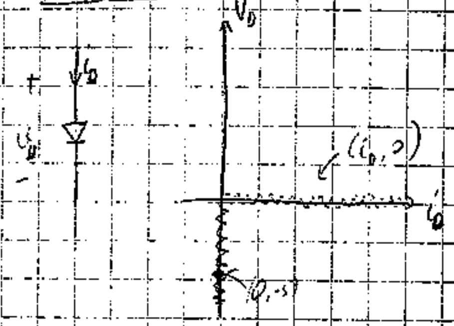


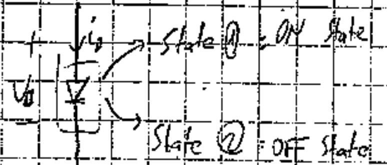
$$\left.\begin{aligned} x+y &= A \\ x-y &= B \end{aligned}\right\} \quad x = \frac{A+B}{2} \qquad y = \frac{A-B}{2}$$

Diodes (Non Linear Resistors)

- Current Controlled Component $(v = f(i))$
- Voltage Controlled Component $(i = g(v))$

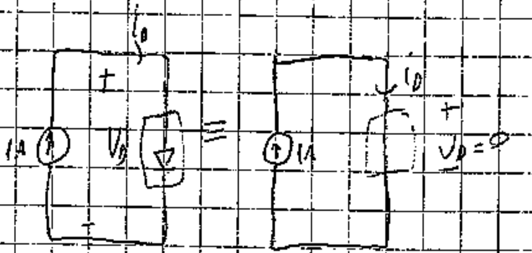




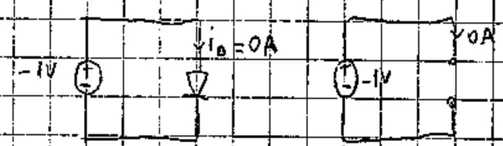


ON State (Conduction State): $i_D \geq 0$ (diode conducts)

$V_D = 0$ (short circuit)

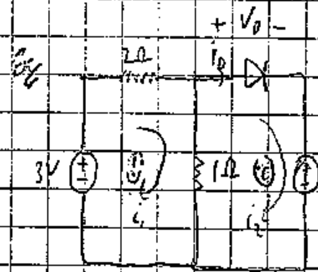


OFF STAGE (No Conduction): $V_D < 0 \longrightarrow i_D = 0$



OFF : Open-Circuit (No current conduction)

Ex



KVL around mesh ②:

$$2 + (i_2 - i_1)1 + f_{Diode}(i_2) = 0$$

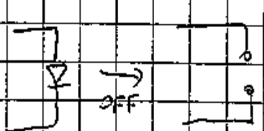
$V_{Diode} = f_{Diode}(i_2)$

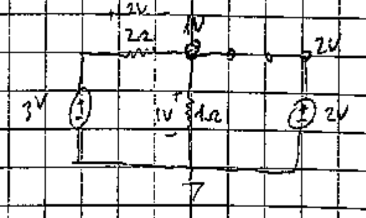
KVL around mesh ①:

$$-3 + 2i_1 + (i_1 - i_2)1 = 0$$

→ What we do is;

Assume diode is OFF (Assume $V_D \leq 0$)

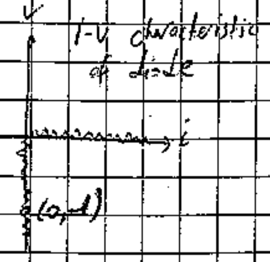




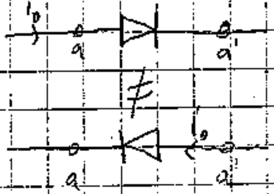
$V_D = (1 - 2)$

$V_D = -1\ V$ Assumption Holds ✓

Analysis complete

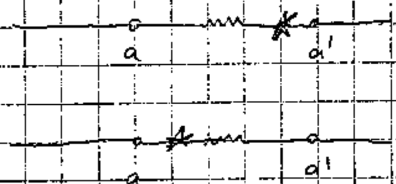


Bilateral Components

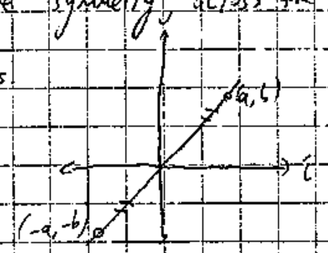


Not equal to each other

Simple LTI resistors (Ohm's Law) are unilateral components.

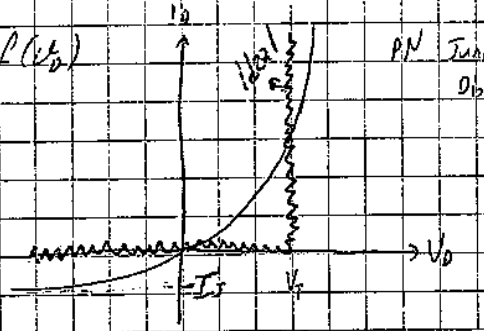


Unilateral components have symmetry across the origin in their i-v characteristics.



Practical Diodes

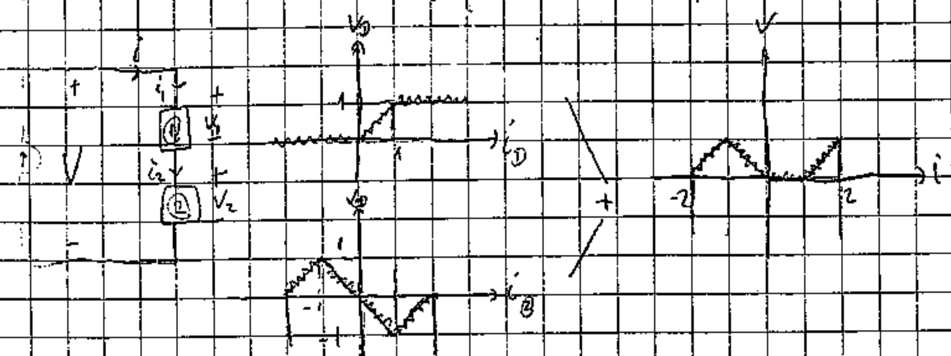
$i_D = f(v_D)$



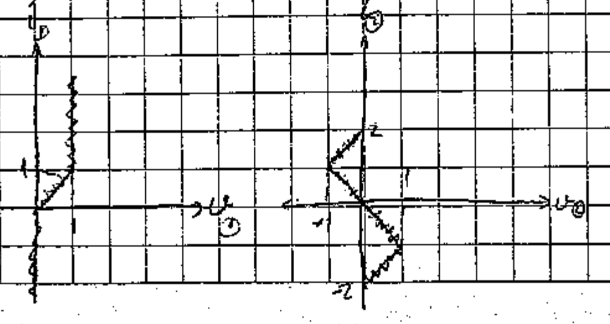
PN Junction Diode

$i_D = I_s(e^{v_D/V_T} - 1)$

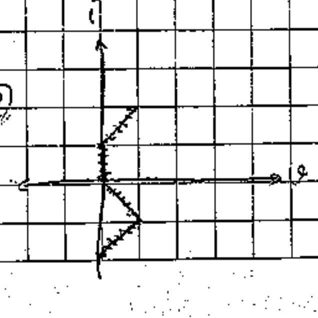
Series and Parallel Combination Of Ideal Diodes (Graphical Solutions)



!!!

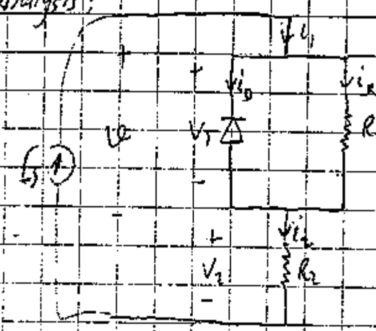


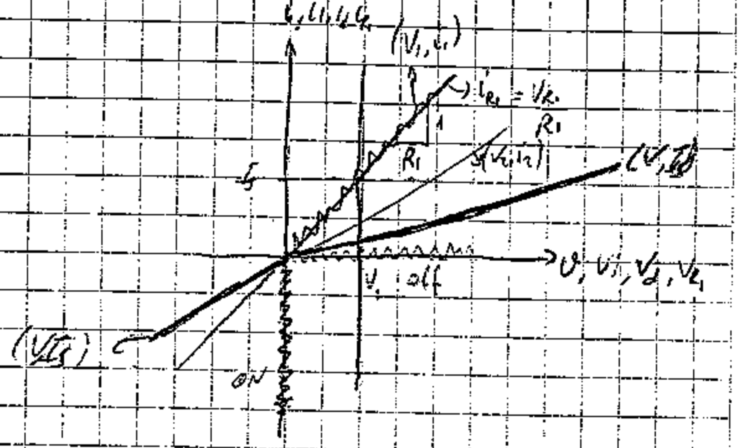
$\Rightarrow$ ① + ②

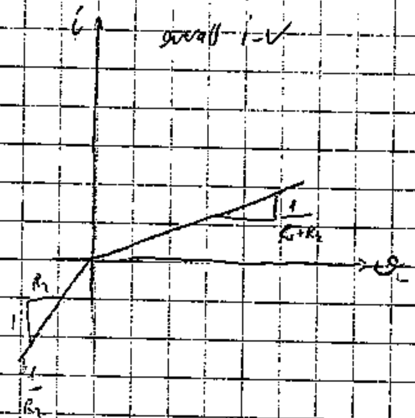


Diodes (Series and Parallel Combination)

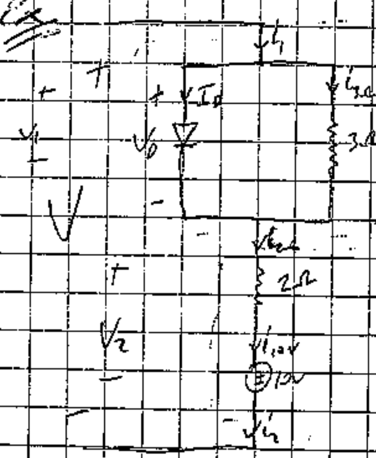
Analysis:

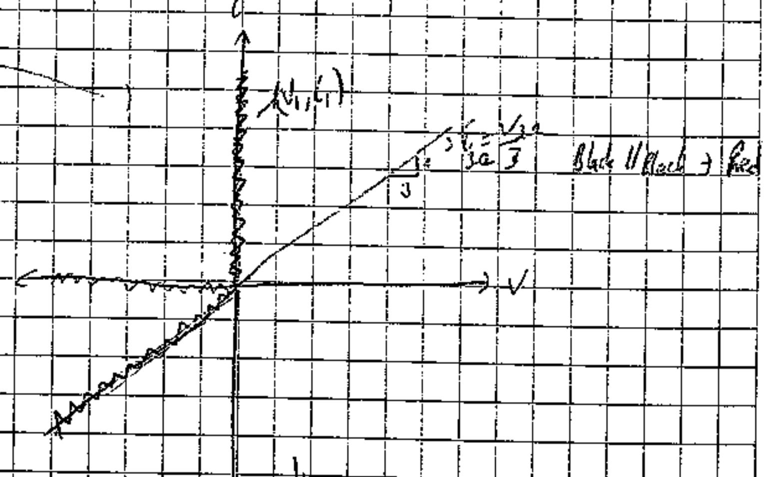


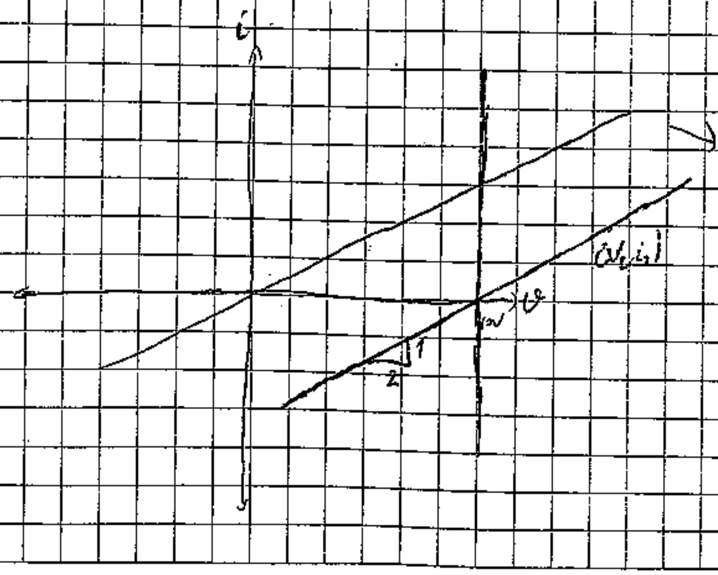


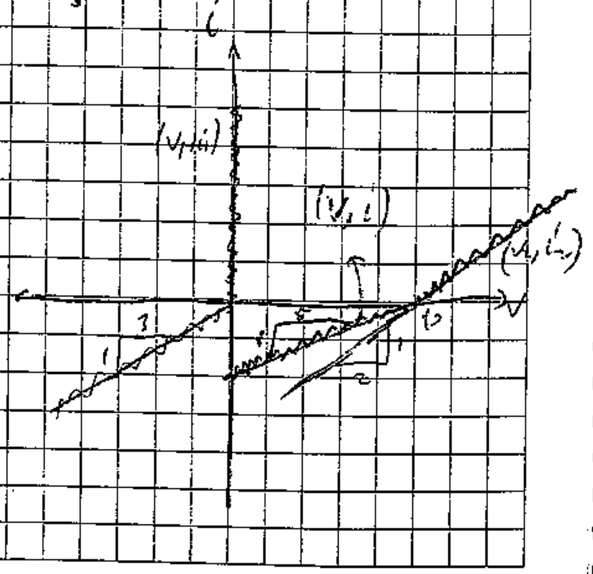


Ex



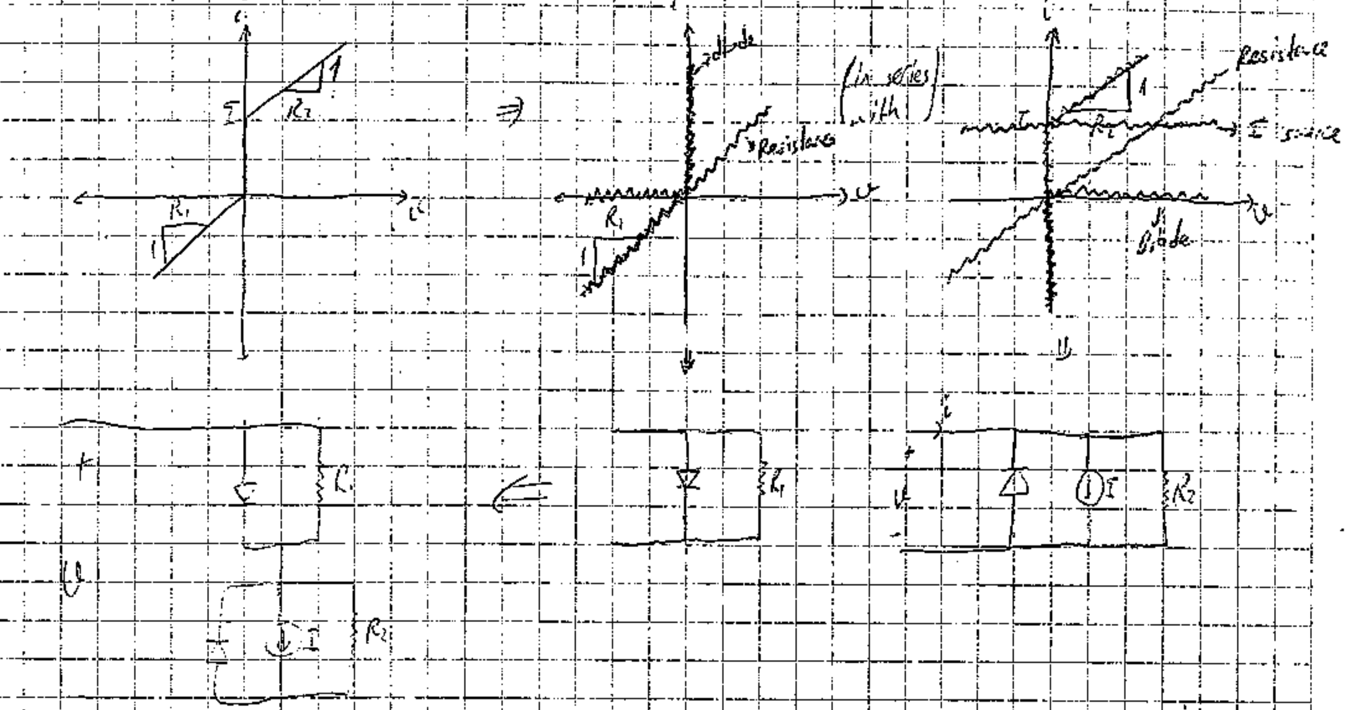




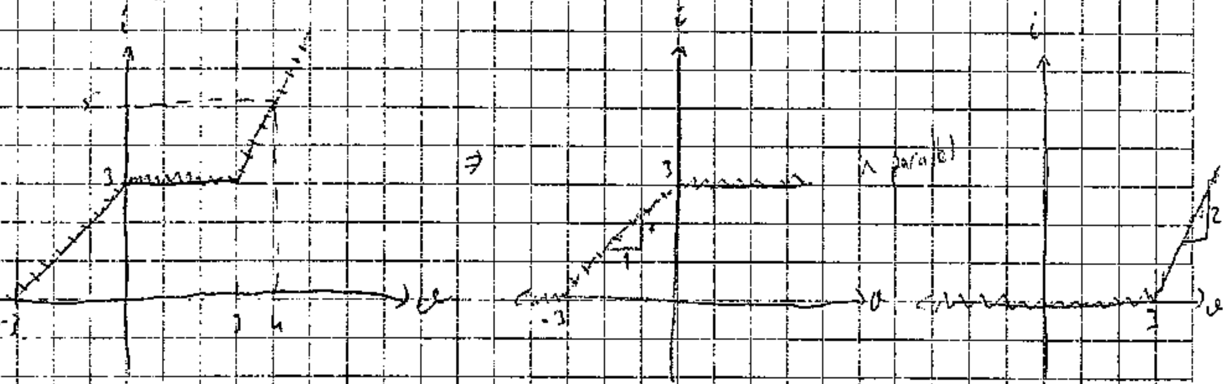


Synthesis (Design)

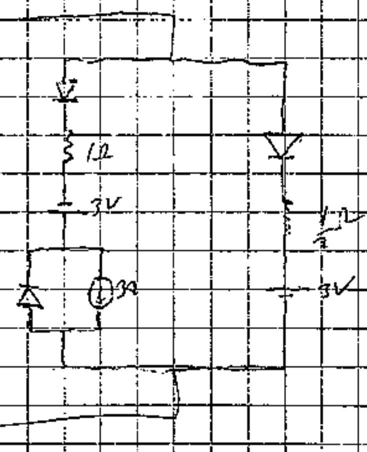
Design a one-port with R, Voltage Source, Current Source and diodes such that



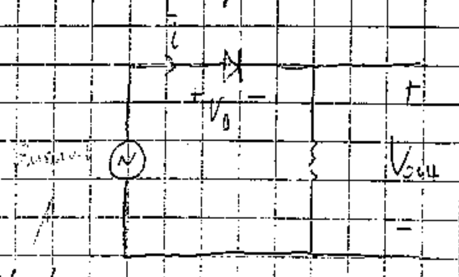
Ex//

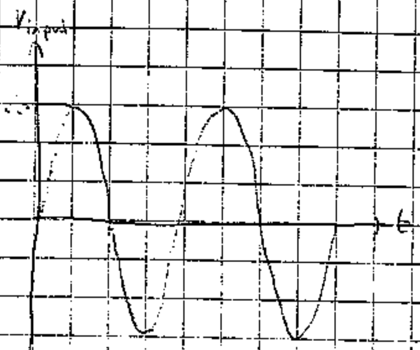


⇒ Answer ⇒

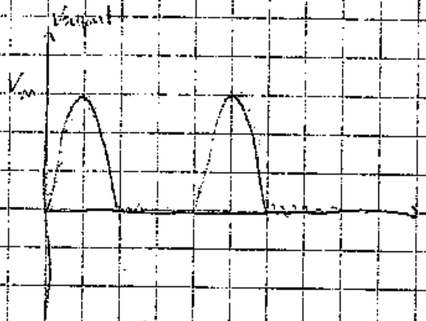


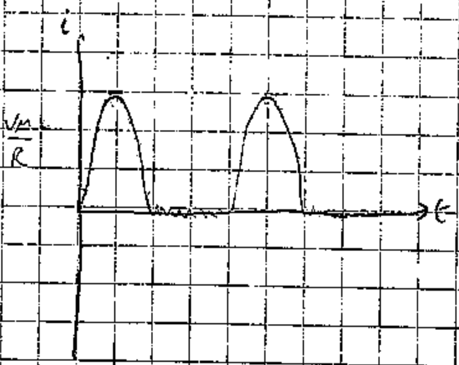
Diode Aplications



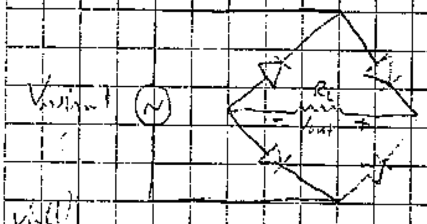


Half Wave Rectifier

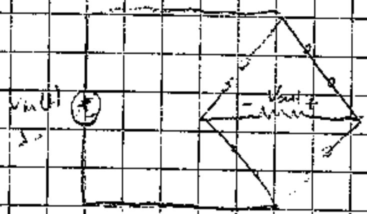




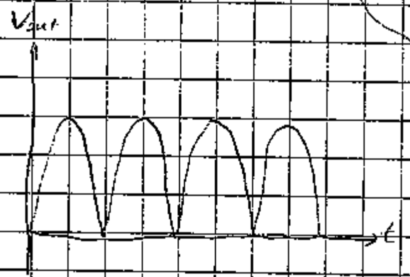
Full Wave Rectifier



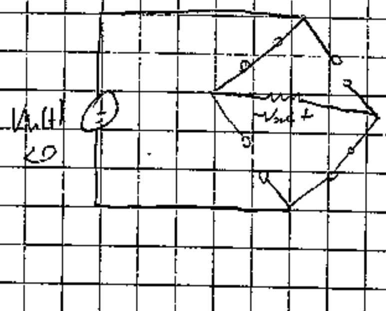
$V_{in}(t) > 0$



$V_{out} = V_{in}$



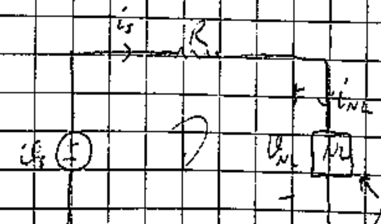
$V_{in}(t) < 0$



$V_{out} = -V_{in}$

Circuits with a single non-linear element

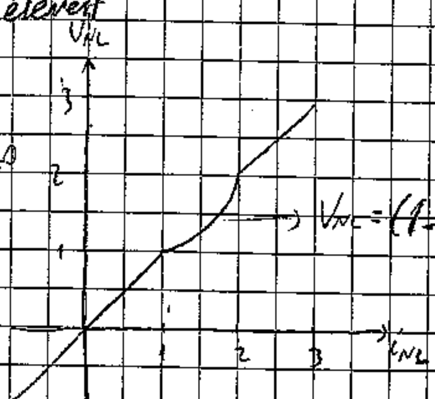
Load Line:



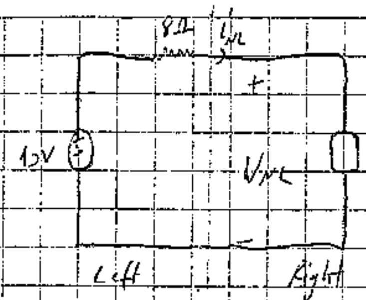
$-V_s + R i_{NL} + V_{NL}(i_{NL}) = 0$

$f_{NL}(i_{NL}) = V_{NL}$

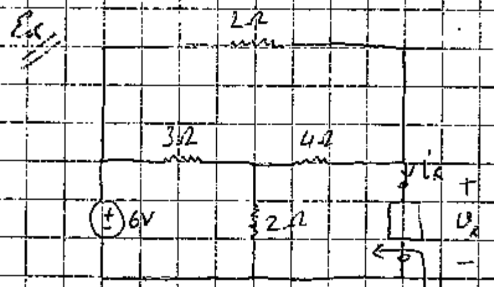
Non-linear memoryless component (non-linear resistor)



$V_{NL} = (1 - i_{NL})^2 + 1$

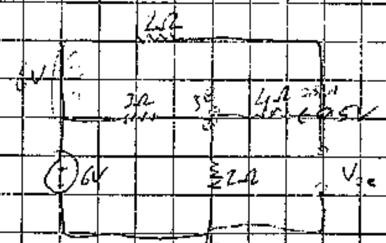


$$0 = -10 + 8 i_{NL} + v_{NL}$$



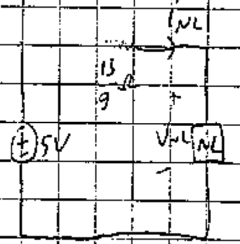
$$i_R = \begin{cases} 0.03\, V_R^2, & V_R \geq 0 \\ 0, & V_R < 0 \end{cases}$$

$$R_{Th} = ((3\|2)+4)\|(2) = \frac{13}{9}$$



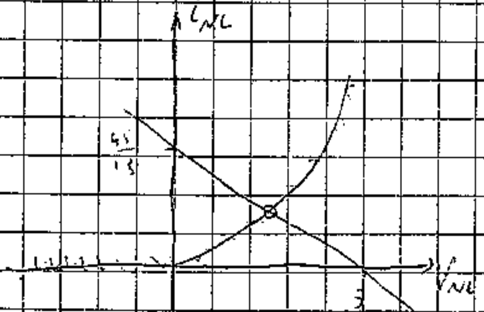
$$\frac{e-6}{3} + \frac{e}{2} + \frac{e-6}{2+4} = 0$$

$$\boxed{e = 3} \Rightarrow \boxed{V_{oc} = 5V}$$



$$-5 + \frac{13}{9} i_{NL} + V_{NL} = 0$$

$$i_{NL} = \frac{45}{13} - \frac{9}{13} V_{NL}$$



$$\frac{45}{13} - \frac{9}{13} V_{NL} = (0.03)(V_{NL})^2$$

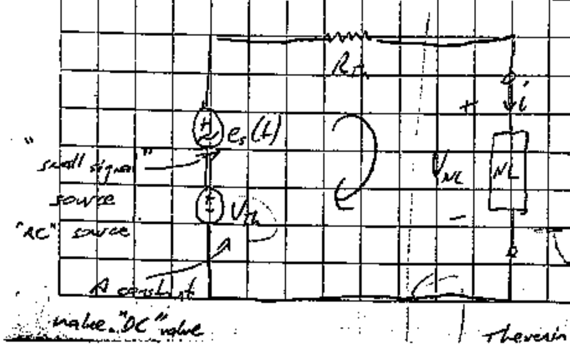
$$45 - 9 V_{NL} = 0.39 (V_{NL})^2 \qquad V_{NL} \geq 0$$

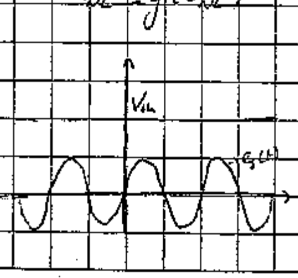
$$V_{NL} = 4.22$$

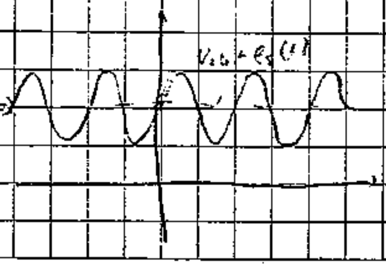
Small Signal Analysis

Let's assume that we have a circuit with a single non-linear element. The thevenin equivalent of this circuit is.

$$i_{NL} = g(v_{NL})$$







$$-(V_{th}+e_s(t)) + R\,i_{NL} + v_{NL} = 0$$

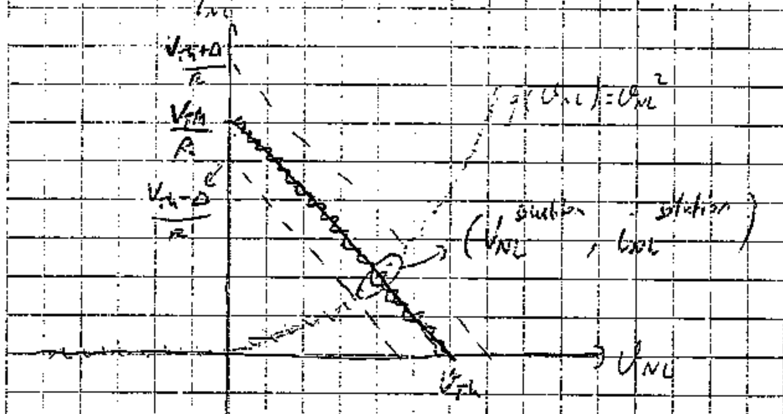
$$i_{NL} = g(v_{NL})$$

$$-(V_{th}+e_s(t)) + R\,g(v_{NL}) + v_{NL} = 0 \quad \leftarrow \text{(KVL) should be satisfied for all } t.$$

Due to small signal assumption: $e_s(t)$ is negligible in comparison with V_{th}

$$-V_{th} + R\,g(v_{NL}) + v_{NL} = 0 \quad \text{(using load lines, solution can be found)}$$

$$\frac{V_{th} - v_{NL}}{R} = g(v_{NL})$$



Then the DC operating point found from "load line" analysis gives us the solution when $e_s(t) = 0$. Since $e_s(t)$ is small, we can expand the $(i_{NL} = g(v_{NL}))$ $g(x)$ function around the operating point using Taylor series

$$g(x) = g(x_0) + \frac{(x-x_0)}{1!} g^{(1)}(x_0) + \frac{(x-x_0)^2}{2!} g^{(2)}(x_0) + \dots$$

Ex: $g(x) = x^2 \longrightarrow g'(x_0) \Rightarrow$ gives us the slope of the black curve.

Then the slope of the linear term can be interpreted as an LTI resistor.

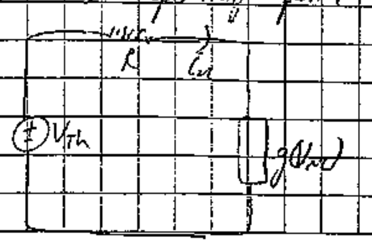
That is

$$g_{NL}^{small}(v) = g'(x_0)$$

small signal model for Nonlinear element

Summarizing:

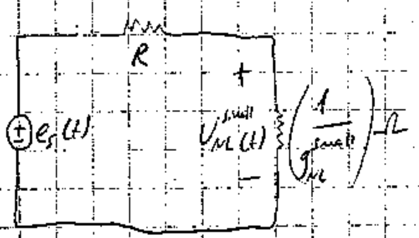
(1) Find DC operating point (using load line)



② Find small signal conductance parameter

$$g_{NL}^{small} = g'(v_{NL}^{DC}) \rightarrow \text{DC op. point}$$

③

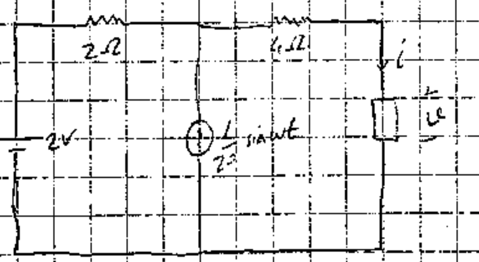


Find $v_{NL}^{small}(t)$

④ Final solution

$$v_{NL}(t) \cong v_{NL}^{DC} + v_{NL}^{small}(t)$$

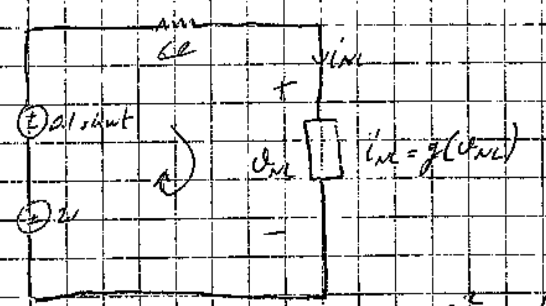
Ex//



Thevenin eq. $R_{eq} = 6\Omega$

$$V_{oc} = 2 - 2 + \frac{1}{20}\sin wt \cdot 2 = (2 + 0.1\sin wt)$$

①



DC op. point: (Neglect AC source)

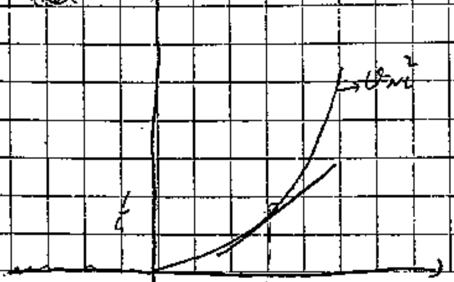
$$-2 + 6 i_{NL} + v_{NL} = 0$$

$$-2 + 6 v_{NL}^2 + v_{NL} = 0 \quad (\text{assuming } i_{NL} = v_{NL}^2 \text{ i.e. } v_{NL} > 0)$$

$$v_{NL}^2 + \frac{1}{6} v_{NL} - \frac{1}{3} = 0 \rightarrow v_{NL} = \left\{\underbrace{\frac{1}{2}}_{\text{op. point}}, -\frac{2}{3}\right\}$$

DC op. point $\Rightarrow (v_{NL}^{DC}, i_{NL}^{DC}) = \left(\frac{1}{2}, \frac{1}{4}\right)$

then

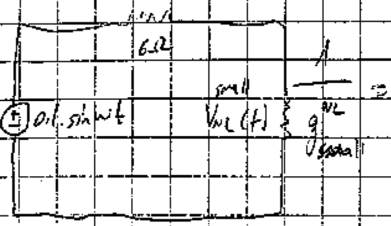


② $i_{NL} = v_{NL}^2 = g(v_{NL})$

$$= \frac{1}{4} + \frac{(v_{NL} - \frac{1}{2})}{1!} g'\left(\frac{1}{2}\right) + \dots$$

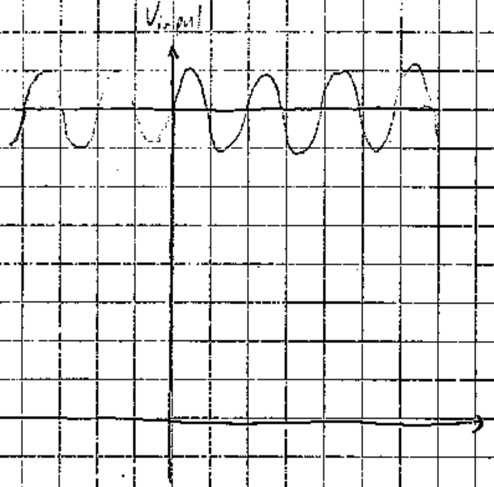
$$= \frac{1}{4} + \left(v_{NL} - \frac{1}{2}\right) \cdot 1$$

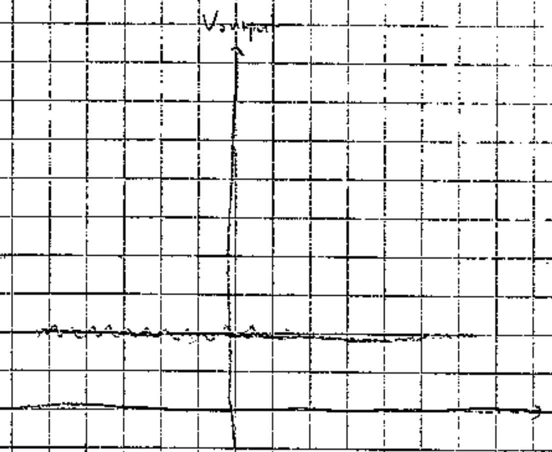
3



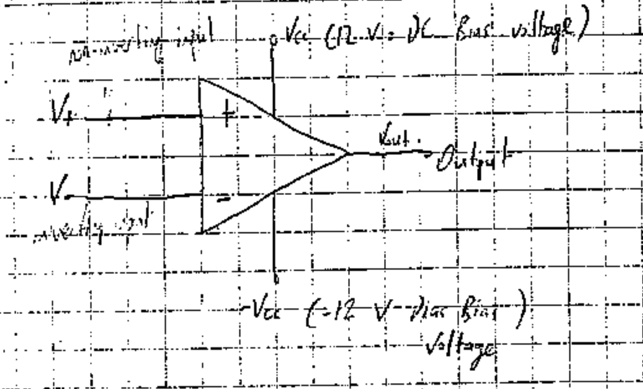
$$\frac{1}{g_{small}^{NL}} = 1\,\Omega \qquad v_{NL}^{small}(t) = \frac{0.1 \sin \omega t}{7}$$

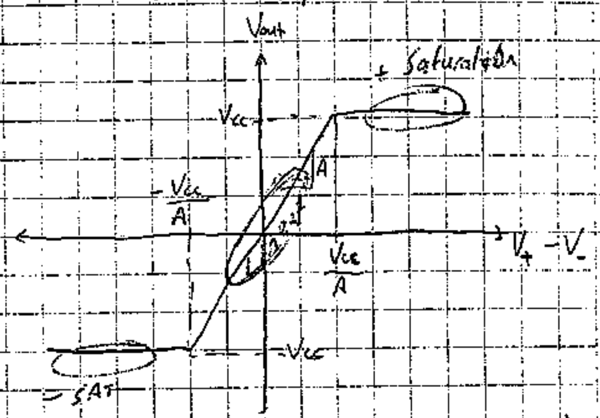
1) $$v_{NL}(t) = \underbrace{\frac{1}{2}}_{DC} + \underbrace{\frac{\sin \omega t}{70}}_{AC}$$





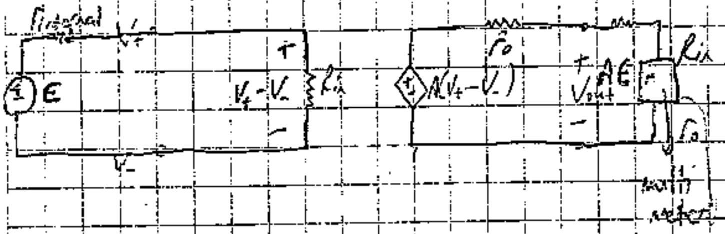
OPERATIONAL AMPLIFIERS





A: Open-loop gain (typically $10^5 - 10^8$)

A little more detailed model



R_{in} : input resistance ~ ($10^6 - 10^{12}$ Ω)

r_o : output resistance ~ (10-100 Ω)

model for only Linear Region

+SAT Region

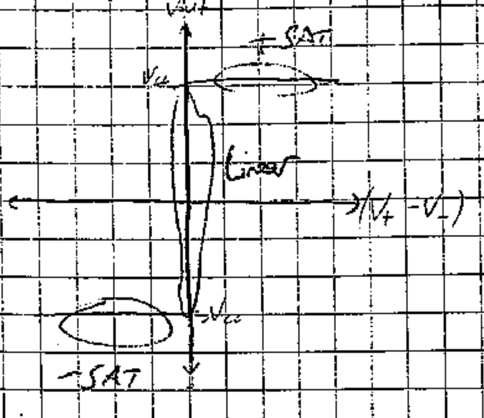
$V_{out} = +E_{sat}$ (Vcc) and $(V_+ - V_-)$ should be positive and sufficiently large $(V_+ - V_-) \geq \frac{E_{sat}}{A}$

−SAT Region

$V_{out} = -E_{sat}$, $(V_+ - V_-) \leq -\frac{E_{sat}}{A}$

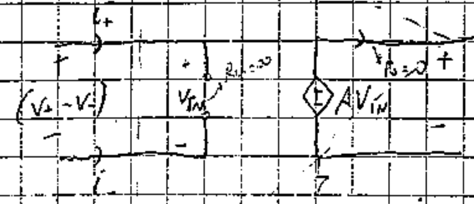
Ideal Op-Amp

For ideal Op-Amp



$R_{in} = \infty$, $r_o = 0$ (Linear region)

Using Ideal Op-Amp assumptions, we have in Linear Region;



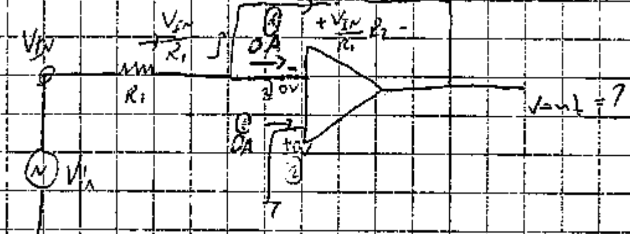
① $i_+ = i_- = 0$ A ← Linear region when

$A = \infty$

$i_+ = i_- = 0$

② $V_+ = V_-$ (for ideal opamp) in linear region

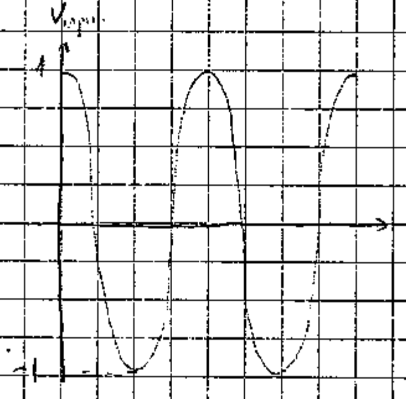
Inverting Amplifier (Ideal Op-Amp)

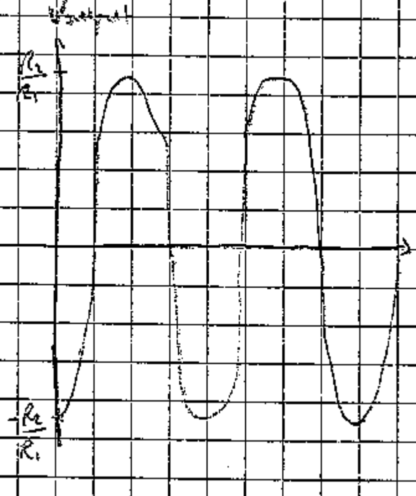


Assume linear region (assume V_{in} is adjusted so that Op-Amp is guaranteed to be in linear region)

$V_{out} = 0 - V_{R_2}$

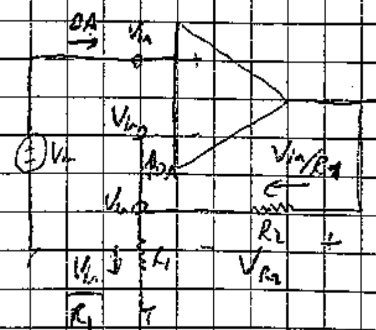
$V_{out} = -V_{IN} \frac{R_2}{R_1} \Rightarrow \boxed{\frac{V_{out}}{V_{IN}} = -\frac{R_2}{R_1}}$





Noninverting Op-Amp

Assume linear (V_{IN} is sufficiently small)



Linear region ($A = \infty$)

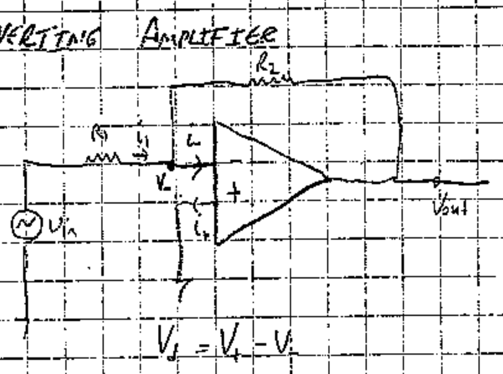
$V_{R_2} = \frac{V_{IN}}{R_1} R_2$

$V_{out} = V_{IN} + V_{R_2}$

$V_{out} = V_{IN} + \frac{V_{IN} R_2}{R_1}$

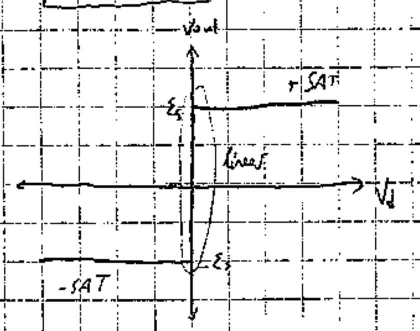
$V_{out} = V_{IN} \left(\frac{R_1 + R_2}{R_1} \right)$ → a positive gain → noninverting OPAMP

INVERTING AMPLIFIER



$V_d = V_+ - V_-$

$\boxed{\text{Ideal Model}}$ $(A = \infty,\ R_{in} = \infty,\ r_o = 0)$



① Assume linear operation $\rightarrow (V_d = 0,\ i_- = i_+ = 0\,A)$

$V_d = 0 \rightarrow V_+ = V_- = 0$

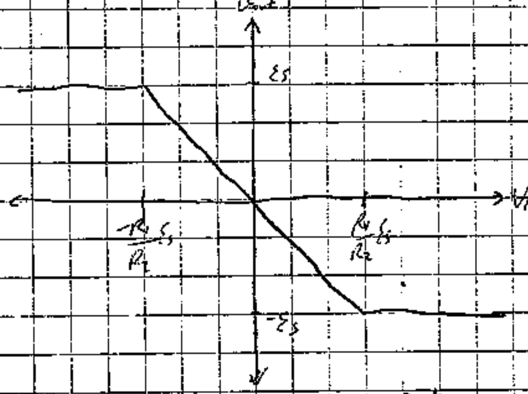
$i_1 = \dfrac{V_{in}}{R_1}$ and $V_{out} = 0 - V_{R_2} = 0 - i_1 R_2 = -\dfrac{R_2}{R_1} V_{in}$

Check the validity of linear region assumption $(V_d = 0,\ -E_s < V_{out} < E_s)$

$-E_s < -\dfrac{R_2}{R_1} V_{in} < E_s \Rightarrow \boxed{\dfrac{R_1}{R_2} E_s > V_{in} > -\dfrac{R_1}{R_2} E_s}$ ← input in this results in op-amp working in linear region.

② Assume $\mp$SAT, $(V_d \lessgtr 0,\ V_{out} = \mp E_s)$

$V_d = \underset{\overset{\uparrow}{0}}{V_+} - V_- \lessgtr 0 \Rightarrow \boxed{V_- \gtrless 0}$

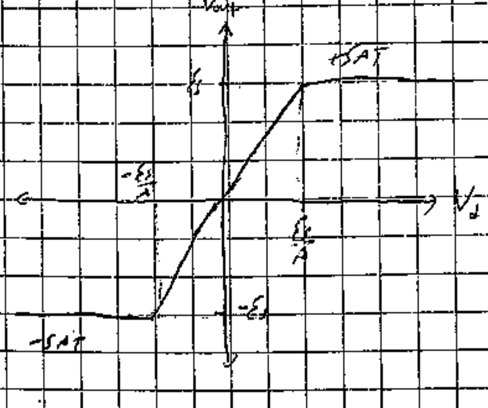


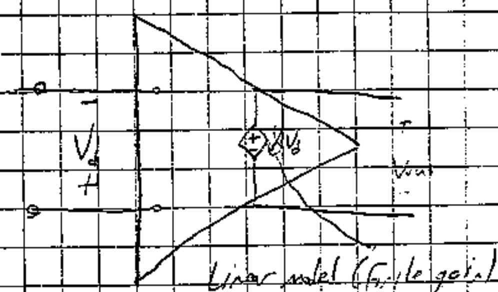
Node equation at V_-:

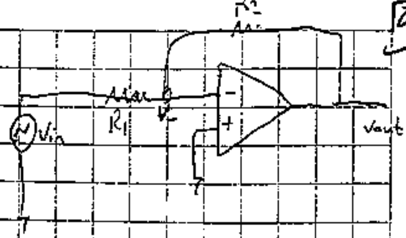
$\dfrac{(V_- - V_{IN})}{R_1} + \dfrac{(V_- - \overset{\mp E_s}{V_{out}})}{R_2} = 0$

$\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right) V_- = \dfrac{V_{IN}}{R_1} + \dfrac{\mp E_s}{R_2} \Rightarrow V_- = \dfrac{R_2}{R_1+R_2} V_{IN} + \dfrac{R_1}{R_1+R_2} \mp E_s \gtrless 0 \Rightarrow V_{IN} \gtrless \pm \dfrac{R_1}{R_2} E_s$

$\boxed{\text{Finite Gain}}$ $(A: \text{finite},\ R_{in} = \infty,\ r_o = 0)$







1) Assume linear: $|V_d| < E_s/A \rightarrow |V_-| < E_s/A$, $|V_{out}| < E_s$, $i_- = i_+ = 0$ ($R_{in} = \infty$)

KCL at V_-: ($AV_d = -AV_-$)

$$\left(\frac{V_- - V_{IN}}{R_1}\right) + \left(\frac{V_- - V_{out}}{R_2}\right) = 0$$

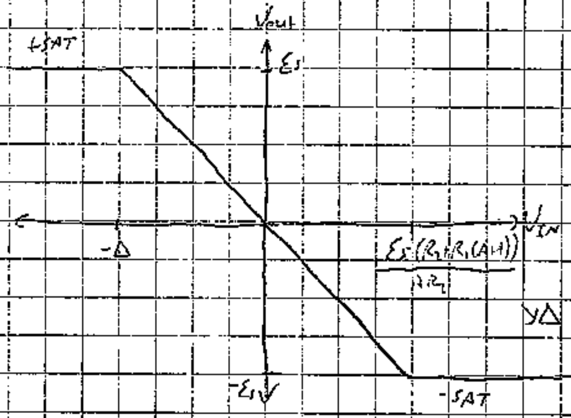
$$R_2(V_- - V_{IN}) + R_1(V_- + AV_-) = 0 \Rightarrow V_- = \frac{R_2}{R_2 + (A+1)R_1} V_{IN}$$

For the validity of linear region assumption

$$|V_-| < E_s/A \qquad |V_{IN}| < \frac{E_s(R_2 + (A+1)R_1)}{AR_2} \qquad \text{if } A \to \infty \quad |V_{IN}| < \frac{E_s R_1}{R_2}$$

$$V_{out} = AV_d = -AV_-$$

$$\boxed{V_{out} = -A\frac{R_2}{R_2 + R_1(A+1)} V_{IN}} \qquad A \to \infty \qquad V_{out} = -\frac{R_2}{R_1} V_{IN}$$



2) Assume $+SAT$

$\left(V_d > \frac{E_s}{A}\right)$, $R_{in} = \infty$ ($i_- = i_+ = 0\,A$)

Write node equation at V_-

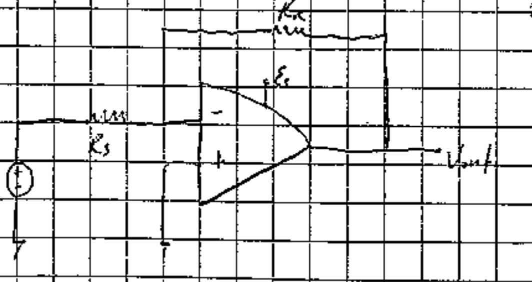
$$\left(\frac{V_- - V_{IN}}{R_1}\right) + \frac{(V_- - V_{out})}{R_2} = 0$$

$$V_- = \frac{R_2 V_{IN} + R_1 E_s}{R_1 + R_2}$$

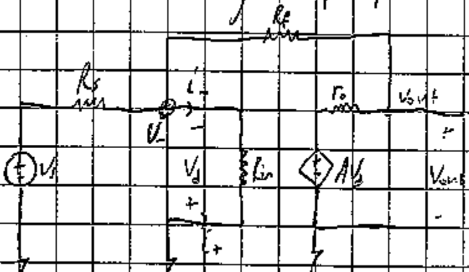
$V_d > \frac{E_s}{A} \rightarrow -V_- > \frac{E_s}{A}$ ($0 - V_-$)

$$V_- < -\frac{E_s}{A} \rightarrow \frac{R_2 V_{IN} + R_1 E_s}{R_1 + R_2} < -\frac{E_s}{A} \Rightarrow V_{IN} < \underbrace{-\left(\frac{E_s}{A}\frac{(R_1+R_2)}{R_2} + \frac{R_1 E_s}{R_2}\right)}_{-\Delta}$$

Inverting Op-Amp



1) Assume linear region of operation



Node equation at V_-:

$$\frac{V_- - V_s}{R_s} + \frac{V_-}{R_{in}} + \frac{V_- - V_{out}}{R_f} = 0$$

For the validity of the linear region model

$|V_{out}| < E_s$, $i_- \neq 0$, $i_+ \neq 0$

Node eqn at V_{out}: $\frac{V_{out} - V_-}{R_f} + \frac{V_{out} - AV_d^{\to -V_-}}{r_o} = 0$

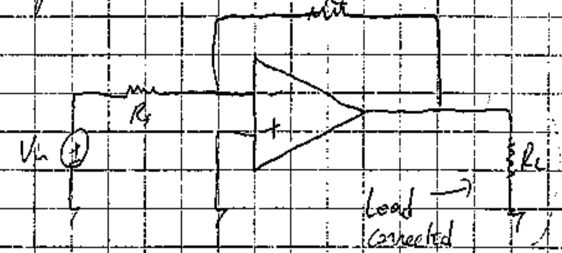
open loop gain: $A = 10^6$, $r_o = 10-100\,\Omega$'s, $R_{in} = M\Omega$'s | $A = 10^5$, $r_o = 100\,\Omega$, $R_{in} = 100\,k\Omega$

$$V_{out} = \frac{-A + (r_o/R_f)}{\frac{R_s}{R_f}\left(1 + A + \frac{r_o}{R_f}\right) + \left(\frac{R_s}{R_{in}} + 1\right) + \frac{r_o}{R_f}}$$

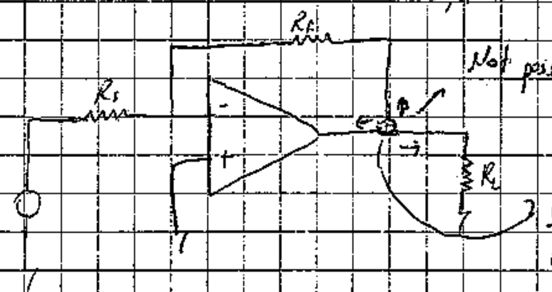
As expected as $\begin{cases} A \to \infty \\ R_{in} \to \infty \\ r_o \to 0 \end{cases} \Rightarrow V_{out} = -\frac{R_f}{R_s} V_s$

Important Note

① Using the model with finite A, R_{in} and $r_o \neq 0$, we can also analyze the following circuit



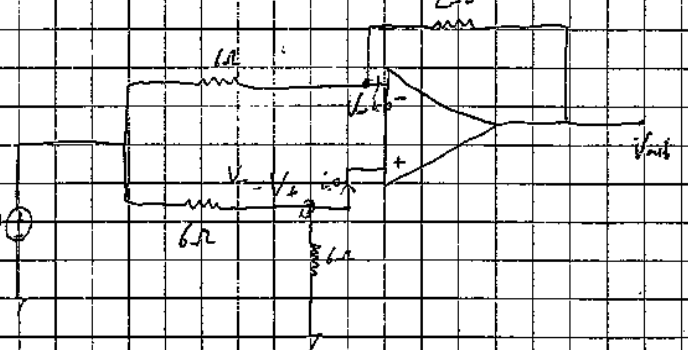
② For the earlier less accurate models, it is not possible to include the effect of R_L, since



Nodal Analysis For Op-Amp Circuits

If ideal model or finite gain model is used then writing node equations except at the output node can easily lead to the solution

Ex:



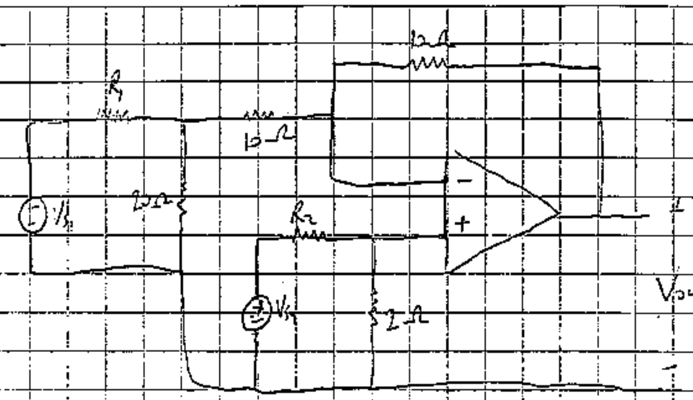
(Assume linear region operation and $A \to \infty$)

KCL at V_-: $\frac{V_- - V_g}{6} + \frac{V_- - V_{out}}{2} = 0 \Rightarrow V_{out} = \frac{4}{3}V_- - \frac{V_g}{3} \Rightarrow V_{out} = \frac{1}{3} V_g$

$V_{out} = 2\cos 2t$

KCL at V_+: $\frac{V_-}{6} + \frac{V_- - V_g}{6} = 0 \Rightarrow V_- = \frac{V_g}{2}$

Ex/

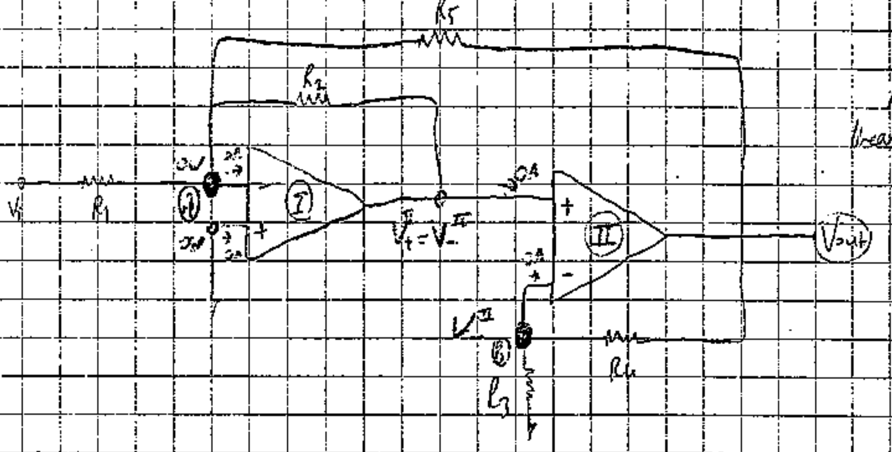


Assuming ideal model and linear operation

Find R_1 and R_2 s.t.

$$V_o = V_{s_2} - \frac{V_{s_1}}{4}$$

Ex/



Assume ideal model and linear region; find V_{out}.

KCL at node A: $\dfrac{0-V_i}{R_1} + \dfrac{0-V_{out}}{R_5} + \dfrac{0-V_-^{II}}{R_2} = 0$

KCL at node B: $\dfrac{V_-^{II}-V_{out}}{R_4} + \dfrac{V_-^{II}}{R_3} = 0$

Mathematical calculations $\rightarrow$ $V_-^{II} = -\dfrac{R_3}{R_1}\,\dfrac{R_5R_2}{R_2(R_3+R_4)+R_3R_5}$ and $V_o = \dfrac{-R_5R_2(R_3+R_4)}{R_2(R_3+R_4)+R_3R_5}\,\dfrac{V_i}{R_1}$

· Let's check the conditions for V_{in} so that both op-amps are in linear region.

Op-Amp (I) $\Rightarrow$ $V_{out}^{I} = V_-^{II}$ $\quad |V_{out}^{I}| < E_s$

Op-Amp (II) $\Rightarrow$ $V_{out}^{II} = V_{out}$ $\quad |V_{out}^{II}| < E_s$

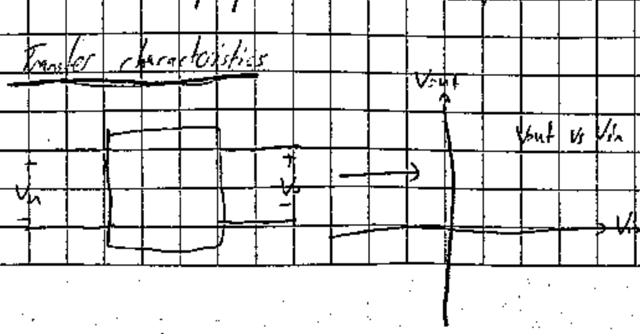
Both conditions should be satisfied at the same time for the validity of analysis.

$V_{in} \in [A^{I}, B^{I}]$ $\qquad V_{in} \in [A^{II}, B^{II}]$

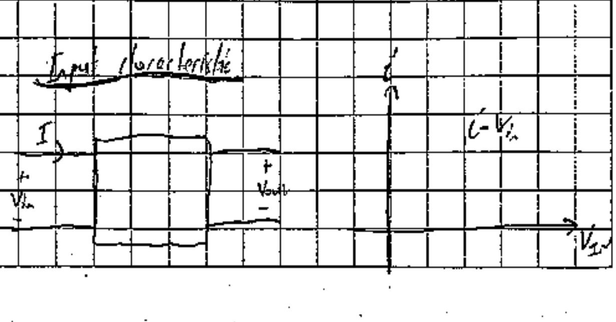
Range for V_{in} for the opamp (I)

V_{in} should be the intersection of these two intervals

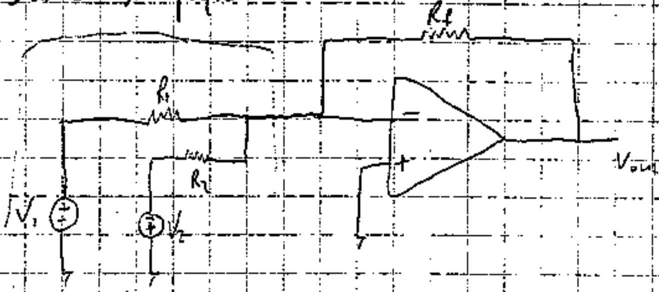
Transfer characteristics



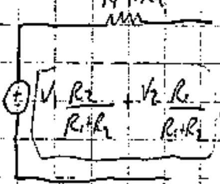
Input characteristic



Summer Amplifier



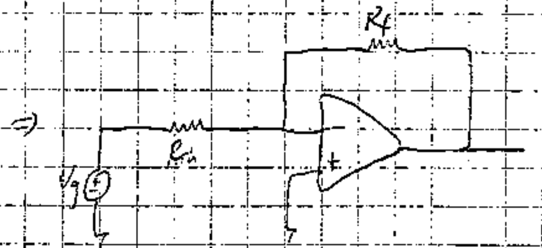
V_{out} ⇒ by Thevenin eq. first.



$R_1 \| R_2 = R_{th}$

$$V_1\frac{R_2}{R_1+R_2} + V_2\frac{R_1}{R_1+R_2} = V_g$$

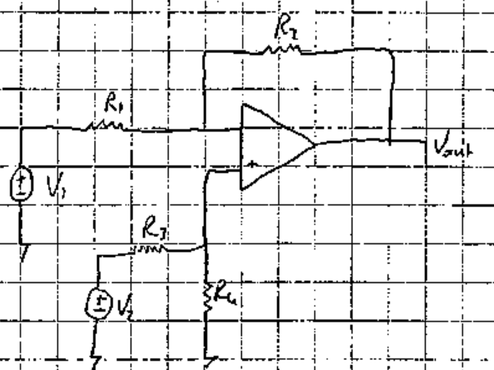
⇒



$$V_{out} = -\frac{R_f}{R_{th}}V_g = -\frac{R_f(R_1+R_2)}{R_1R_2}\left(\frac{V_1R_2+V_2R_1}{R_1+R_2}\right)$$

$$V_{out} = -\frac{R_f}{R_1}V_1 - \frac{R_f}{R_2}V_2$$

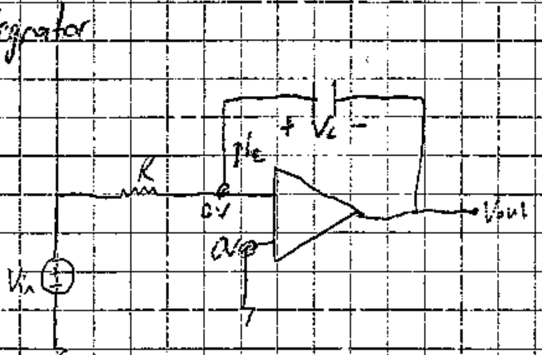
Difference Amplifier



$$V_{out} = \left(\frac{R_1+R_2}{R_1}\right)\left(\frac{R_4}{R_3+R_4}\right)V_2 - \frac{R_2}{R_1}V_1$$

if $\frac{R_3}{R_4} = \frac{R_1}{R_2}$ ⇒ $V_{out} = \frac{R_2}{R_1}(V_2 - V_1)$

Integrator



$$i_c = C\frac{dV_c(t)}{dt}$$

$$V_{out} = -V_c$$

$$\frac{0-V_{in}}{R} + i_c = 0$$

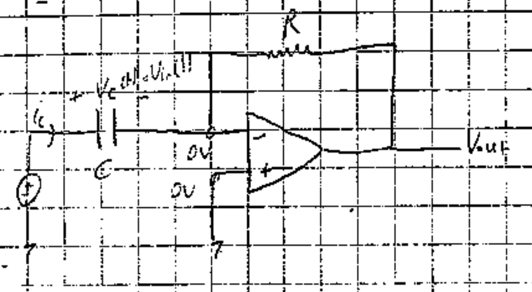
$$\frac{V_{in}(t)}{R} = C\frac{dV_c(t)}{dt}$$

$$\Rightarrow \frac{dV_c(t)}{dt} = \frac{1}{RC}V_{in}(t)$$

$$\int_0^t dV_c(t) = \int_0^t \frac{1}{RC}V_{in}(t)dt \Rightarrow V_c(t) - V_c(0^-) = \frac{1}{RC}\int_0^t V_{in}(\tau)d\tau$$

$t=0^-$ value is not given, assume $V_c(-\infty) = 0V$ ⇒ $V_c(t) = \frac{1}{RC}\int_{-\infty}^t V_{in}(\tau)d\tau$

Differentiator



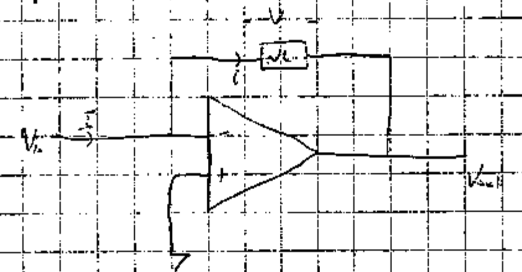
differentiable

$$i_c = C\dot{V}_c(t) = C\frac{dV_c(t)}{dt}$$

$$i_c = C\dot{V}_{in}(t)$$

$$V_{out} = -Ri_c = -RC\dot{V}_{in}(t)$$

Opamps & Nonlinear Elements

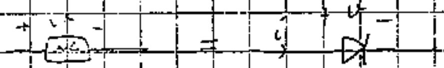


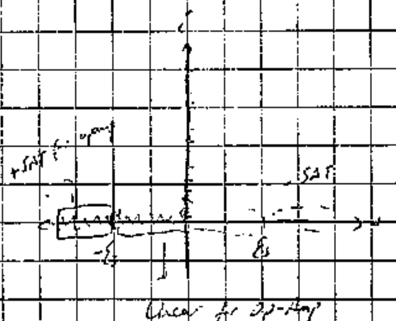
+SAT: $V_{out} = +E_s$, $V_+ > V_-$ } $V_- < 0$, $V_{in} < 0$; $v = V_{in} - V_{out} - E_s$ (negative); $v < -E_s$

−SAT: $V_{out} = -E_s$, $V_+ < V_-$ } $V_- > 0$; $V_i > 0$; $v = V_{in} - V_{out} \cdot -E_s$ (positive); $v > E_s$

Linear: $V_+ = V_- = 0$, $-E_s < V_{out} < E_s$ → $V_- = 0$, $V_{in} = 0$; $V_{out} = -v$; $V_{out} = -v$; $-E_s < v < E_s$

Ex:

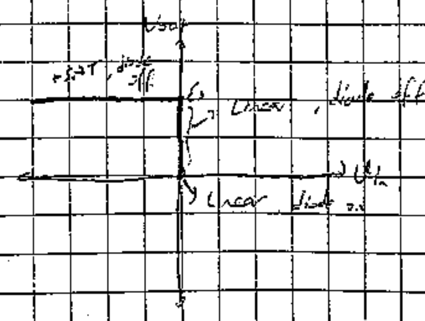




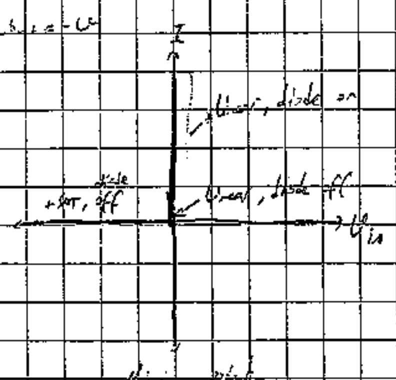
+SAT (Op-Amp) → ($v < -E_s$) → Diode characteristic $i = 0$
$V_{out} = E_s$
$V_{in} < 0$

−SAT (Op-Amp) → $v > E_s$ → not possible to have a current i so that $v > E_s$
$V_{out} = -E_s$
$V_i > 0$
→ Op-Amp does not enter into −SAT region

Linear (Op-Amp) → $-E_s < v < E_s$ → Only possible for ① $-E_s < v < 0$ → $i = 0$; ② $v = 0$ → $i > 0$
$V_i = 0$
$-E_s < V_{out} < E_s$
$V_{out} = -v$



Transfer char

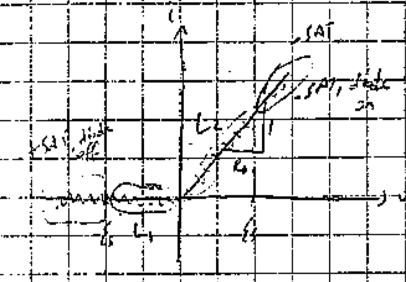


Driving point plot

Ex 2

$+SAT: \; v < -\mathcal{E}_s \rightarrow i = 0$

$V_{out} = \mathcal{E}_s$



$-SAT: \; v > \mathcal{E}_s \rightarrow i = \frac{v}{R_0} = \frac{V_i - (-\mathcal{E}_s)}{R_0} \Rightarrow V_i = iR_0 - \mathcal{E}_s$

$V_{out} = -\mathcal{E}_s$

Linear: $-\mathcal{E}_s < V_{out} < \mathcal{E}_s \rightarrow -\mathcal{E}_s < v < \mathcal{E}_s \xrightarrow{L_1} -\mathcal{E}_s \le v \le 0$, $i = 0$

$V_i = 0$

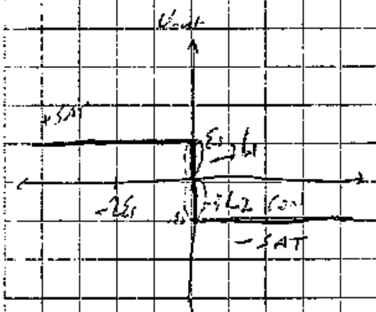
$\xrightarrow{L_2} 0 \le v \le \mathcal{E}_s$

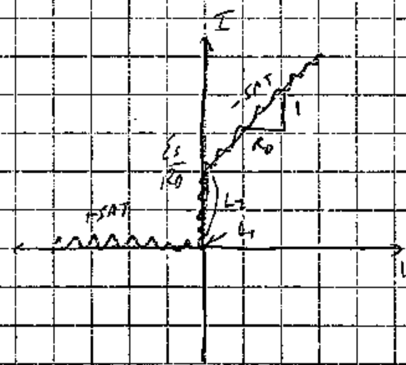
$i = \frac{v}{R_0} = \frac{-V_{out}}{R_0}$

$-\mathcal{E}_s < V_{out} < 0$

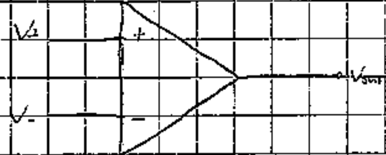
V_i

$-\mathcal{E}_s < V_{out} < \mathcal{E}_s$





The Common Mode Rejection Ratio (CMRR)



Ideal Op-Amp (Linear Region).

$$V_{out} = A(V_+ - V_-) = AV_+ - AV_-$$

Practical Op-Amp

$$V_{out} = A_+V_+ - A_-V_- \quad \text{and} \quad A_+ \neq A_-$$

$(V_{common}) \rightarrow V_c = \frac{V_+ + V_-}{2}$ $\quad$ $(V_{diff}) \rightarrow V_d = V_+ - V_-$ $\quad \Rightarrow \quad V_+ = V_c + V_d/2$, $\quad V_- = V_c - V_d/2$

Then practical Op-Amp:

$$V_{out} = A_+V_+ - A_-V_-$$
$$= A_+\left(V_c + \frac{V_d}{2}\right) - A_-\left(V_c - \frac{V_d}{2}\right)$$
$$V_{out} = (A_+ - A_-)V_c + \left(\frac{A_+ + A_-}{2}\right)V_d$$

$A_d(\underbrace{V^+ - V^-}_{V_d}) + A_c \underbrace{\frac{(V^+ + V^-)}{2}}_{V_c}$

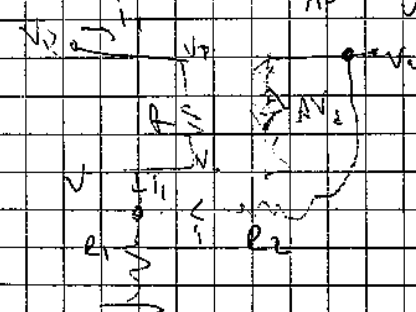
$A_c \triangleq A_+ - A_-$ (Common Mode Gain)

$A_d \triangleq \frac{A_+ + A_-}{2}$ (Differential Mode Gain)

Ideally: $A_c = 0$ $\quad A_d = A$

$CMRR \triangleq \left|\frac{A_d}{A_c}\right|$ (Ideally ∞)

$(CMRR)_{dB} = 20\log_{10}\left|\frac{A_d}{A_c}\right|$

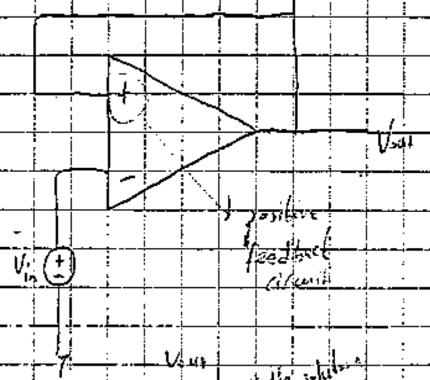


$$\frac{V_1 - V}{R} + \frac{V_2 - V}{R_2} = \frac{V}{R_1}$$

$A_d V$

$$\frac{V_1 - V}{R} + \frac{V_2 - V}{R_2} =$$

Positive - Negative feedback

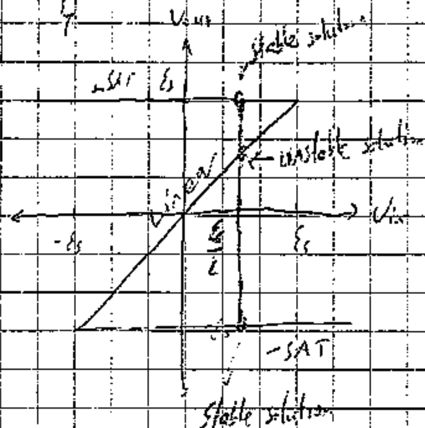


(1) +SAT : $V_{out} = E_s$ $\quad V_d > 0 \Rightarrow V_{in} < E_s$

$V_d = E_s - V_{in}$

(2) −SAT : $V_{out} = -E_s$ $\quad V_d < 0 \quad V_{in} > -E_s$

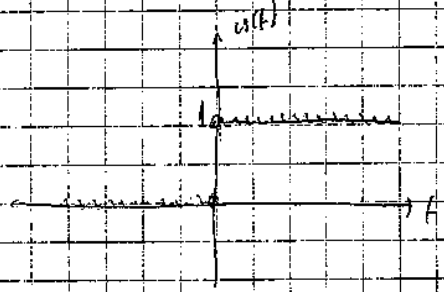
$V_d = -E_s - V_{in}$



WAVEFORMS

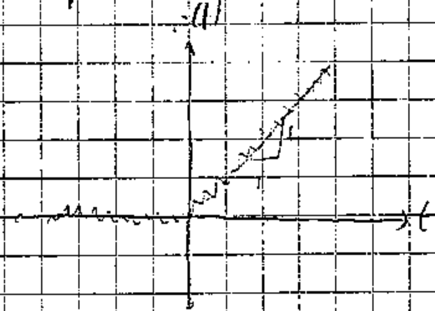
* Unit step, ramp, impulses

1) Unit Step function



$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & \text{other} \end{cases} \quad t \neq 0$$

2) Ramp function



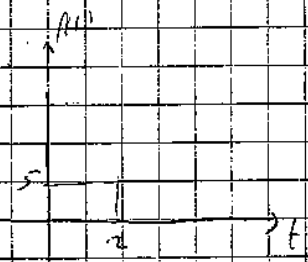
$r(t) = t\,u(t)$

Properties

① $\frac{dr(t)}{dt} = u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \quad (t \neq 0)$

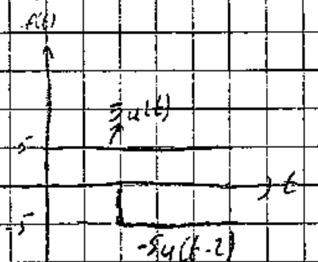
② $r(t) = \int_{-\infty}^{t} u(t)\,dt$

Ex/

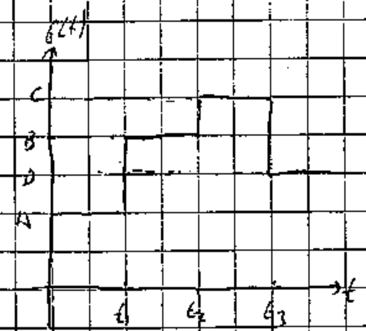


Express $f(t)$ as a linear combination of unit step functions

$f(t) = 5u(t) - 5u(t-2)$



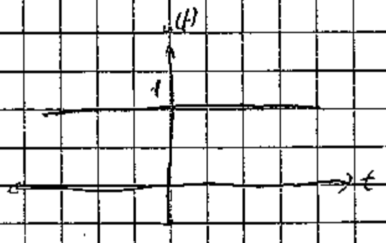
Ex/



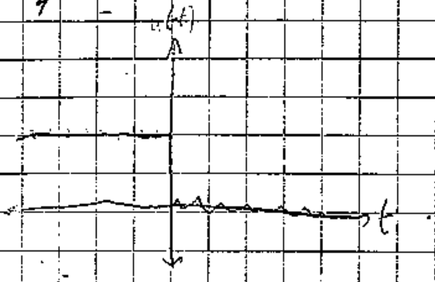
$f(t) = Au(t) + (B-A)u(t-t_1) + (C-B)u(t-t_2) + (D-C)u(t-t_3)$

Ex $u(t^2) \stackrel{?}{=} 1$

$\hookrightarrow t \in$ Real number

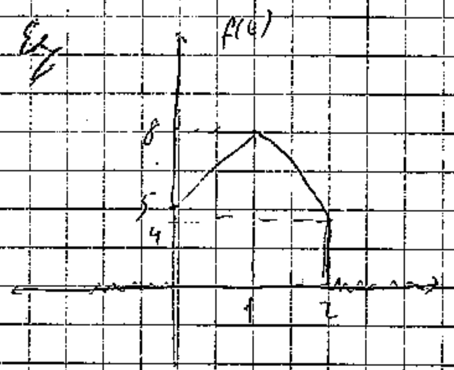


Ex: $u(-t) = ?$



$u(-t)$ is the reflection of $u(t)$ w.r.t. y-axis.

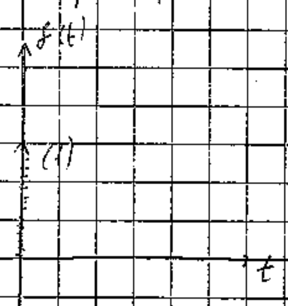
Ex:

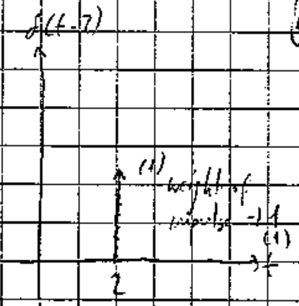


$5u(t)+3r(t)-7r(t-1)+4r(t-2)-4u(t-2)$

Impulse function

$\delta(t)$: impulse function





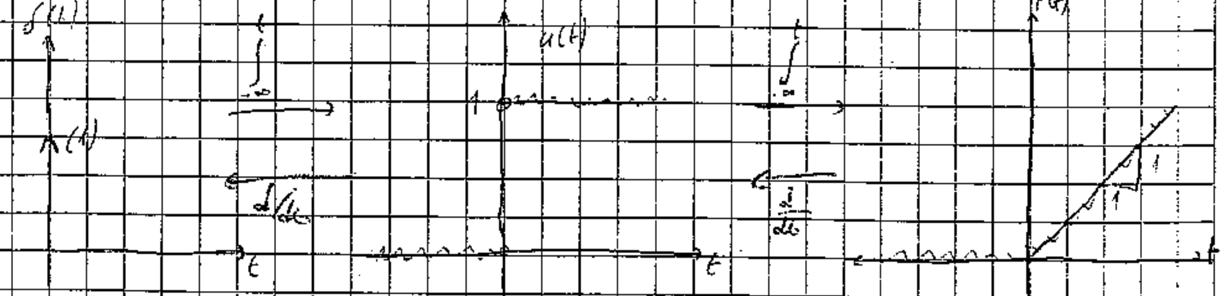
Properties

① Sifting property: $\int_{-\epsilon}^{\epsilon}\delta(t)dt = 1 \quad \forall \epsilon > 0$

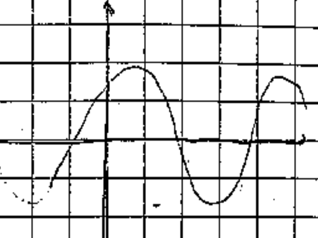
$\delta(t)$: Generalized function (function of ...)

② $u(t) = \int_{-\infty}^{t}\delta(\tau)d\tau$

③ $\frac{d}{dt}u(t) = \delta(t)$ (using previously discussed Leibnitz's $u(t) = \frac{dr(t)}{dt}$ rule.



Sinusoidal Waveform



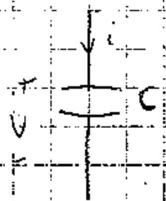
$s(t) = \underbrace{A}_{\text{Amplitude}}\cos(2\pi f t + \underbrace{\theta}_{\text{phase (radians)}})$

Hz = 1/s seconds

$w = 2\pi f \Rightarrow$ angular freq (rad/sec)

$T = \frac{1}{f} \Rightarrow$ T: period (sec)

CAPACITORS



$Q = CV$

$Q(t) = C(t)\,V(t)$

$\frac{dQ(t)}{dt} = \frac{dC(t)}{dt} V(t) + C(t)\frac{dV_c(t)}{dt} = i(t)$

For LTI Capacitors $\Rightarrow C(t) = C$

$$\boxed{i(t) = C\frac{dV_c(t)}{dt} = C\,D V_c(t)}$$

$p_{cap}(t) = V_c(t)\,i_c(t)$ (instantaneous power)

Energy $\Rightarrow \int_{-\infty}^{t} p_{cap}(\tau)\,d\tau = \int_{-\infty}^{t} V_c(\tau)\, C \frac{dV_c(\tau)}{d\tau}\,d\tau$

$= C\int_{-\infty}^{t} \underbrace{V_c(t)}_{u}\,\underbrace{dV_c(t)}_{du} = C\int_{V_{-\infty}}^{V_t} u\,du$

$= C\frac{u^2}{2}\Big|_{V_{-\infty}}^{V_t} = \frac{C}{2}\left(V_c^2(t) - \cancel{V_c^2(-\infty)}^{\,0}\right)$

$E_{cap}(t) = \frac{C}{2} V_c(t)$

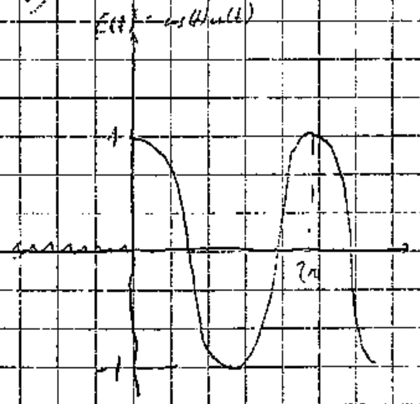
Ex: $f(t) = f_1(t) f_2(t)$

$\frac{d}{dt} f(t) = \dot f_1(t) f_2(t) + f_1(t)\dot f_2(t)$

$\dot f_2(t) = \frac{d f_2(t)}{dt}$

Ex: $f(t) = \cos(t)u(t)$

$f(t) = \cos(t)u(t)$



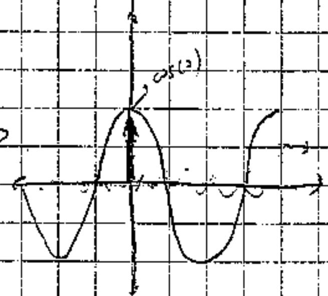
$f(t) = \cos(t)u(t)$

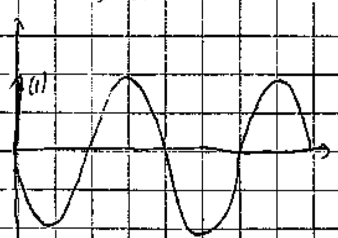
$\dot f(t) = -\sin(t)u(t) + \cos(t)\frac{d\,u(t)}{dt}$

$\dot f(t) = -\sin(t)u(t) + \cos(t)\,\delta(t)$

$\dot f(t) = -\sin(t)u(t) + \cos(0)\,\delta(t)$

$\dot f(t) = -\sin(t)u(t) + \delta(t)$



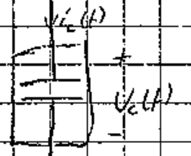


LTI Capacitors

$Q = CV$ — C is not a function of time

$\frac{d}{dt}$ $\Downarrow$

$i_c(t) = C\frac{dV_c(t)}{dt}$



Most Important Fact

Capacitor voltage is a continuous function if input (current through the capacitor) is bounded (not infinity)

True since:

$$i_c(t) = C\frac{dv_c(t)}{dt}$$

$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i_c(\tau)\,d\tau = \frac{1}{C}\int_{-\infty}^{t_0} i_c(\tau)\,d\tau + \frac{1}{C}\int_{t_0}^{t} i_c(\tau)\,d\tau \qquad t_0 < t$$

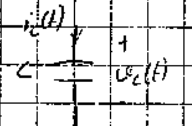
$$v_c(t) = v_c(t_0) + \frac{1}{C}\int_{t_0}^{t} i_c(\tau)\,d\tau$$

finite $\Rightarrow |i_c(\tau)| < M^+$

$$v_c(t_0^+) = v_c(t_0) + \frac{1}{C}\int_{t_0}^{t_0^+} i_c(\tau)\,d\tau \qquad t_0^+ = t_0 + \epsilon \quad \epsilon > 0 \text{ (arbitrarily small)}$$

$$v_c(t_0^+) \le v_c(t_0) + \frac{1}{C}M\epsilon \Rightarrow v_c(t_0^+) = v_c(t_0) \quad \text{as } \epsilon \to 0$$

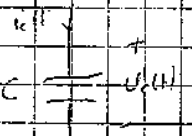
Initial Condition Models for Capacitors



$v_c(t_0) = V_0$

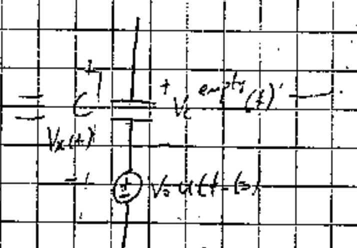
$E_{cap}(t_0) = \frac{1}{2}CV_0^2$ Joules

$$v_c(t) = v_c(t_0) + \frac{1}{C}\int_{t_0}^{t} i_c(\tau)\,d\tau \qquad t > t_0$$



$v_c(t_0) = V_0$

$=$

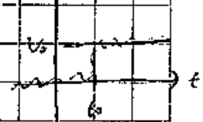


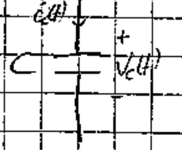
$v_c^{empty}(t_0) = 0V$

$$\to v_c^{empty}(t) + V_0 u(t-t_0) \Rightarrow v_x(t) \stackrel{?}{=} v_c(t)$$

$$v_c^{empty}(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau + V_0 u(t-t_0) \stackrel{?}{=} v_c(t)$$

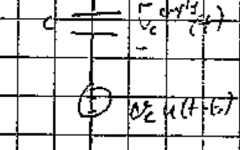
$$V_0 + \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau \quad (t > t_0)$$



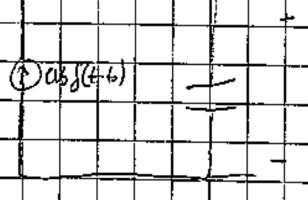


$v_c(t_0) = V_0$

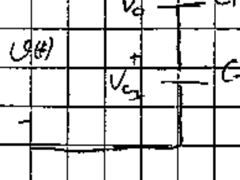
$=$



$V_c^{empty}(t_0) = 0V$



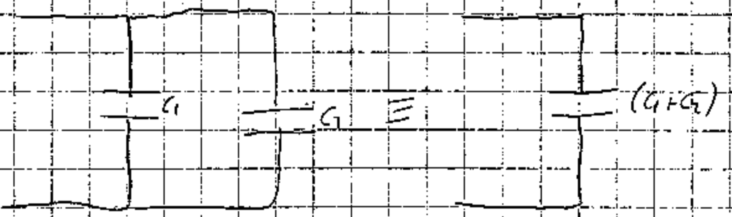
Series and Parallel Combination of Capacitors



$$\to v(t) = v_{c_1}(t) + v_{c_2}(t)$$

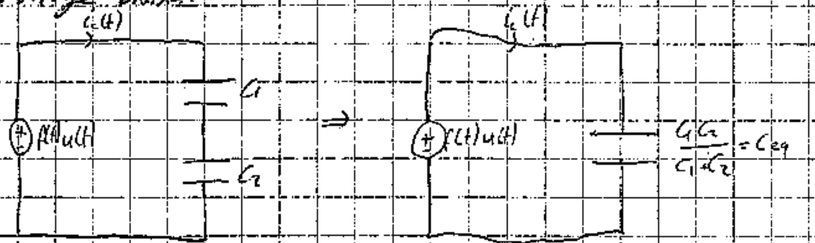
$$= \frac{1}{C_1}\int_{t_0}^{t} i(\tau)\,d\tau + \frac{1}{C_2}\int_{t_0}^{t} i(\tau)\,d\tau = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\int_{t_0}^{t} i(\tau)\,d\tau = \frac{1}{C_{eq}}\int_{t_0}^{t} i(\tau)\,d\tau$$

$$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}$$



$\Leftarrow$ parallel

Voltage division



$V_{c_1}(0) = 0$

$V_{c_2}(0) = 0$

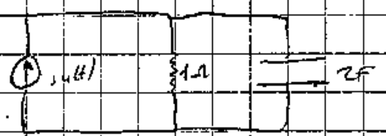
$i_c(t) = C_{eq}\dfrac{d(V_c(t))}{dt} = C_{eq}\left[f'(t)u(t) + f(t)\delta(t)\right]$

$V_1(t) = \dfrac{1}{C_1}\displaystyle\int_0^t i_c(\tau)d\tau = \dfrac{C_{eq}}{C_1}\int_{\tau=0}^{\tau=t}\left[f'(\tau)u(\tau) + \underbrace{f(\tau)\delta(\tau)}_{f(0)\delta(t)}\right]d\tau$ $\quad t>0$, $1 \quad 0<\tau<t$

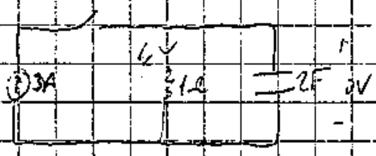
$V_1(t) = \dfrac{C_{eq}}{C_1}\left[\displaystyle\int_{\tau=0}^{t} f'(\tau)d\tau + f(0)\int_0^t \delta(\tau)d\tau\right] \quad t>0$

$$\boxed{V_1(t) = \frac{C_2}{C_1+C_2}f(t)u(t)} \Leftarrow V_1(t) = \frac{C_{eq}}{C_1}\left(f(t) - f(0) + f(0)\right)$$

Ex

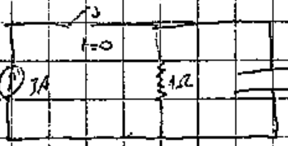


Find $t=0^+$ solution



$i_{1\Omega} = \frac{0}{1} = 0\ A$

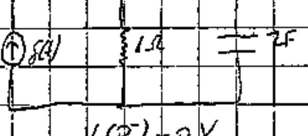
$V_c(0^-) = 0$



$V_c(0^+) = V_c(0^-) = 0V$ Continuity of cap. voltage

$i_{1\Omega}(0^+) = \frac{0}{1} = 0\ A \qquad i_c(0^+) = 3 - i_{1\Omega} = 3\ A$

Ex

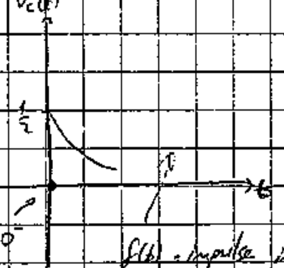


$\left.\begin{array}{l} i_c(t) = \delta(t) \\ i_{1\Omega}(t) = 0\ A \end{array}\right\} 0^- < t < 0^+$

$V_c(0^-) = 0\ V$

$V_c(0^+) = \dfrac{1}{C}\displaystyle\int_{0^-}^{0^+} i_c(t)d\tau = \frac{1}{C}\int_{0^-}^{0^+}\delta(\tau)d\tau = \frac{1}{C} \Rightarrow \boxed{V_c(0^+) = \frac{1}{2}V}$

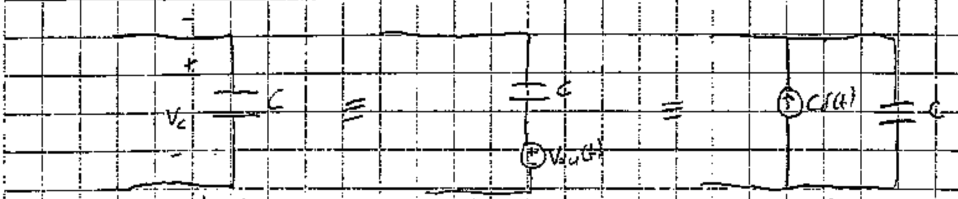
$i_c = -\frac{1}{2}\ A$



$\delta(t)$ - impulse input

$V_c(t)$ is not continuous at $t=0$

Revision



$V_c(0^-) = V_0$

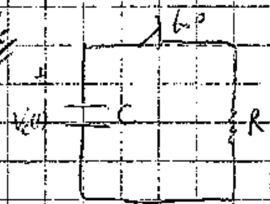
$E_{cap}(t) = \frac{1}{2} C V_0^2$

empty cap

empty cap

Initial condition model

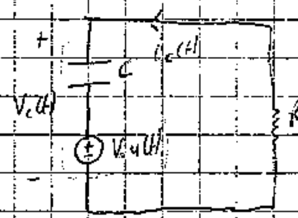
Ex



$V_c(0^-) = V_0$

Analyze the circuit for $t \geq 0$

At $t \geq 0$



$i_c = \frac{C\, dV_c(t)}{dt}$

$V_x(t) + V_0 u(t) + i_c(t) R = 0$

$V_x(t) + V_0 u(t) + RC \frac{d}{dt} V_x(t) = 0$

$\left(D + \frac{1}{RC}\right) V_x(t) = \frac{-V_0}{RC} \quad t \geq 0$

$\Rightarrow \left(D + \frac{1}{RC}\right) V_x(t) = \frac{-V_0}{RC}$ $\qquad \frac{d}{dt}$

$V_x(0^-) = 0V$

$V_x(t) = \alpha e^{-t/RC} + A \quad t \geq 0$

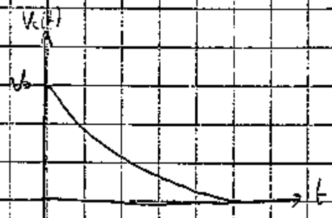
$V_x(t) = \alpha e^{-t/RC} - V_0 \quad t \geq 0 \qquad V_x(0^-) = V_x(0^+) = V_x(0) = 0 \Rightarrow \alpha = V_0$

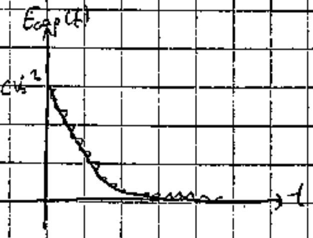
$\Rightarrow V_x(t) = V_0 \left(e^{-t/RC} - 1\right) u(t)\ t \geq 0$

$u(t)$ probably $t \geq 0$ ... using the ... between

$\Rightarrow V_c(t) = V_0 u(t) + V_x$

$V_c(t) = V_0 e^{-t/RC} u(t) \qquad V_c(0^-) = V_c(0^+) = V_c(0) = V_0$

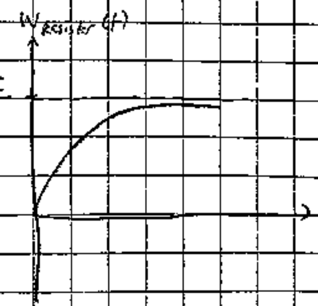




What has happened to the energy initially stored $\left(\frac{1}{2} C V_0^2\right)$?

$$W_{resistor}(t) = \int_0^t i_c(t') v_x(t')\, dt' = \int_0^t \frac{V_c^2(t')}{R} dt' = \frac{V_0^2}{R} \int_0^t e^{-2t'/RC} dt' = \frac{V_0^2}{R} \left(\frac{e^{-2t'/RC}}{\frac{-2}{RC}} \Bigg|_{t'=0}^{t'=t} \right)$$

$W_{resist}(t) = \frac{CV_0^2}{2} \left[1 - e^{-2t/RC}\right]$ Joule



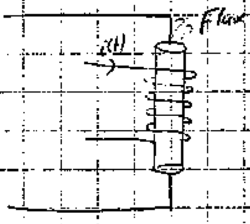
Total energy in Cap at $t = t_x$

+

Total energy consumed by R in $[0, t_x]$

= Initial energy

INDUCTORS



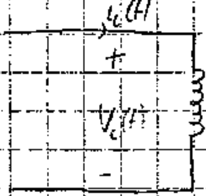
webers

$$\phi(t) = L(t)\,i(t)$$

$$V_L(t) = \frac{d\phi(t)}{dt} = i(t)\frac{dL(t)}{dt} + L(t)\frac{di(t)}{dt}$$

LTI Inductors ⇒ $V_L(t) = L\dfrac{di_L(t)}{dt}$

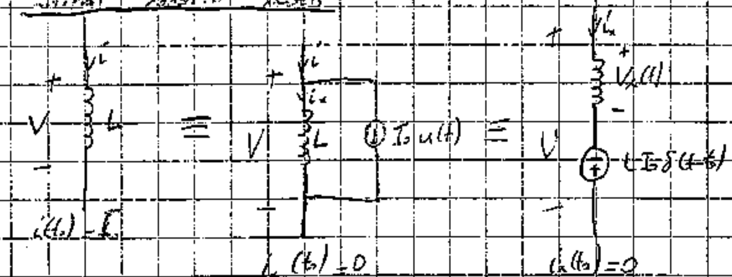
$$E_{ind}(t) = \frac{1}{2} L\, i_L^2(t)$$



$i_L(0^-) = i_0\,A$ $\qquad V_L(t) = L\dfrac{di_L(t)}{dt}$

$$i_L(t) = i_0 + \frac{1}{L}\int_0^t V_L(\tau)\,d\tau \qquad t \geq 0$$

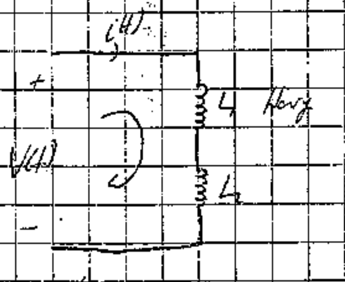
Initial condition models

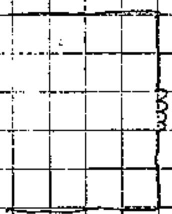


$$V_L(t) = L\frac{d}{dt}i_L(t) = L\frac{d}{dt}\left(I_0 u(t-t_0) + \frac{1}{L}\int_{t_0}^{t} V_x(\tau)\,d\tau\right)$$

$$V_L(t) = L I_0\,\delta(t-t_0) + V_x(t)$$

Inductors In Series





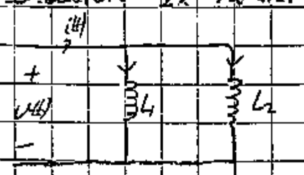
$$V(t) = V_{L_1}(t) + V_{L_2}(t)$$

$$= L_1\frac{d}{dt}i(t) + L_2\frac{d}{dt}i(t)$$

$$V(t) = (L_1 + L_2)\frac{d}{dt}i(t)$$

$i_{L_1}(0^-) = i_{L_2}(0^-) = 0$

Inductors In Parallel



$$i(t) = i_{L_1}(t) + i_{L_2}(t)$$

$$= \frac{1}{L_1}\int_0^t v(\tau)\,d\tau + \frac{1}{L_2}\int_0^t v(\tau)\,d\tau$$

$i_{L_1}(0) = i_{L_2}(0) = 0A$

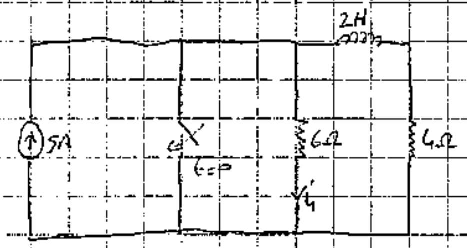
$$= \left(\frac{1}{L_1} + \frac{1}{L_2}\right)\int_0^t v(\tau)\,d\tau$$

$$L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^{-1}$$

Inductors has $i_L(t)$ as a continuous function of time (unless there is an impulse source in the system)

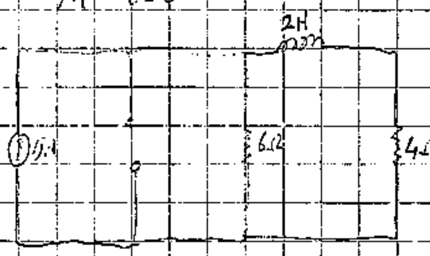
$$i_L(t) = i_L(t_0) + \frac{1}{L}\int_{t_0}^{t_0^+} V_L(\tau)\,d\tau \Rightarrow i_L(t_0^+) = i_L(t_0) \quad \text{unless } V_L(t) = \delta(t-t_0) \Rightarrow \text{impulse at } t=t_0$$

Ex



$i_L(0^-) = 2A$ find $i_L(0^+)$, $i_1(0^+)$, $\frac{d}{dt} i_L(t)\Big|_{t=0^+}$

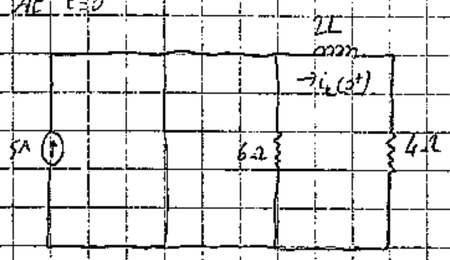
At $t=0^-$



$i_L(0^-) = 3A$

$V_{4\Omega}(0^-) = 12V$

At $t=0^+$

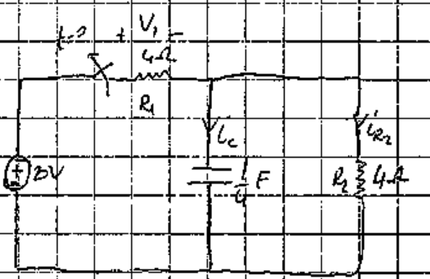


$i_L(0^+) = 3A \quad i_1(0^+) = 0\,A$

$$L\frac{di_L(0^+)}{dt} + 4i_L(0^+) = 0$$

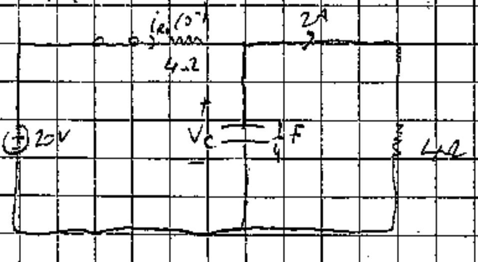
$$\frac{d\,i_L(0^+)}{dt} = \frac{-4i_L(0^+)}{L} = -\frac{4.3}{2} = -6$$

Ex



$i_{R_2}(0^-) = 2A$ Find $q(0^-)$, $q(0^+)$, $i_{R_1}(0^-)$, $i_{R_1}(0^+)$, $i_c(0^-)$, $i_c(0^+)$

At $t=0^-$

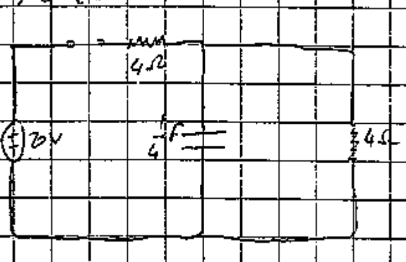


$V_c(0^-) = 8V$; $Q(0^-) = CV_c = 2$ Coulomb

$i_{R_1}(0^-) = \frac{20-8}{4} = 3A$

$i_c(0^-) = 1A$

At $t=0^+$

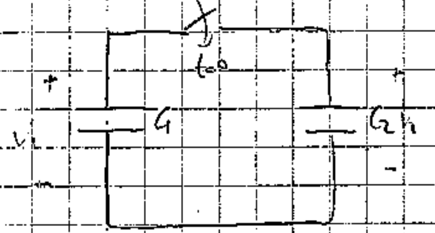


$V_c(0^+) = 8V$; $Q(0^+) = 2$ coulombs

$i_{R_2}(0^+) = 2A \quad i_{R_1}(0^+) = 0\,A$;

$i_c(0^+) = -2A) \quad C\dot{V}_c(0^+) = -2 \Rightarrow \frac{d}{dt}V_c(0^+) = -8$

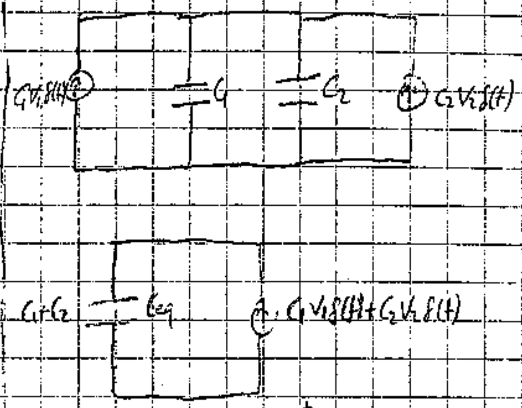
Some Pathological Cases



$V_{c_1}(0^-) = V_1$

$V_{c_2}(0^-) = V_2$

$(V_1 \neq V_2)$



Explanation at $t = 0^+$

$Q_{tot}(0^-) = Q_{tot}(0^+)$

$C_1V_1 + C_2V_2 = (C_1+C_2)\, V_{common}(0^+)$

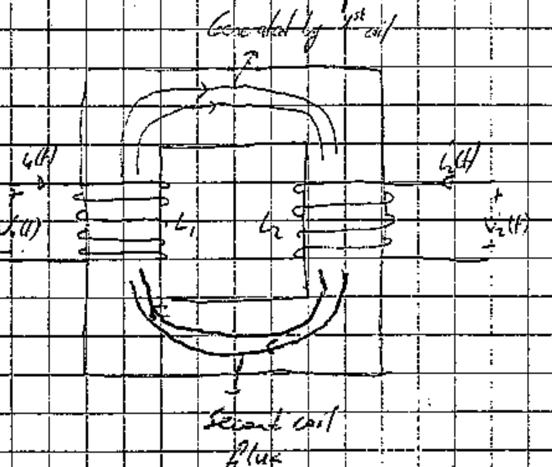
$$V_{common}(0^+) = \frac{C_1}{C_1+C_2}V_1 + \frac{C_2}{C_1+C_2}V_2$$

$$V_c(0^+) = V_{C_{eq}}(0^-) + \frac{1}{C_{eq}}\int_{0^-}^{0^+}\left(C_1V_1\delta(t) + C_2V_2\delta(t)\right)dt$$

Total energy at $t=0^-$: $\frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2$

Total energy at $t=0^+$: $\frac{1}{2}(C_1+C_2)\left(\frac{C_1}{C_1+C_2}V_1 + \frac{C_2}{C_1+C_2}V_2\right)^2$

Mutual Inductor (Coupled Inductor)



$\phi_1(t) = L_1 i_1(t) + M i_2(t)$

$\phi_2(t) = M i_1(t) + L_2 i_2(t)$

$$\begin{bmatrix}\phi_1(t)\\ \phi_2(t)\end{bmatrix} = \begin{bmatrix}L_1 & M\\ M & L_2\end{bmatrix}\begin{bmatrix}i_1(t)\\ i_2(t)\end{bmatrix}$$

$$\frac{d\phi(t)}{dt} = v(t) \Rightarrow \begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix} = \begin{bmatrix}L_1 & M\\ M & L_2\end{bmatrix}\begin{bmatrix}\frac{di_1(t)}{dt}\\ \frac{di_2(t)}{dt}\end{bmatrix}$$

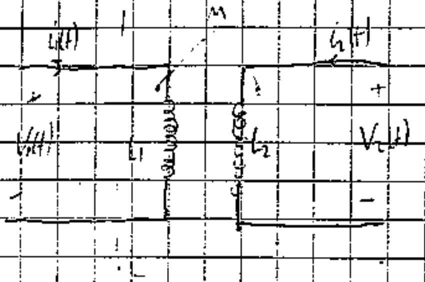
① Special case $M=0 \to$ Self Induction

$$\begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix} = \begin{bmatrix}L_1 & 0\\ 0 & L_2\end{bmatrix}\frac{d}{dt}\begin{bmatrix}i_1(t)\\ i_2(t)\end{bmatrix}$$

② $M = \sqrt{L_1L_2}$ then $k = \frac{M}{\sqrt{L_1L_2}}$, $k=1$ coupling coefficient

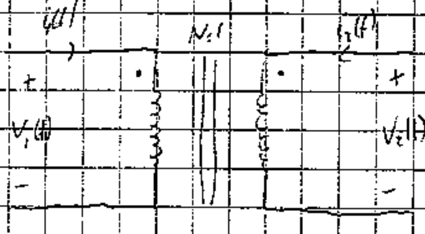
Then we say that coils are perfectly coupled → No loss in power → Ideal transformer.

(More on perfect coupling / coupling coefficient at a later time)



$$\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}$$

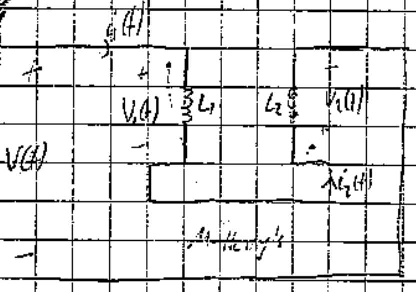
Note the dot convention!



$$\frac{V_1(t)}{V_2(t)} = N \qquad \frac{i_1(t)}{i_2(t)} = \frac{1}{N}$$

Same dot convention

Ex



Express $V(t)$ in terms of $i_1(t)$

$i_1(t) = i_2(t)$

$$\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}$$ ← Since I have exactly obeyed dot convention.

$$V(t) = V_2(t) + V_1(t)$$

$$= \left(L_1 \frac{d}{dt} i_1(t) + M \frac{d}{dt} i_2(t)\right) + \left(M \frac{d}{dt} i_1(t) + L_2 \frac{d}{dt} i_2(t)\right)\Big|_{i_2(t) = -i_1(t)}$$

$$\underline{V(t) = \left(L_1 + L_2 - 2M\right) \frac{d}{dt} i_1(t)}$$

FIRST ORDER CIRCUITS

$$\underbrace{(D+\underset{\text{constant real number}}{\gamma})}_{d/dt}\ \overset{\text{unknown function}}{x(t)} = f(t) \rightarrow \text{forcing term}$$

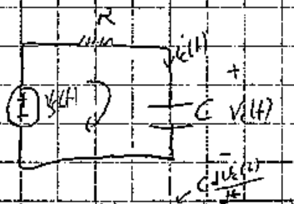
$x(0) = x_0 \leftarrow$ the solution at $t=0!$

Find $x(t)$ for $t \geq 0$

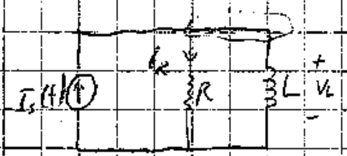
Solution terminology

1. Homogeneous, particular solutions
2. Zero-input, zero-state solutions
3. Transient, steady-state solutions
4. Complete solution

RC Series $\xleftrightarrow{\text{Dual}}$ RL Parallel



$V_c(0^-) = V_0$



$I_L(0^-) = I_0$

KVL: $-V_s(t) + R\underbrace{i_c(t)}_{C\frac{dV_c(t)}{dt}} + V_c(t) = 0 \quad t \geq 0$

$$\left(RC\frac{d}{dt}+1\right)V_c(t) = V_s(t) \quad t \geq 0$$

$$\left(D+\frac{1}{RC}\right)V_c(t) = \frac{V_s(t)}{RC}, \quad t \geq 0$$

KCL: $-I_s(t) + \frac{V_L}{R} + i_L = 0$

$$\left(\frac{L}{R}\frac{d}{dt}+1\right)i_L(t) = I_s(t), \quad t \geq 0$$

$$\left(D+\frac{R}{L}\right)i_L(t) = \frac{R}{L}I_s(t), \quad t \geq 0$$

(1) Homogeneous, particular : (RC Series)

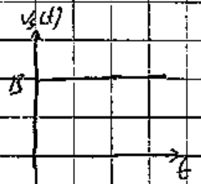
$$\left(D+\frac{1}{RC}\right)V_c(t) = \frac{V_s(t)}{RC} \quad ; t \geq 0$$

Homogeneous $\Rightarrow \left(D+\frac{1}{RC}\right)\underbrace{V_c(t)}_{\text{homogeneous}} = 0 \rightarrow V_c^h(t) = \alpha e^{-\frac{1}{RC}t} \quad ; t \geq 0$

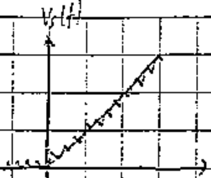
$\forall \alpha \in \mathbb{R}$

Particular $\Rightarrow \left(D+\frac{1}{RC}\right)V_c^p(t) = \frac{V_s(t)}{RC} \longrightarrow V_c^p(t) = Bu(t)$

Let's say that $V_s(t) = Bu(t)$



$V_s(t) = \beta t u(t)$



$\left(D+\frac{1}{RC}\right) V_c^p(t) = \frac{\beta t}{RC} \quad t \geq 0$

$V_c^p(t) = A + Bt$ after substitution

$\left(B + \frac{A}{RC}\right) + \frac{Bt}{RC} = \frac{\beta t}{RC} \quad \forall t$

$V_c^p(t) = -\beta RC + \beta t \quad t \geq 0$ $\qquad B = \beta \Rightarrow A = -\beta RC$

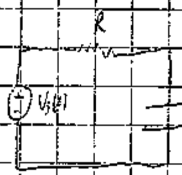
Complete solution

$V_c^{complete}(t) = V_c^h(t) + V_c^p(t)$ Remember "α" is yet to be determined "α" is set such that

$V_c^{complete}(0) = V_c(0^-) = 0$

② Zero Input / Zero State Solutions

Zero State:



State: (state vector, state variables)

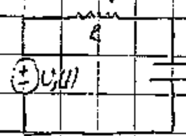
— State variables allow us to exactly characterize the circuit.

$V_c(t)$, $I_c(t)$

Zero state ⇒ All state variables ($V_c(t)$) are equal to zero at $t = 0$

$V_c(0) = 0\,V$; → No initial energy in the circuit.

Zero state solution →



$V_c(0) = 0\,V$

$V_c^{zs}(t) = \_\_\_\_ \quad t \geq 0$

① $V_s(t) = \beta u(t) \rightarrow V_c^{zs}(t) = \left(\beta - \beta e^{-\frac{1}{RC}t}\right) u(t) \quad t \geq 0$

$V_c^{zs}(0) = 0$ and $V_c^{zs}(t)$ satisfies $\left(D + \frac{1}{RC}\right) V_c(t) = \frac{\beta}{RC} \quad t \geq 0$

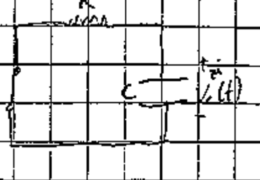
⇒ it's the complete solution for zero initial condition.

② $V_s(t) = \beta t u(t)$

$V_c^{zs}(t) = \beta RC e^{-\frac{1}{RC}t} + \beta(t - RC) \Rightarrow V_c^{zs}(0) = 0 \rightarrow V_c^{zs}(t)$ satisfies

$$\left(D + \frac{1}{RC}\right) V_c^{zs}(t) = \frac{\beta t}{RC} \quad t \geq 0$$

Zero-Input solution The solution when $V_s(t) = 0\,V$ (all independent sources are killed there is no forcing term in the differential equation)



$V_c(0^-) = V_0$

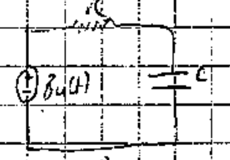
$\left(D + \frac{1}{RC}\right) V_c^{zi}(t) = 0 \leftarrow \left(\frac{V_s(t)}{RC}\Big|_{V_s(t)=0}\right)$

$V_c^{zi}(0) = V_0 \Rightarrow V_c^{zi}(t) = V_0 e^{-\frac{1}{RC}t}$

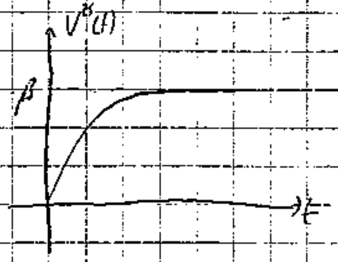
Complete solution $= \boxed{V_c(t) = V_c^{zs}(t) + V_c^{zi}(t)}$

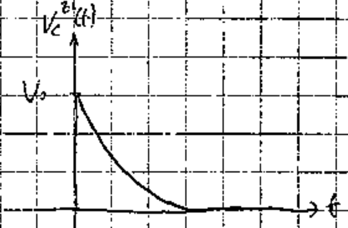
Transient / Steady - State

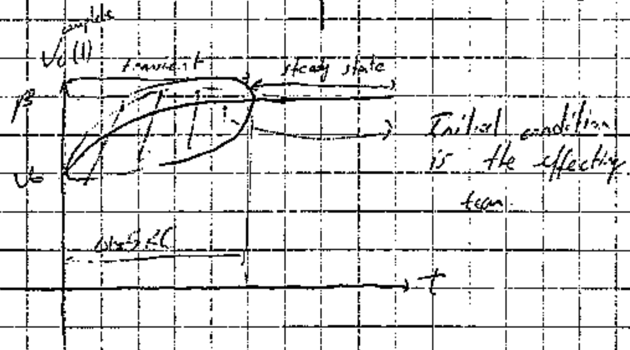
$V_c^{complete}(t) = \left(\beta - \beta e^{-t/RC}\right) + \left(V_0 e^{-t/RC}\right) \quad t \geq 0$



$V_c(0^-) = V_0$



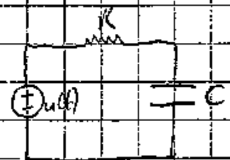




$$V^{complete}(t) = \underbrace{\beta}_{\text{steady state}} + \underbrace{(V_0 - \beta) e^{-t/RC}}_{\text{transient}} \quad t \geq 0$$

Transient part of the complete solutions $\to 0$ as $t \to \infty$ (decays to zero)

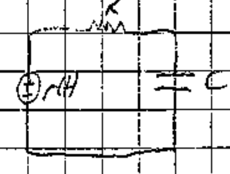
Step Response



$V_c(0^-) = V_0$

$$V^{step}(t) = 1 + (V_0 - 1) e^{-\frac{t}{RC}} \quad t \geq 0$$

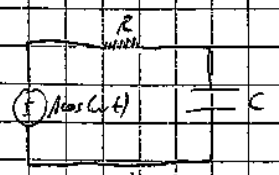
Ramp Response



$$\to \quad V^{ramp}(t) = (t - RC) + (V_0 + RC) e^{-t/RC}$$

$V_c(0^-) = V_0$

Sinusoidal Response:



$V_c(0^-) = V_0$

$V^{zi}(t) = V_0 e^{-t/RC}$

$V^{zs}(t) = ?$

$\left(D + \frac{1}{RC}\right) V_c^{zs}(t) = \frac{V_s(t)}{RC}$ ← $A\cos\omega t$

$V^{zs}(0) = 0$

$V^{zs}(t) = \alpha\cos(\omega t) + \beta\sin(\omega t)$

$\frac{d}{dt} V^{zs}(t) = -\alpha\omega\sin\omega t + \beta\omega\cos\omega t$

$+ \quad \frac{1}{RC} V_c^{zs}(t) = \frac{\beta}{RC}\sin(\omega t) + \frac{\alpha}{RC}\cos(\omega t)$

$\left(D + \frac{1}{RC}\right) V_c^{zs}(t) = \left(-\alpha\omega + \frac{\beta}{RC}\right)\sin\omega t + \left(\beta\omega + \frac{\alpha}{RC}\right)\cos(\omega t)$

$= A\cos\omega t$

$$\begin{bmatrix} \frac{1}{RC} & w \\ -w & \frac{1}{RC} \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{A}{RC} \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\frac{1}{(RC)^2}+w^2}\begin{bmatrix} \frac{1}{RC} & -w \\ w & \frac{1}{RC} \end{bmatrix}\begin{bmatrix} \frac{A}{RC} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{A}{(RC)^2} \\ \frac{A}{RC}w \end{bmatrix}\frac{1}{\left(\frac{1}{RC}\right)^2+w^2}$$

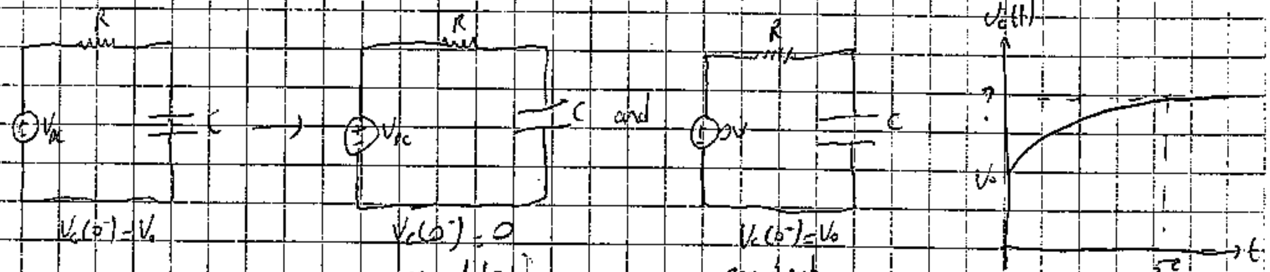
$$v^{zs}(t) = \alpha\cos(wt) + \beta\sin(wt) - \alpha e^{-t/RC} \quad t>0$$

Since $v^{zs}(0) = 0$

Complete solution

$$v^{com}(t) = v^{zs}(t) + v^{zi}(t)$$

DC Excitation



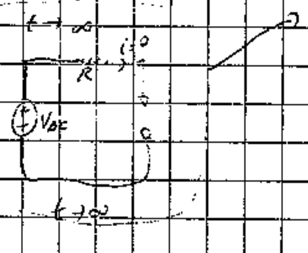
$V_c(0^-) = V_0$ $\qquad$ $V_c(0^-) = 0$ zero-state response $\qquad$ $V_c(0^-) = V_0$ zero-input response

complete response = ( ) + ( )

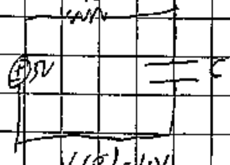
$$V_c^{complete}(t) = (A + Be^{-\frac{t}{RC}})\,u(t) \quad \text{for a DC input}$$

$A, B = ?$ $\quad V_c(0^+) = V_0$, $\quad V_c(\infty) = V_{DC}$

$$V_c(t) = \left[V_c(\infty) + (V_c(0) - V_c(\infty))e^{-\frac{t}{RC}}\right]u(t)$$ General solution for DC input



Ex/



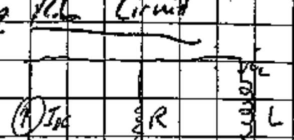
$V_c(0) = 4V$

$V_c(t) = ?$

$V_c(\infty) = 5V \Rightarrow V_c(t) = \left[5 + [4-5]e^{-\frac{t}{RC}}\right]u(t)$

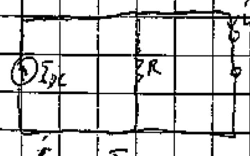
$V_c(t) = \left[5 - 1e^{-\frac{t}{RC}}\right]u(t)$

RL Circuit



$i_L(0^-) = I_0$

As $t \to \infty$



$i_L = I_{DC}$

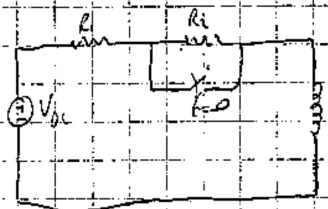
$$I_L^{complete} = \left[I_L(\infty) + (I_L(0) - I_L(\infty))e^{-\frac{t}{RC}}\right]u(t)$$

For DC inputs

Capacitor $\xrightarrow[t\to\infty]{}$ open circuit

Inductor $\xrightarrow[t\to\infty]{}$ short circuit

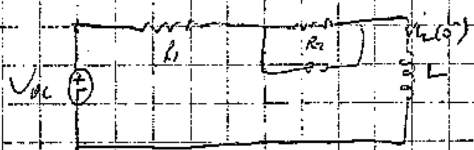
Ex



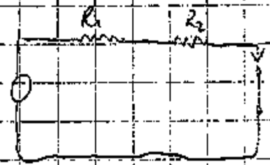
Assume switch is open for a long time, then closes at $t=0$

Find $i_L(t)$, $t \geq 0$

At $t=0^-$



Since circuit is in this config for a long time we expect $i_L(0^-)$ to reach the steady state value



$$I_L(0^-) = \frac{V_{DC}}{R_1+R_2}$$

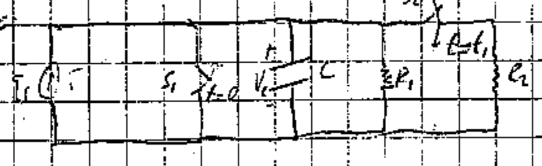
At $t \geq 0$

$$I_L(t) = \left[I_L(\infty) + \left(I_L(0) - I_L(\infty)\right)e^{-\frac{t}{\tau}}\right]u(t) \qquad \tau = \frac{R}{L}$$

$$I_L(0^-) = I_L(0^+) = I_L(0) = \frac{V_{DC}}{R_1+R_2}$$

$$I_L(\infty) = \frac{V_{DC}}{R_1}$$

Ex

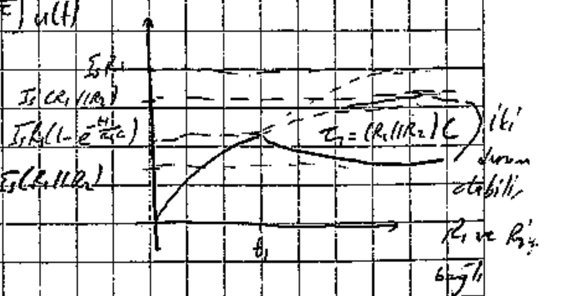


Switch S_1 is opened at $t=0$ $\qquad V_C(0) = 0V$

Switch S_2 is closed at $t = t_1 > 0$ $\qquad V_C(t) = ?$ $\quad t \geq 0$

$0 < t < t_1$

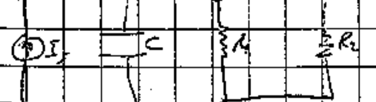
$$V_C(t) = \left[\underbrace{V_C(\infty)}_{I_S R_1} + \left(\underbrace{V_C(0)}_{0} - \underbrace{V_C(\infty)}_{I_S R_1}\right)e^{-\frac{t}{\tau}}\right]u(t)$$



iki durum stabili, R_1 ve R_2'ye bağlı

τ = (equivalent resistance seen by capacitor) x Capacitor = RC

$t_1 < t < \infty$

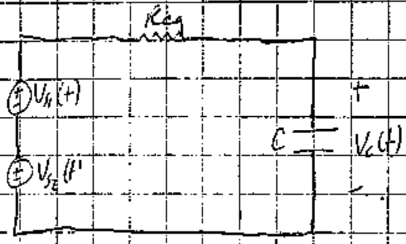


$$V_C(t_1^-) = V_C(t_1^+) = V_C(t_1)$$

$$V_C(t_1^-) = I_S R_1 \left(1 - e^{-\frac{t_1}{R_1 C}}\right)$$

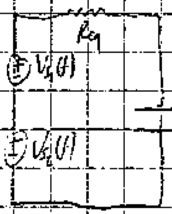
$$V_C(\infty) = I_S (R_1 // R_2)$$

Linearity of Zero-State Response.



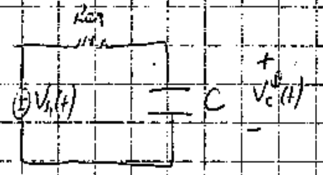
$$\left(D + \frac{1}{R_{eq}C}\right) V_c(t) = \frac{V_{s_1}(t) + V_{s_2}(t)}{R_{eq}C} \qquad V_c(0^-) = V_0$$

Zero-state response



$V_c(0) = 0$!! zero state

(1)



$V_c^{(1)}(0) = 0$

$$V_c^{zs}(t)\Big|_{t=0} = V_c^{(1)zs}\Big|_{t=0} + V_c^{(2)zs}(t)\Big|_{t=0}$$

$$0 = 0 + 0$$

(2)

$V_c^{(2)}(0) = 0$

Whether

$V_c^{zs}(t) = V_c^{(1)zs}(t) + V_c^{(2)zs}(t)$ satisfies the diff. eqn. or not

Then from (1)st circuit diff eqn. is Then from (2)nd circuit

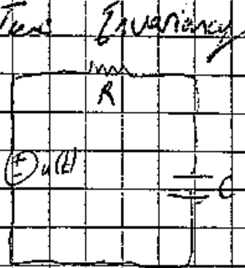
$$\left(D + \frac{1}{R_{eq}C}\right) V_c^{(1)}(t) = \frac{V_{s_1}(t)}{R_{eq}C} \qquad \left(D + \frac{1}{R_{eq}C}\right) V_c^{(2)}(t) = \frac{V_{s_2}(t)}{R_{eq}C}$$

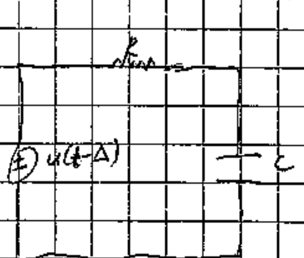
Adding two diff eqn's

$$\left(D + \frac{1}{R_{eq}C}\right)\left(V_c^{(1)}(t) + V_c^{(2)}(t)\right) = \frac{V_{s_1}(t) + V_{s_2}(t)}{R_{eq}C} \Bigg\} \text{ The original diff eqn.}$$

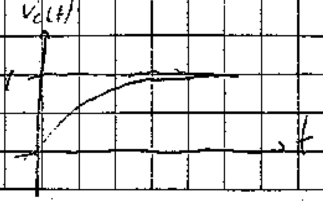
Finally the superposition solution for ZERO-STATE satisfies the original diff eqn. and the initial condition for zero-state solution! → So superposition solution is the solution of the ...

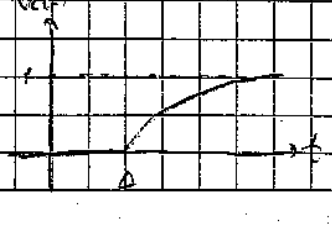
Time Invariancy





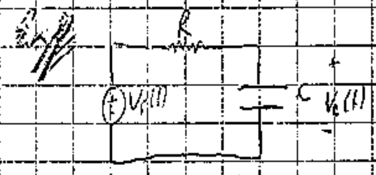
$V_c(0^-) = 0V$ $V_c(\Delta) = 0$



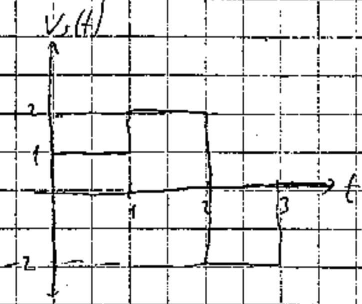


$\left(D + \frac{1}{RC}\right) v_c(t) = f(t) \leftarrow$ forcing term $v_c(t)$ is solution for $f(t)$

$\left(D + \frac{1}{RC}\right) x(t) = f(t-\Delta) \Rightarrow x(t) \stackrel{?}{=} v_c(t-\Delta)$ Put it in diff. eqn. See it is equal ✓



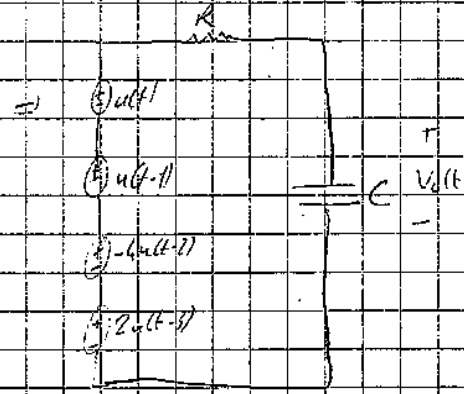
$v_c(0) = 0\,V$



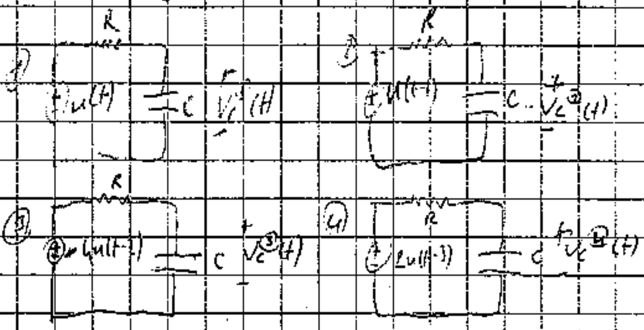
Find $v_c(t)$

$v_s(t) = u(t) + u(t-1) - 4u(t-2) + 2u(t-3)$

$\Rightarrow$



$$v_c^{(t)}(t) = \sum_{k=1}^{4} v_c^{(k)}(t)$$



$$v_c^{(1)}(t) = \left(1 - e^{-\frac{t}{RC}}\right) u(t)$$

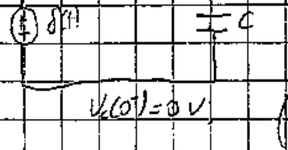
$$v_c^{(2)}(t) \stackrel{\text{invariancy}}{=} v_c^{(1)}(t-1) = \left(1 - e^{-\frac{(t-1)}{RC}}\right) u(t-1)$$

$$v_c^{(3)}(t) = -4\, v_c^{(1)}(t-2)$$

$$v_c^{(4)}(t) = 2\, v_c^{(1)}(t-3)$$

$$v_c(t) = \left(1 - e^{-\frac{t}{RC}}\right) u(t) + \left(1 - e^{-(t-1)/RC}\right) u(t-1) - 4\left(1 - e^{-\frac{(t-2)}{RC}}\right) u(t-2) + 2\left(1 - e^{-(t-3)/RC}\right) u(t-3)$$

Impulse Response (Always analyzed under zero-state conditions)

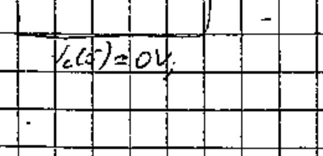


* When we consider impulse / step / ramp response, we always talk about zero-state solution. (that is the initial conditions are all zero)

$v_c(0^-) = 0\,V$ $\left(D + \frac{1}{RC}\right) v_c(t) = \frac{\delta(t)}{RC}$ Since $v_s(t) = \delta(t)$; $v_c(0^-) \neq v_c(0^+)$

$0^- < t < 0^+$

$$i_c(t) = \frac{\delta(t)}{R} \quad v_s(t)$$



$v_c(0^-) = 0\,V$

$$v_c(0^+) = v_c(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i_c(\tau)\,d\tau = \frac{1}{RC}\int_{0^-}^{0^+} \delta(\tau)\,d\tau = \frac{1}{RC}$$

$$i_c(\tau) = \frac{\delta(\tau)}{R}$$

$t > 0^+$

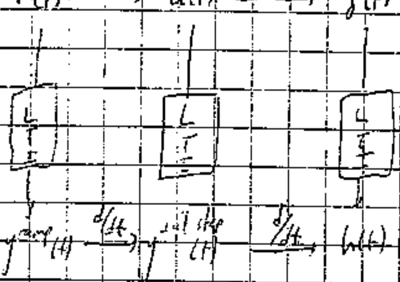
$V_c(0^+) = \frac{1}{RC}$ $V_c(t) = \frac{1}{RC} e^{-t/\tau}$ $t > 0^+$ $\tau = RC$

$h(t)$: impulse response

$\delta(t)$ → LTI circuit → $h(t)$ capacitor voltage

$u(t)$ → LTI circuit → $y^{\text{unit step}}(t)$

$r(t)$ → LTI circuit → $y^{\text{ramp}}(t)$



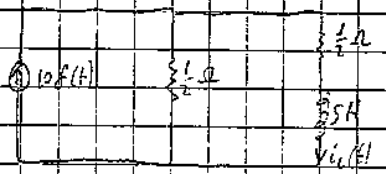
$r(t) \xrightarrow{d/dt} u(t) \xrightarrow{d/dt} \delta(t)$

$y^{\text{ramp}}(t) \xrightarrow{d/dt} y^{\text{unit step}}(t) \xrightarrow{d/dt} h(t)$

Ex RC circuit impulse response $h(t) = \frac{1}{RC} e^{-t/RC} u(t)$. Unit step response = ?

$$y^{\text{unit-step}}(t) = \int_{-\infty}^{t} h(\tau) d\tau = \int_{-\infty}^{t} \frac{1}{RC} e^{-\tau/RC} u(\tau) d\tau = \int_{0}^{t} \frac{1}{RC} e^{-\tau/RC} d\tau = -e^{-\frac{\tau}{RC}} \Big|_{\tau=0}^{\tau=t} = 1 - e^{-\frac{t}{RC}}$$

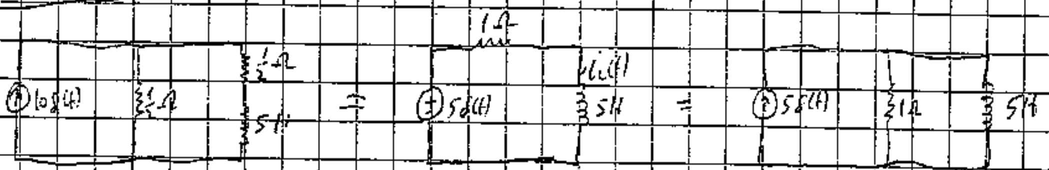
Ex



Find $i_L(t)$, $t > 0$

$i_L(0^-) = 2A$

Zero state



$i_L(0^-) = 0A$

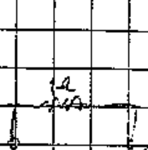
$i_L(0^-) = 0A$ $i_L(0^+) = ?$

$\Rightarrow 0^- < t < 0^+$

$$i_s(t) = 5\delta(t) \rightarrow V_L(t) = 5\delta(t); \; i_L(0^+) = \underbrace{i_L(0^-)}_{0} + \underbrace{\frac{1}{5H}}_{5H} \int_{0^-}^{0^+} \underbrace{V_L(\tau)}_{5\delta(\tau)} d\tau = 1A$$

for $t > 0^+$ $i_L^{zs}(t) = 1 e^{-t/\tau} u(t) \Leftrightarrow \tau = \frac{L}{R} = 5$ secs

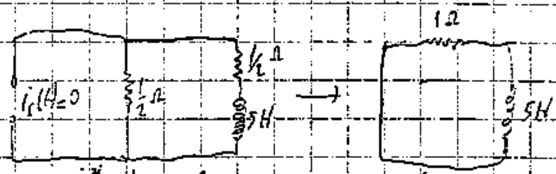
for $t > 0^+$



$i_L(0^+) = 1A$ $i_L(t) = (\underbrace{i_L(\infty)}_{0} + (\underbrace{i_L(0^+)}_{1} - \underbrace{i_L(\infty)}_{0}) e^{-\frac{t}{5}}) u(t)$

$\boxed{i_L(t) = e^{-\frac{t}{5}} u(t)}$

Zero Input

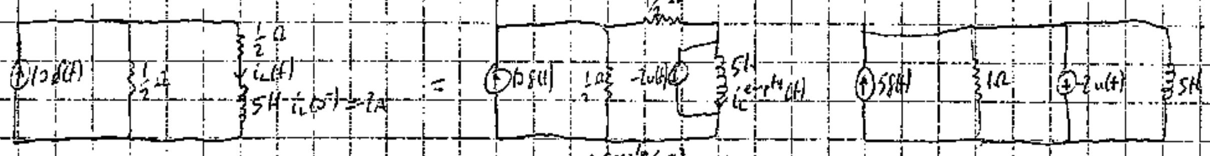


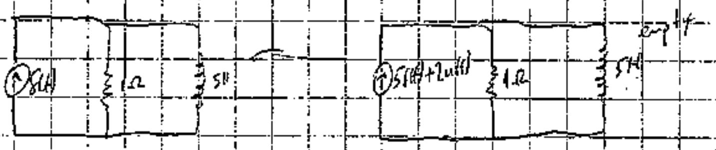
$i_L^{zi}(0) = -2A$ $\qquad$ $i_L^{zi}(0) = -2A$

$i_L^{zi}(t) = -2e^{-t/\tau}\,u(t)$ $\qquad$ $\tau = \frac{L}{R} = 5$

$$i_L^{complete}(t) = i_L^{zs}(t) + i_L^{zi}(t) = e^{-t/\tau}u(t) - 2e^{-t/\tau}\,u(t) = \boxed{-e^{-t/\tau}u(t)}$$

2nd way

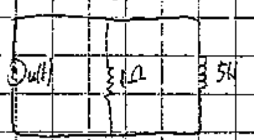




5h(t)

$$i_L^{empty}(t) = 2\,i_L^{unit\,step}(t) + 5\frac{d}{dt}i_L^{unit\,step}(t)$$

$$i_L^{empty}(t) = 2(1-e^{-t/\tau}) + \frac{5}{\tau}e^{-t/\tau} = 2 - e^{-t/\tau}$$

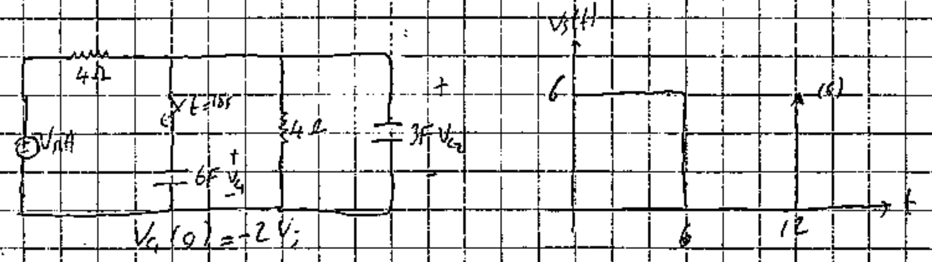


$i_L(t) = i_L^{empty}(t) - 2u(t)$

$= -e^{-t/\tau}u(t)$

$i_L^{unit\,step}(t) = (1-e^{-t/\tau})u(t)$ $\quad$ $(\tau = 5)$

Ex

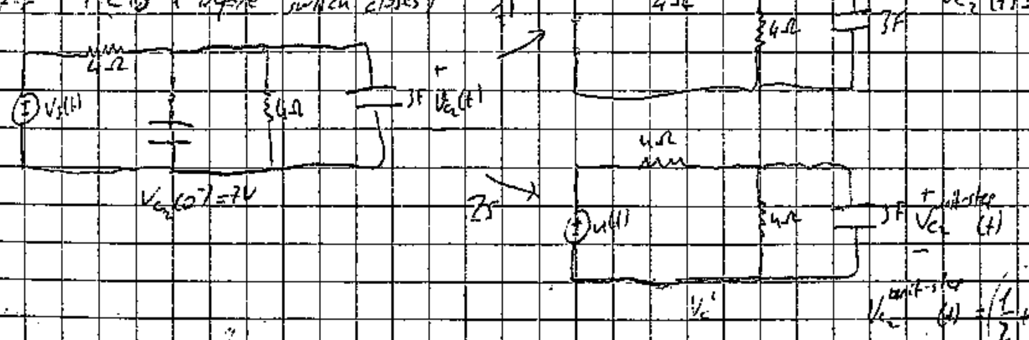


$V_{c_1}(0) = -2V;$

$V_{c_2}(0) = 7V;$

$V_{c_2}(t) = ?$

I- $A < 15$ (Before switch closes)



$V_{c_2}^{zi}(t) = 7e^{-t/\tau}u(t)$

$\tau = (4\|4)\,3 = 6$ sec

V_c'

$V_{c_2}^{unit\,step}(t) = \left(\frac{1}{2} - \frac{1}{2}e^{-t/\tau}\right)u(t)$

$\tau = 6$ sec

$$V_{c_2}^{zs}(t) = 6\, V_{c_2}^{us}(t) - 6\, V_{c_2}^{us}(t-6) + 6\left(\frac{d}{dt} V_{c_2}^{us}(t-12)\right)$$

$$= 3(1-e^{-\frac{t}{\tau}})u(t) - 3(1-e^{-(t-6)/\tau})u(t-6) + 6\frac{d}{dt}\left\{\frac{1}{2}(1-e^{-t/\tau})u(t)\right\}$$

$$+6\left(\frac{1}{2\tau}e^{-t/\tau}u(t) + \frac{1}{2}\underbrace{(1-e^{-t/\tau})\delta(t)}_{0\cdot\delta(t)=0}\right)$$

$$= +\frac{3}{\tau}e^{-t/\tau}u(t)$$

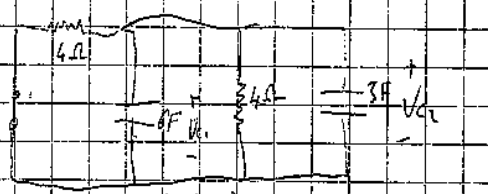
$$V_{c_2}^{zs}(t) = 3(1-e^{-\frac{t}{\tau}})u(t) - 3(1-e^{-\frac{(t-6)}{\tau}})u(t-6) + \frac{1}{2}e^{-\frac{(t-12)}{\tau}}u(t-12), \quad t \geq 0$$

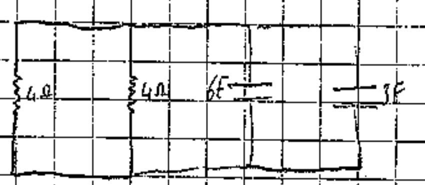
At $t = 18^-$

$$v_{c_2}(t) = V_{c_2}^{zi}(t) + V_{c_2}^{zs}(t)\Big|_{t=18^-}$$

$$V_{c_2}(18^-) = 2.14\,V$$

$t \geq 18$





$V_{c_2}(18^-) = 2.14\,V$

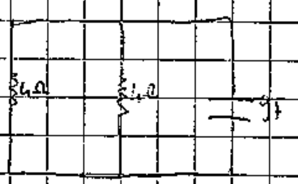
$V_{c_1}(18^-) = V_{c_1}(0) = -2V$

Total charge at $t = 18^-$ = Total charge at $t^+ = 18$

$$6F \cdot V_{c_1}(18^-) + 3F\, V_{c_2}(18^-) = V_{common}(18^+)(6+3)F$$

$$V_{common} = \frac{6(-2) + 3\times 2.14}{9} = -1.07\,V$$

$t \geq 18^+$

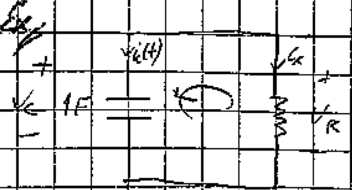


$$V_{common}(t) = -1.07\, e^{-\frac{(t-18)}{18}}\, u(t-18) \qquad t \geq 18$$

$V_{common}(18^+) = -1.07\,V$

Time Varying Circuits and Non-Linear Elements

Ex/



$V_c(0) = 1V$

$V_c(t) = ?$

a) $R = 1\,\Omega$

b) $R(t) = \frac{1}{(1+0.5\cos t)}\,\Omega$

c) Non linear resistor with $i_R = V_R^2$

a) $i_c(t) = \boxed{1 e^{-\frac{t}{\tau}} u(t)}$; $\tau = RC = 1$ sec.

b) $V_c(t) + R(t)\, i_c(t) = 0$

$V_c(t) + R(t)\, C \dfrac{dV_c(t)}{dt} = 0 \qquad V_c(0) = 1V$

$\dfrac{dV_c(t)}{dt} = \dfrac{-V_c(t)}{R(t)C} \rightarrow \dot{V}_c(t) = -(1+0.5\cos t)\, V_c(t) \Rightarrow \dfrac{dV_c(t)}{V_c(t)} = -(1+0.5\cos t)\, dt$

$\ln V_c(t) = -(t + 0.5\sin t) + K$

$V_c(t) = A e^{-(t+0.5\sin t)} \qquad V_c(0) = 1V \Rightarrow 1 = A \Rightarrow \boxed{V_c(t) = e^{-(t+0.5\sin t)}\, u(t)}$

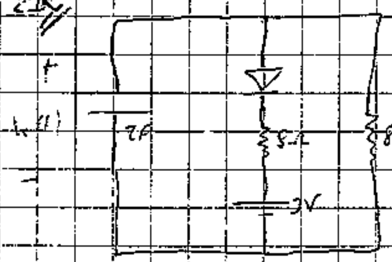
c) $C \dfrac{dV_c(t)}{dt} = -i_R(t) = -(V_c(t))^2$

$\dfrac{dV_c(t)}{(V_c(t))^2} = -dt \rightarrow -\dfrac{1}{V_c(t)} = -t + K \Rightarrow V_c(t) = \dfrac{1}{t-K}$

$V_c(0) = \dfrac{1}{-K} = 1 \Rightarrow K = -1$

$V_c(t) = \dfrac{1}{t+1} u(t)$

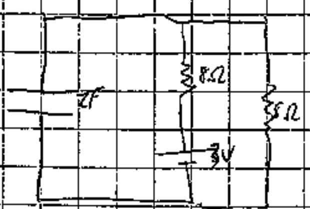
Ex



$V_c(0) = 12V \quad V_c(t) = ?$

method 1 Diode on when $V_c(t) \geq 3V$

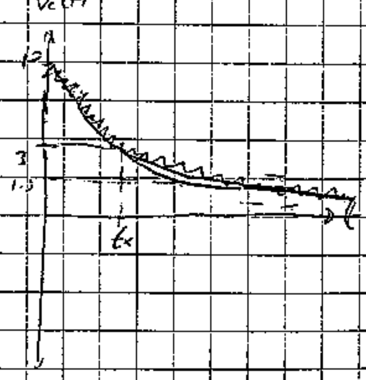
For $V_c(t) \geq 3V$



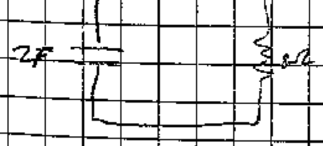
$V_c(\infty) = \frac{3}{2}$ V

$V_c(t) = \left(\frac{3}{2} + \left(12 - \frac{3}{2}\right) e^{-\frac{t}{8}}\right)$

$V_c(t) = \frac{3}{2} + \frac{17}{2} e^{-\frac{t}{8}}$



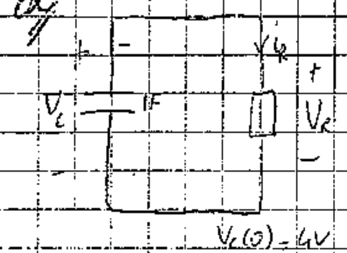
For $V_c(t) < 3V$

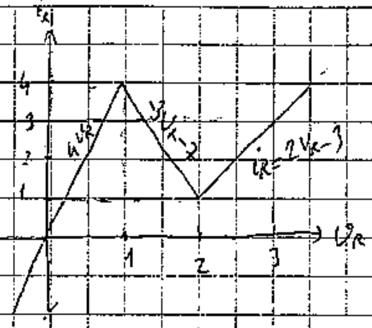


$V_c(t) = 3 e^{-\frac{(t-t_x)}{16}} u(t-t_x)$

32)

Ex//





Find $V_c(t)$

$V_c(0^+) = V_R(0^+) = 4V$

$C\frac{dV_c}{dt} = -i_R \quad i_R > 0$ when $V_R > 0$
1F $\quad i_R < 0$ when $V_R < 0$

$\frac{dV_c}{dt} < 0$ when $V_c(t) > 0$

when $V_c(t) > 0$, then $V_c(t)$ is decreasing.

a) $2 < V_c(t) < 4 \rightarrow i_R = 2V_R - 3$

$\frac{dV_c(t)}{dt} = -2(V_c - \frac{3}{2}) \rightarrow DV_c = -2V_c + 3$

$(D+2)V_c = 3$

$V_c(t_x) = 2 \rightarrow 2 = \frac{5e^{-2t_x} + 3}{2} \qquad V_c = \alpha e^{-2t} + \frac{3}{2}$

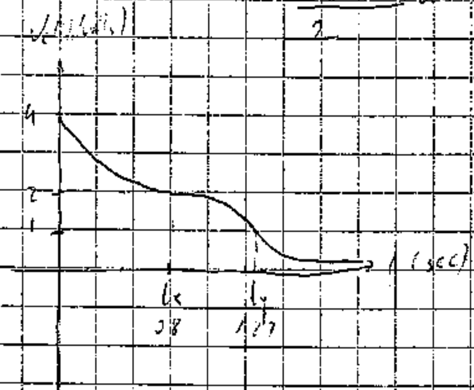
$t_x = \frac{\ln 5}{2} = 0.8$ sec $\qquad V_c(0^+) = 4 \rightarrow \alpha = \frac{5}{2}$

$V_c(t) = \frac{5e^{-2t} + 3}{2} u(t)$

b) $1 < V_c(t) < 2$; $\frac{dV_c}{dt} = -(-3(V_c - \frac{2}{3}))$ $\qquad V_c(t_x^+) = 2V$

$V_c = \frac{2}{3} - \frac{4}{3}e^{3(t - t_x)}$

$V_c(t_y) = 1V \Rightarrow t_y = 1.27$ sec

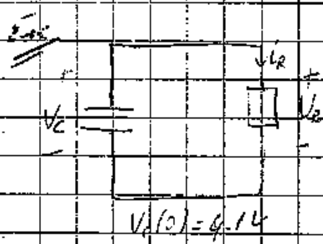


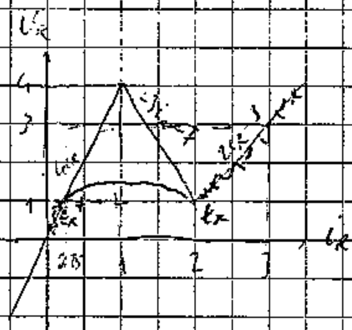
c) $0 < V_c < 1$

$\frac{dV_c}{dt} = -4V_c, \quad V_c(t_y) = 1$

$V_c(t) = e^{-4(t - t_y)} \quad t > t_y$

Ex//





Find $V_c(t)$

$\frac{dV_c(t)}{dt} = -i_R \rightarrow i_R > 0 \rightarrow V_c(t)$ decreases

a) $i_R > 2 \longrightarrow \frac{dV_c(t)}{dt} = -(i_R) = -\left(\frac{V_c(t) + 3}{2}\right) \qquad V_c(0) = 4.1V \Rightarrow V_c(t) = \alpha e^{-\frac{t}{2}} - 3$

$i_R = \frac{V_R + 3}{2} \qquad V_c(t) = 7.1e^{-t/2} - 3$;

b) $V_c(t_x^+) = 1V$

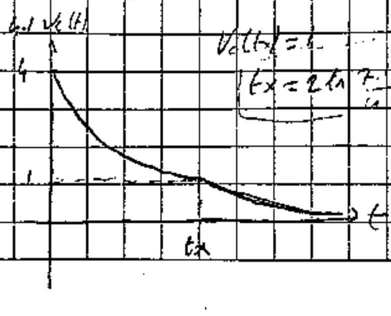
$i_R < 0.25 \rightarrow$ on the sequel $\frac{-dV_c(t)}{dt} = -\frac{1}{4}V_c(t)$

$V_R = 4i_R$

$\Rightarrow V_c(t) = 1e^{-\frac{(t - t_x)}{4}} \quad t \geq t_x$

$V_c(t_x) = 1$

$\boxed{t_x = 2\ln\frac{7.1}{4}}$



SECOND ORDER CIRCUITS

Parallel RLC (Series RLC)

$i_L(0^-) = I_0$
$V_C(0^-) = V_0$

$i_s(t) = i_R(t) + i_C(t) + i_L(t)$

$i_s(t) = i_R(t) + i_L(t) + C\frac{dV_C(t)}{dt}$; $V_C(t) = V_R(t) = V_L(t)$

$i_s(t) = \frac{V_C(t)}{R} + i_L(t) + C\frac{dV_C(t)}{dt}$

$i_L(0^-) + \frac{1}{L}\int_0^t V_C(\tau)\,d\tau \Rightarrow D^{-1}V_C(t) = \int_0^t V_C(\tau)\,d\tau$

$\frac{d}{dt}\Big(\quad i_s(t) = \frac{V_C(t)}{R} + i_L(0^-) + \frac{1}{L}D^{-1}V_C(t) + C\frac{dV_C(t)}{dt}$

$\frac{di_s(t)}{dt} = \frac{1}{R}\frac{dV_C(t)}{dt} + \frac{V_C(t)}{L} + C\frac{d^2V_C(t)}{dt^2}$

$\left(CD^2 + \frac{1}{R}D + \frac{1}{L}\right)V_C(t) = \frac{d}{dt}i_s(t)$ forcing term.

$V_C(0) = V_0$

$\frac{d}{dt}V_C(0) = ?$ ← need to find this from given initial conditions

$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)V_C(t) = \frac{\frac{d}{dt}i_s(t)}{C}$ ← parallel RLC

Write a 1st order matrix dif eqn. for RLC circuit

State variables: $(V_C(t), i_L(t))$

$\frac{dU(t)}{dt} = (\alpha V_C(t) + \beta I_L(t) + \gamma i_s(t))$ ← in terms a linear combination of state variables and input.

$C\frac{dV_C(t)}{dt} = i_s(t) - \underbrace{i_R(t)}_{V_C/R} - i_L(t)$

(1) $\frac{dV_C(t)}{dt} = -\frac{1}{RC}V_C(t) - \frac{1}{C}I_L(t) + \frac{1}{C}i_s(t)$

$L\frac{dI_L(t)}{dt} = V_L(t) = V_C(t)$

(2) $\frac{dI_L(t)}{dt} = \frac{V_C(t)}{L}$

$$\begin{bmatrix} \frac{d}{dt}V_C(t) \\ \frac{d}{dt}i_L(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} i_s(t)$$

$V_C(0^-) = V_0$
$I_L(0^-) = I_0$

$$\frac{d^2 V_c(t)}{dt^2} = -\frac{1}{RC}\frac{dV_c(t)}{dt} - \frac{1}{C}\frac{d I_L(t)}{dt} + \frac{1}{C}\frac{d}{dt} i_s(t)$$

$\rightarrow$ (3rd) state equation $= \frac{V_c(t)}{L}$

$$D^2 V_c(t) = -\frac{1}{RC} D V_c(t) - \frac{1}{LC} V_c(t) + \frac{1}{C} D\, i_s(t)$$ $\leftarrow$ equivalent to the earlier form

Type of Zero-Input Solutions (Parallel RLC)

$(D^2 + 2\alpha D + \omega_0^2)\, V_c(t) = 0$ $\leftarrow$ zero input (as forcing term)

Parallel RLC $\quad 2\alpha = \frac{1}{RC} \quad \omega_0^2 = \frac{1}{LC}$

Characteristic polynomial $= s^2 + 2\alpha s + \omega_0^2 = 0$

$$s_{1,2} = \frac{-2\alpha \mp \sqrt{4\alpha^2 - 4\omega_0^2}}{2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2}$$

$V_c(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$

3 cases of s_1 and s_2

1. Real and distinct $(\Delta > 0)$ $\Longrightarrow$ Overdamped $\quad V_c(t) = C_1 e^{-2t} + C_2 e^{-3t}$ $\quad s_1 = -2,\ s_2 = -3$
2. Real and same $(\Delta = 0)$ $\Longrightarrow$ Critically damped $\quad V_c(t) = C_1 e^{-t} + C_2 t e^{-t}$ $\quad s_1 = -1,\ s_2 = -1$
3. Imaginary roots $(\Delta < 0)$ $\Longrightarrow$ Underdamped $\quad V_c(t) = 2e^{-t}(C_R \cos t - C_I \sin t)$ $\quad s_1 = -1+j,\ s_2 = -1-j$

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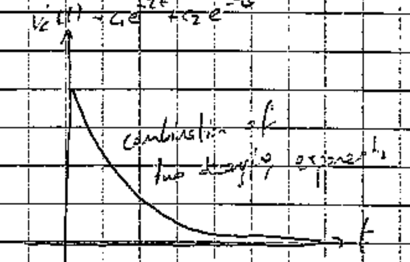
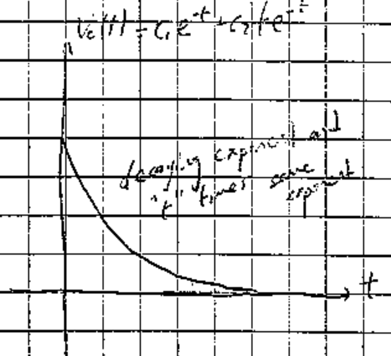
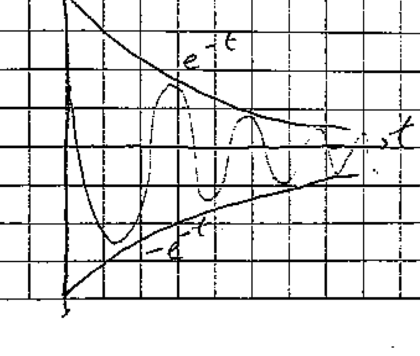
(1) Overdamped $\longrightarrow$



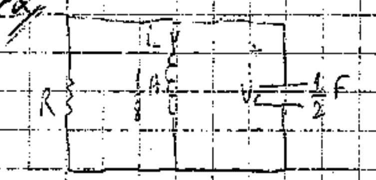
(2) Critically Damped $\longrightarrow$



(3) Underdamped $\longrightarrow$



Ex/



$i_L(0^-) = -4A$

$v_C(0^-) = 5V$

$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right) v_C(t) = \frac{1}{C} D\, i(t)$

$v_C(0^-) = V_0$

$\frac{d}{dt} v_C(0^-) = \ldots$

a) $R = \frac{1}{5}\Omega$

$(D^2 + 10D + 16)\, v_C(t) = 0$

$(D^2 + (2\alpha)D + w_0^2) \rightarrow \lambda^2 + (2\alpha)\lambda + w_0^2 = 0 \Rightarrow \lambda_{1,2} = -\alpha \mp \sqrt{\alpha^2 - w_0^2}$

α : Damping ratio

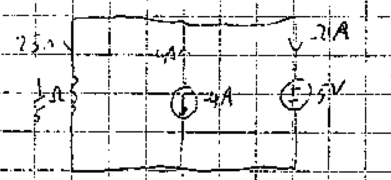
w_0 : Natural frequency

$R = \frac{1}{5}\Omega$ $\begin{cases} |\alpha| = 5 \\ w_0 = 4 \end{cases}$ $|\alpha| > w_0 \rightarrow$ overdamped response

$V_c(t) = (A e^{-2t} + B e^{-8t})\, u(t) \quad t > 0$

$\lambda_{1,2} = \{-2, -8\}$

$V_c(0^+) = 5V \Rightarrow A + B = 5$



$\frac{d}{dt} V_c(0^+) = -21A \Rightarrow \frac{d}{dt} V_c(0^+) = -42 = -2A - 8B$

$A + B = 5$

$A + 4B = 21$

$B = \frac{16}{3} \quad A = -\frac{1}{3}$

$V_c(t) = -\frac{1}{3} e^{-2t} + \frac{16}{3} e^{-8t}$

b) $R = \frac{1}{4} \Rightarrow \alpha = 4 \quad w_0 = 4$

$\alpha = w_0 \Rightarrow$ critically damped response.

$(D^2 + 2\alpha D + w_0^2) = (D + w_0)^2$

$\Rightarrow V_c(t) = (5 - 12t)\, e^{-4t}\, u(t)$

c) $R = \frac{1}{3} \Rightarrow |\alpha| < w_0$ (λ - complex roots for characteristic equation) (under damped)

$\lambda_{1,2} = \{-3 \pm j\sqrt{7}\}$

$V_c(t) = c e^{\lambda_1 t} + c^* e^{\lambda_2 t}$

$= c e^{\lambda_1 t} + c^* e^{\lambda_1^* t}$

$= \|c\| e^{j\angle c} e^{\lambda_1 t} + \|c^*\| e^{j\angle c^*} e^{\lambda_1^* t}$

$= \|c\| \left[e^{\lambda_1 t + j\angle c} + e^{\lambda_1^* t + j\angle c^*} \right]$

$= \|c\| \left[e^{-3t + j\sqrt{7}t + j\angle c} + e^{-3t - j\sqrt{7}t - j\angle c} \right]$

$= \|c\| e^{-3t} \left[e^{j(\sqrt{7}t + \angle c)} + e^{-j(\sqrt{7}t + \angle c)} \right]$

$\Rightarrow V_c(t) = \|c\| e^{-3t}\, 2\cos(\sqrt{7}t + \angle c)$ real number in radians

$V_c(0^+) = 5$

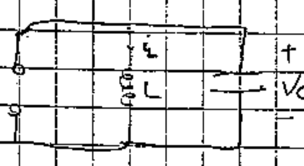
$\frac{d}{dt} V_c(0^+) = -22 \Rightarrow V_c(t) = e^{-3t}\, 4\sqrt{2} \cos\left(\sqrt{7}t + \tan^{-1}\frac{\sqrt{7}}{5}\right)$

Burada ne yaptık?

Bu notasyonu bilmemize gerek var mı?

örnek sorularda kullanmamışız

d) $R \rightarrow \infty$



(Lossless System)

$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right) V_c(t) = 0$

$2\alpha \rightarrow 0$ as $R \rightarrow \infty$

$\left(D^2 + \frac{1}{LC}\right) V_c(t) = 0$

$(D^2 + 16)\, V_c(t) = 0$

$V_c(t) = c e^{-j4t} + d e^{+j4t}$

$V_c(t) = A\cos(4t) + B\sin 4t$

$V_c(t) = C\cos(4t + \angle C)$

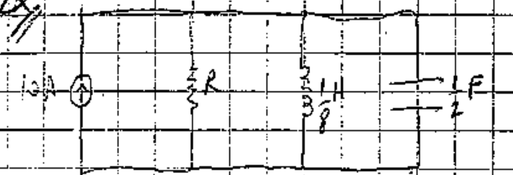
$V_c(0) = 5$

$\frac{d}{dt} V_c(0) = 8$

$V_c(t) = 5\cos 4t + 2\sin 4t$

Ramp / Unit Step / Impulse Response (Elementary Zero-State Response)

Ex)



a) $R = \frac{1}{2}\Omega \rightarrow V_c^{zi} = -\frac{1}{3}e^{-2t} + \frac{16}{3}e^{-8t}$

$(D^2 + 10D + 16)V_c(t) = \frac{1}{C}\frac{d}{dt}i_s(t) = 0$

$V_c(t) = a + \alpha e^{-2t} + \beta e^{-8t} \rightarrow V_c(\infty) = 0 \rightarrow a = 0$

$\underbrace{V_c(0^+),\ \frac{d}{dt}V_c(0^+)}_{\alpha,\beta}$

$\Rightarrow V_c(t) = 3e^{-2t} + 2e^{-8t}$

$i_L(t) = 10 - 12e^{-2t} - 2e^{-8t}$

$(t \to \infty)$ analysis, analysis at $t = 0^+$

Ramp response

$i_s(t) = r(t)$ $\quad u(t)$

$(D^2 + 2\alpha D + \omega_0^2)V_c(t) = \frac{1}{C}\frac{d}{dt}i_s(t)$

$\left.\begin{array}{l} V_c(0^-) = 0 \\ \frac{d}{dt}V_c(0^-) = 0 \end{array}\right\}$ Zero-State!

$V_c(0^+) = ?$

$\frac{d}{dt}V_c(0^+) = ?$

$(D^2 + 2\alpha D + \omega_0^2)V_c(t) = \frac{1}{C} \quad t \geq 0$

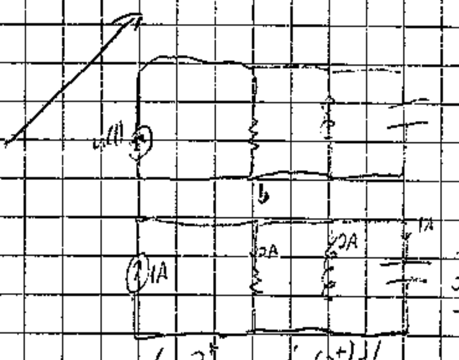
$V_c(t) = A + Be^{-\lambda_1 t} + Ce^{-\lambda_2 t}$

Ramp ve unit response da mutlaka zero state mi olmalı (Impulse gibi)

$V_c(0^+) = V_c(0^-)$

$\frac{d}{dt}V_c(0^+) = \frac{d}{dt}V_c(0^-)$

since there is no impulse



$V_c(0^-) = 0V$

$i_L(0^-) = 0V$

Unit Step Response

$(D^2 + 2\alpha D + \omega_0^2)V_c(t) = \frac{1}{C}\frac{d}{dt}u(t)$ $\quad \delta(t)$

$V_c(0^+) = 0V$

$\frac{d}{dt}V_c(0^+) = \frac{1}{C}$

$t = 0^+$ $\quad i_c(0^+) = 1$

Then for $t > 0$

$(D^2 + 2\alpha D + \omega_0^2)V_c(t) = 0 \quad V_c(0^+) = 0 \quad \frac{d}{dt}V_c(0^+) = \frac{1}{C}$

$(D^2 + 2\alpha D + \omega_0^2)V_c(t) = \frac{\delta(t)}{C} \rightarrow \ddot{V}_c(t) = \frac{1}{C}\delta(t)$

$\dot{V}_c(0^+) = \frac{1}{C}$

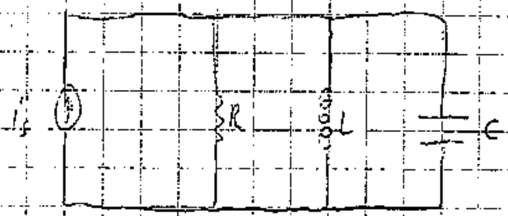
$V_c(0^+) = 0$

$\int_{0^-}^{0^+}\ddot{V}_c(t)dt + 2\alpha\int_{0^-}^{0^+}\dot{V}_c(t)dt + \omega_0^2\int_{0^-}^{0^+}V_c(t)dt = \frac{1}{C}\int_{0^-}^{0^+}\delta(t)dt$

$[\dot{V}_c(0^+) - \dot{V}_c(0^-)] + 2\alpha[V_c(0^+) - V_c(0^-)] + \omega_0^2\int_{0^-}^{0^+}V_c(t)dt = \frac{1}{C}$

$\dot{V}_c(0^+) + 2\alpha(0-0) + \omega_0^2(0) = \frac{1}{C}$

Impulse Response (Zero-State Analysis)

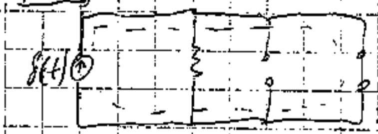


$(D^2 + 2\alpha D + w_0^2)\, V_c(t) = \frac{1}{C}\frac{d}{dt} i_s(t)$

$\left.\begin{array}{l} V_c(0^-) = 0 \\ \dot{V}_c(0^-) = 0 \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} V_c(0^-) = 0 \\ I_L(0) = 0 \end{array}\right.$

$i_s(t) \rightarrow \delta(t) \Rightarrow (D^2 + 2\alpha D + w_0^2)\, V_c(t) = \frac{1}{C}\underbrace{\dot{\delta}(t)}_{\text{doublet}}$

maths / circuit

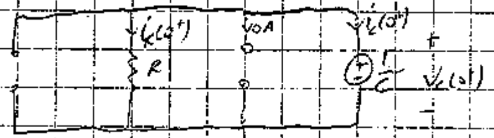


$0^- < t < 0^+$

$i_c(t) = \delta(t) \rightarrow V_c(0^+) = \frac{1}{C}\int_{0^-}^{0^+} i_c(\tau)\,d\tau = \frac{1}{C}$

$V_L(t) = 0V \rightarrow I_L(0^+) = \frac{1}{L}\int_{0^-}^{0^+} V_L(\tau)\,d\tau = 0$

at $t = 0^+$



$i_R(0^+) = \frac{1}{CR}$

$i_c(0^+) = -\frac{1}{CR}$

$C\dot{V}_c(0^+) = -\frac{1}{CR} \Rightarrow \dot{V}_c(0^+) = \frac{-1}{C^2 R}$

for $t > 0^+$

$(D^2 + 2\alpha D + w_0^2)\, V_c(t) = 0$

$V_c(0^+) = \frac{1}{C} \qquad \dot{V}_c(0^+) = \frac{-1}{RC^2}$

$V_c^{\text{impulse}}(t) = h(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

Maths $(D^2 + 2\alpha D + w_0^2)\, V_c(t) = \frac{1}{C}\dot{\delta}(t) \quad (I)$

$\ddot{V}_c(t) \propto \dot{\delta}(t) \qquad \dot{V}_c(t) \propto \delta(t) \rightarrow \dot{V}_c(t) \propto \delta(t) \rightarrow V_c(t) \propto u(t)$ Bu integral neden sıfır!

Integrate (I) in between $\int_{0^-}^{0^+}$

$\int_{0^-}^{0^+} \ddot{V}_c(t)\,dt + 2\alpha\int_{0^-}^{0^+} \dot{V}_c(t)\,dt + w_0^2 \int_{0^-}^{0^+} V_c(t)\,dt = \frac{1}{C}\int_{0^-}^{0^+} \dot{\delta}(t)\,dt$

Impulse olduğu için $V_c(0^+) \neq V_c(0^-)$

bu bu integrali sıfırdan farklı yapmaz mı?

$[\dot{V}_c(0^+) - \dot{V}_c(0^-)] + 2\alpha[V_c(0^+) - V_c(0^-)] + w_0^2[0] = \frac{1}{C}\,0$

$\dot{V}_c(0^+) + \underbrace{\frac{1}{RC}}_{\rightarrow 2\alpha} V_c(0^+) = 0 \quad (1)$

$(D^2 + 2\alpha D + w_0^2)\, V_c(t) = \frac{1}{C} D\delta(t) \quad \Rightarrow$ Apply D^{-1} $\quad (D^{-1} = \int^t (\cdot)\,d\tau)$

$D^{-1}(D^2 + 2\alpha D + w_0^2)\, V_c(t) = \frac{1}{C} D^{-1} D\,\delta(t)$

$(D + 2\alpha + w_0^2 D^{-1})\, V_c(t) = \frac{1}{C}\delta(t)$ (New diff eqn)

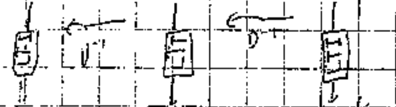
- Integrate the new DE between $\int_{0^-}^{0^+}$

$$\int_{0^-}^{0^+} \ddot{V}_C(t)\,dt + 2\alpha \int_{0^-}^{0^+} \dot{V}_C(t)\,dt + \omega_0^2 \int_{0^-}^{0^+}\int_{0^-}^{0} (V_C(\tau)\,d\tau)\,dt = \frac{1}{C}$$ $\quad \ddot{V}_C(t) \propto u(t)$

$\dot{V}_C(0^+) - \dot{V}_C(0^-) = \frac{1}{C}$

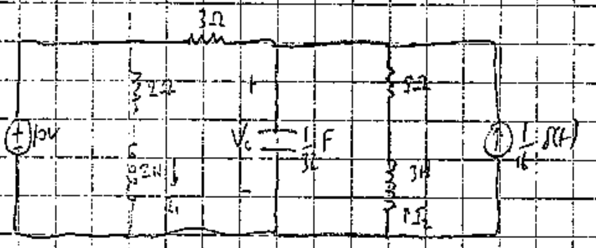
$\dot{V}_C(0^+) = \frac{1}{C}$ $\rightarrow$ $\boxed{V_C(0^+) = -\frac{1}{RC^2}}$ (from (1))

$r(t) \xrightarrow{D} u(t) \xrightarrow{D} \delta(t)$



$y^{ramp}(t) \xrightarrow{D} y^{step}(t) \xrightarrow{D} y^{impulse}(t)$

Ex//



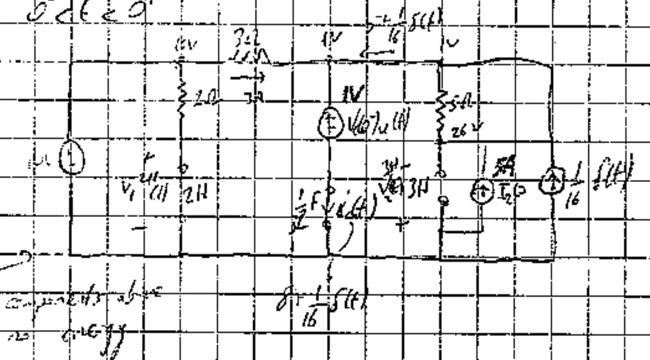
$V_C(0^-) = 1V$

$I_1(0^-) = 0A$

$I_2(0^-) = 5A$

Find all currents at $t=0^+$ and $t \to \infty$

$0^- < t < 0^+$



Dynamic components above have no energy

$V_1^{2H}(t) = 10\,V$

$V_2^{3H}(t) = 26\,V$

$V_C^{empty}(0^+) = \frac{1}{C}\int_{0^-}^{0^+} i_C(\tau)\,d\tau = 2\,V$

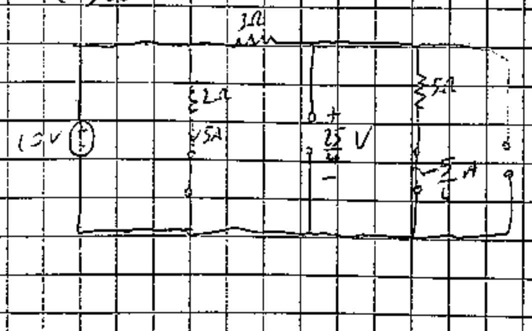
$i_1^{2H\,empty}(0^+) = \frac{1}{L_1}\int_{0^-}^{0^+} 10\,d\tau = 0\,A$

$i_2^{3H\,empty}(0^+) = \frac{1}{L_2}\int_{0^-}^{0^+} 26\,d\tau = 0\,A$

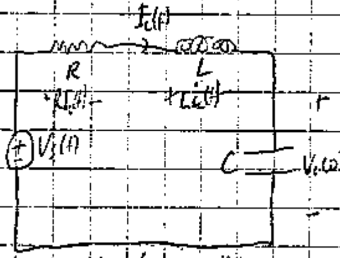
$V_C(0^+) = V_C^{empty}(0^+) + V_C(0^-) = 3V$

$I_1(0^+) = 0A \qquad I_2(0^+) = 5A$

$t \to \infty$



Series RLC



Dual of parallel RLC

Parallel RLC $\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)V_c(t) = \frac{1}{C}D\, i_s(t)$

dual

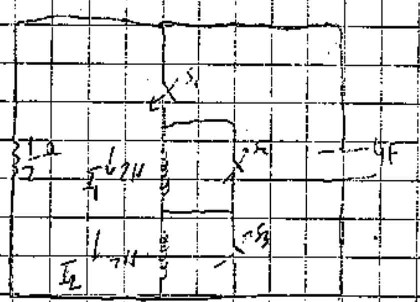
$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right)I_L(t) = \frac{1}{L}D\,V_s(t)$

$D \cdot \left| V_R + V_L + V_C = V_s(t) \right.$

$DV_R(t) + DV_L(t) + DV_C(t) = DV_s(t)$

$R\dot{I}_L(t) + L\ddot{I}_L(t) + \frac{1}{C}I_L(t) = \dot{V}_s(t)$

Ex:



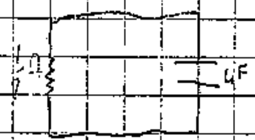
$I_1(0) = 1A$ $\quad I_2(0) = -2A$ $\quad V_c(0) = 5V$ $\quad V_c(t) = ?$

S_1 closes at $t = 4$ sec

S_2 opens at $t = 8$ sec

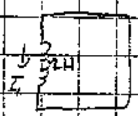
S_3 opens at $t = 12$ sec

1) $0 < t < 4$



$V_c(0^+) = 5V$

$V_c(t) = 5e^{-t/2}$



$I_1(t) = 1A$

$I_2(t) = -2A$

2) $4 < t < 8$

$V_c(4^-) = 5e^{-2} = \frac{5}{e^2}$

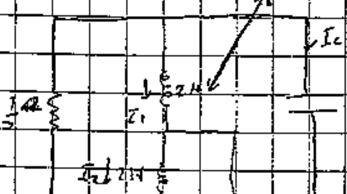
$V_c(4^+) = 0V$

$I_1(4^+) = I_1(4^-) = 1A$

$I_2(4^+) = I_2(4^-) = -2A$

S_1'in kapanması impulse yaratırken

3) $8 < t < 12$ $\quad S_2$'nin açılması neden yaratmıyor? $\rightarrow (D^2 + 2\alpha D + \omega_0^2)V_c(t) = 0$



$V_c(8^+) = 0V$

$I_1(8^+) = 1A$

$I_2(8^+) = -2A$

$V_c(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ $\quad \lambda_{1,2} = \left\{-\frac{1}{4} \mp j\frac{1}{4}\right\}$

$V_c^{general}(t) = e^{-\frac{t}{4}}\left(A\cos\left(\frac{t}{4}\right) + B\sin\left(\frac{t}{4}\right)\right)$

$V_c(t) = e^{-\frac{(t-8)}{4}}\left(A\cos\left(\frac{t-8}{4}\right) + B\sin\left(\frac{t-8}{4}\right)\right)u(t-8)$

parallel RLC

$2\alpha = \frac{1}{RC} = \frac{1}{2} \rightarrow \alpha = \frac{1}{4}$

$|\alpha| < \omega_0 \rightarrow$ underdamped response

$\omega_0^2 = \frac{1}{LC} = \frac{1}{8} \Rightarrow \omega_0 = \frac{1}{2\sqrt{2}}$

$A, B = ?$ $\quad$ Need $V_c(8^+) = ? = 0V$

$\dot{V}_c(8^+) = ?$

$4\dot{V}_c(8^+) = -I_1(8^+) = -1$

$\dot{V}_c(8^+) = -\frac{1}{4}$

(computation) $\Rightarrow A = 0$

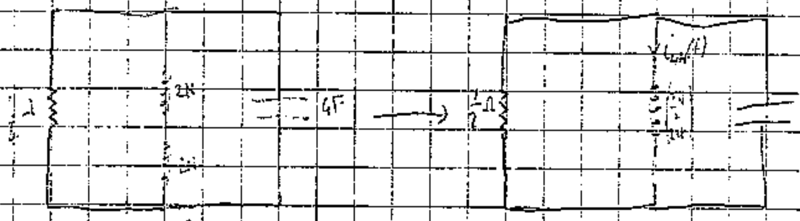
$B = -1 \Rightarrow V_c(t) = e^{-\frac{(t-8)}{4}}\left(-\sin\left(\frac{t-8}{4}\right)\right)$

$\dot{V}_c(12^-) = -e^{-1}\left(\sin\left(\frac{12-t}{4}\right)\right) = -0.31$

$I_1(12^-) = -C\dot{V}_c(12^-) - \frac{V_c(12^-)}{R}$

$I_1(12^-) = -4\dot{V}_c(12^-) - 2V_c(12^-) = \boxed{-\frac{1}{e}}$

$t > 12$



$i_{4H}(12^+) = \frac{L_1 I_{2H}(12^-) + L_2 I_{2H}(12^-)}{L_1 + L_2}$

$i_{4H}(12^+) = \frac{-\frac{1}{e} - 2}{2} = -\frac{1}{2e} - 1$

$V_c(12^+) = -0.31\,V$

$V_c(12^-) = -0.31\,V$

$I_1(12^-) = -\frac{1}{e}$ A

$I_2(12^-) = -2$ A

$(D^2 + 2\alpha D + \omega_0^2)\, V_c(t) = 0$

$\frac{1}{RC} = \frac{1}{2} \qquad \frac{1}{LC} = \frac{1}{16}$

$\alpha = \frac{1}{4} \qquad \omega_0 = \frac{1}{4}$ critically damped

$V_c(t) = Ae^{-\frac{1}{4}t} + Bt\, e^{-\frac{1}{4}t}$ $\qquad \dot{V}_c(12^+) = 0.45 \qquad V_c(12^+) = -0.31\,V$

$\boxed{V_c(t) = \left[Ae^{-\frac{1}{4}(t-12)} + B(t-12)e^{-\frac{1}{4}(t-12)}\right] u(t-12)} \longleftarrow A = -0.31 \qquad B = 0.37$